First Look: Boundary Conditions

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Boundary Conditions for DEs

When solving physical differential equations we recall that it is necessary to specify "boundary conditions".

Examples:

- Specifying temperature at ends of a thermally-conducting rod.
- Specifying heat flow rate (via gradient of temperature) on a rod.
- Specifying electric potential at a conducting wall.

Steady-state solutions, for example for Laplace or Poisson equations, can form a pure boundary-value problem.

Time dependent initial value-boundary value problems may also be solved, for example to the heat equation.

Dirichlet Conditions

Simply, Dirichlet Conditions involve direct specification of a quantity at a boundary x_0. I.e., temperature at a boundary.

$$\boxed{u(x_0, t) = T(t)}$$

Alternatively, for a steady-state problem:

$$\boxed{u(x_0) = T_0}$$

Neumann Conditions

Simply, Neumann Conditions involve specification of a quantity's derivative at a boundary x_0. I.e., gradient of temperature at a boundary.

$$\boxed{\frac{\partial u}{\partial x}(x_0, t) = \gamma(t)}$$

Alternatively, for a steady-state problem:

$$\boxed{\frac{\partial u}{\partial x}(x_0) = \gamma}$$

Waves on a String

As noted by *Potter*, a system of two PDEs describing advection of quantities *velocity* ($v = \partial\xi/\partial t$) and *angular displacement* ($\theta = \partial\xi/\partial t$) can be specified:

$$\frac{\partial v}{\partial t} + c\frac{\partial \theta}{\partial x} = 0$$
$$\frac{\partial \theta}{\partial t} + c\frac{\partial v}{\partial x} = 0$$

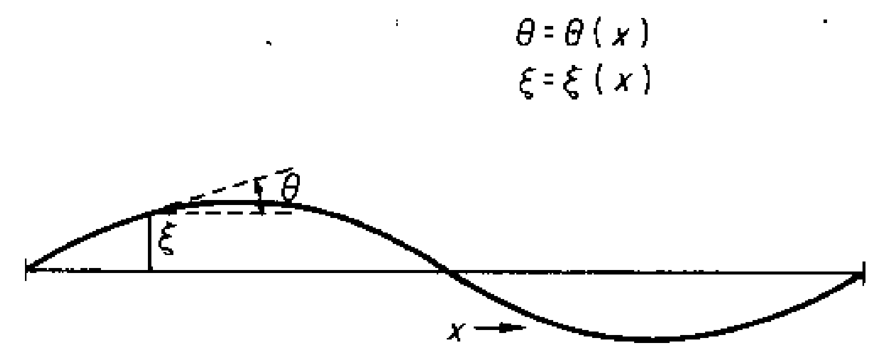

Figure 3.5. Wave on a string. $\xi(x, t)$ is the displacement from equilibrium while $\theta(x, t)$ is the angular displacement

For our coded examples, quantity *u* may refer to the "velocity" on the string, where quantity *v* may refer to the "angular displacement".

Linear Acoustics Equations

The equations of linear acoustics describe conservation of mass density (continuity) and conservation of linear momentum, which in the following system have been coupled via an ideal gas equation of state:

$$\frac{\partial p}{\partial t} + \gamma p_0 \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + \frac{1}{\rho_0} \frac{\partial p}{\partial x} = 0$$

Here, *p* is pressure, *u* is velocity, *rho*$_0$ is density, and *gamma* is the ratio of specific heats.

Physical Boundary Conditions

For waves on a stretched string:

- **Velocity** at fixed end points must go to zero, hence the string is not free to move at these boundaries.
- **Angular displacement** may be non-zero at fixed end points, hence this quantity can vary freely.

For acoustic waves in a confined space:

- **Velocity** normal to surface at the fixed boundary must go to zero, hence the gas is not free to pass.
- **Pressure** may be non-zero at the fixed boundary, hence this quantity may vary freely.

Numerical Boundary Conditions

Reflective Boundary:
Example 1D waves on a string, or acoustic waves:

- For quantity representing pressure (or angular displacement), allow boundary cell or grid-point value to be equal to first inner cell or grid-point value.
- For quantity representing velocity (or velocity), allow boundary cell to be equal in magnitude, opposite in sign (direction) to the first inner cell or grid-point.

$$\boxed{\begin{aligned} p_0^n &= p_1^n \\ v_0^n &= -v_1^n \end{aligned}}$$

Effectively and intuitively, this "launches" an opposing reflection wave to replace the incident wave.

Numerical Boundary Conditions

Open Boundary:
Example 1D waves on a string, or acoustic waves:

- For quantity representing pressure (or angular displacement), allow boundary cell or grid-point value to be equal to first inner cell or grid-point value.
- For quantity representing velocity (or velocity), allow boundary cell to be equal to the first inner cell or grid-point.

$$\boxed{\begin{aligned} p_0^n &= p_1^n \\ v_0^n &= v_1^n \end{aligned}}$$

Effectively, this prevents influx of additional information as the wave flows outward.

These "extrapolated" conditions work well for practical solutions in FVMs, as we'll see later.

Numerical Boundary Conditions

Periodic Boundaries:
No boundaries – Infinite domain

- Left-most boundary cells equal to right-most domain cells, for all quantities solved.
- Right-most boundary cells equal to left-most domain cells, for all quantities solved.

$$\boxed{\begin{aligned} p_0^n &= p_{J-1}^n \\ v_0^n &= v_{J-1}^n \\ p_J^n &= p_1^n \\ v_J^n &= v_1^n \end{aligned}}$$

Effectively, what goes out on one side comes back in the other. Add additional “ghost” cells as needed.

These work brilliantly for testing and ideal studies, but are not “physical” in many systems.

Numerical Boundary Conditions

Forced Boundaries:
A boundary is a great place to inject source terms!

$$u(x_0, t) = U_0 \sin(\omega t)$$

Or:

$$p(x_0, t) = P_0 \sin(\omega t)$$

These can be combined, linearly, with any of the above.