Time-Splitting Methods; Two-Dimensional Problems

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Advection Equation

The general continuity equation (and conservative advection equation is given by:

$$\frac{\partial Q}{\partial t} + \nabla \cdot (Q\vec{v}) = 0$$

For non-divergent flow where $\nabla \cdot \vec{v} = 0$, the simple advection equation may be used:

$$\frac{dQ}{dt} = \boxed{\frac{\partial Q}{\partial t} + \vec{v} \cdot \nabla Q = 0}$$

Or, in one dimension: $$\frac{\partial Q}{\partial t} + u\frac{\partial Q}{\partial x} = 0$$

Diffusion and Heat Equations

The diffusion equation describes the diffusive flux of some quantity over time. Physically, it is a result of Fick's law:

$$\vec{F} = -\kappa \nabla u$$

This flux of the quantity being diffused is proportional to the gradient of the quantity as it varies in space. Therefore, after substitution into a continuity equation:

$$\frac{\partial u}{\partial t} = -\nabla \cdot (-\kappa \nabla u) = \nabla \cdot (\kappa \nabla u)$$

If κ does not vary spatially, the heat equation may be used:

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u$$

$$\boxed{\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0}$$

Advection-Diffusion Equation

Many simple physical systems involve both advective transport terms (also referred to as “convective” terms) and diffusion terms. They do not behave as specifically “hyperbolic” or “parabolic”.

$$\frac{\partial Q}{\partial t} - \nabla \cdot (-\kappa \nabla Q + \vec{v} Q) = 0$$

In 1D, and assuming κ and *u* do not vary, this becomes:

$$\frac{\partial Q}{\partial t} + u \frac{\partial Q}{\partial x} = \kappa \frac{\partial^2 Q}{\partial x^2}$$

(Note that *u, v, w* are East-West, North-South, and Up-Down velocities, respectively.)

Time-Splitting Methods

The system can be split into hyperbolic (advection) and parabolic (diffusion) equations. **Why?** Each equation may then be solved in an optimal manner.

Step One, *Solve*:

$$\boxed{\frac{\partial Q}{\partial t} + u\frac{\partial Q}{\partial x} = 0}$$

Step Two, *Solve*:

$$\boxed{\frac{\partial Q}{\partial t} = -\kappa\frac{\partial^2 Q}{\partial x^2}}$$

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Advection Equation in 2D

The general continuity equation (and conservative advection equation is given by:

$$\frac{\partial Q}{\partial t} + \nabla \cdot (Q\vec{v}) = 0$$

For non-divergent flow where $\nabla \cdot \vec{v} = 0$, the simple advection equation may be used:

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \vec{v} \cdot \nabla Q = 0$$

Or, in two dimensions: $$\boxed{\frac{\partial Q}{\partial t} + v_x \frac{\partial Q}{\partial x} + v_y \frac{\partial Q}{\partial y} = 0}$$

Heat Equations

Starting with the diffusion equation:

$$\frac{\partial u}{\partial t} = -\nabla \cdot (-\kappa \nabla u) = \nabla \cdot (\kappa \nabla u)$$

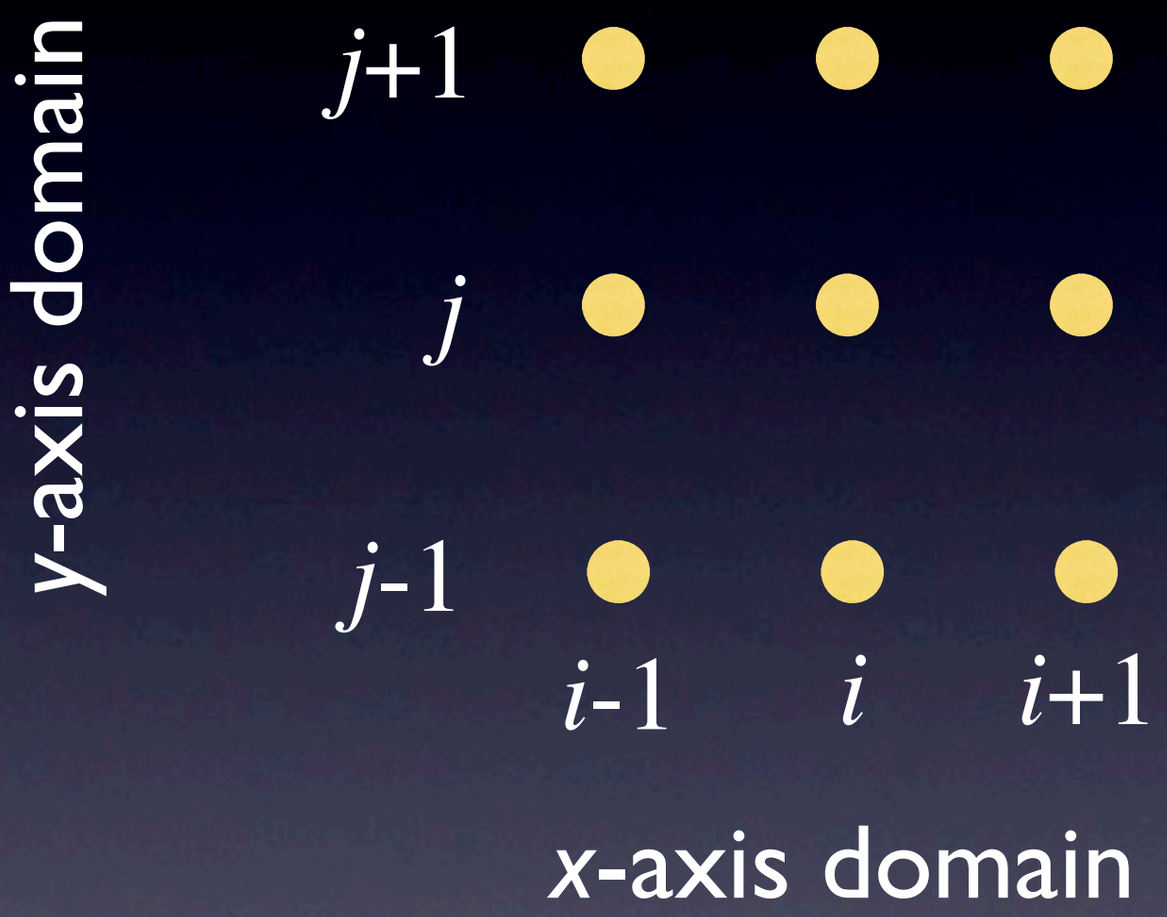
If κ does not vary spatially, the heat equation is given:

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u$$

Or, in two dimensions:

$$\boxed{\frac{\partial Q}{\partial t} - \kappa \frac{\partial^2 Q}{\partial x^2} - \kappa \frac{\partial^2 Q}{\partial y^2} = 0}$$

2D Finite Difference Methods



Indices are given by: n, i, j, k

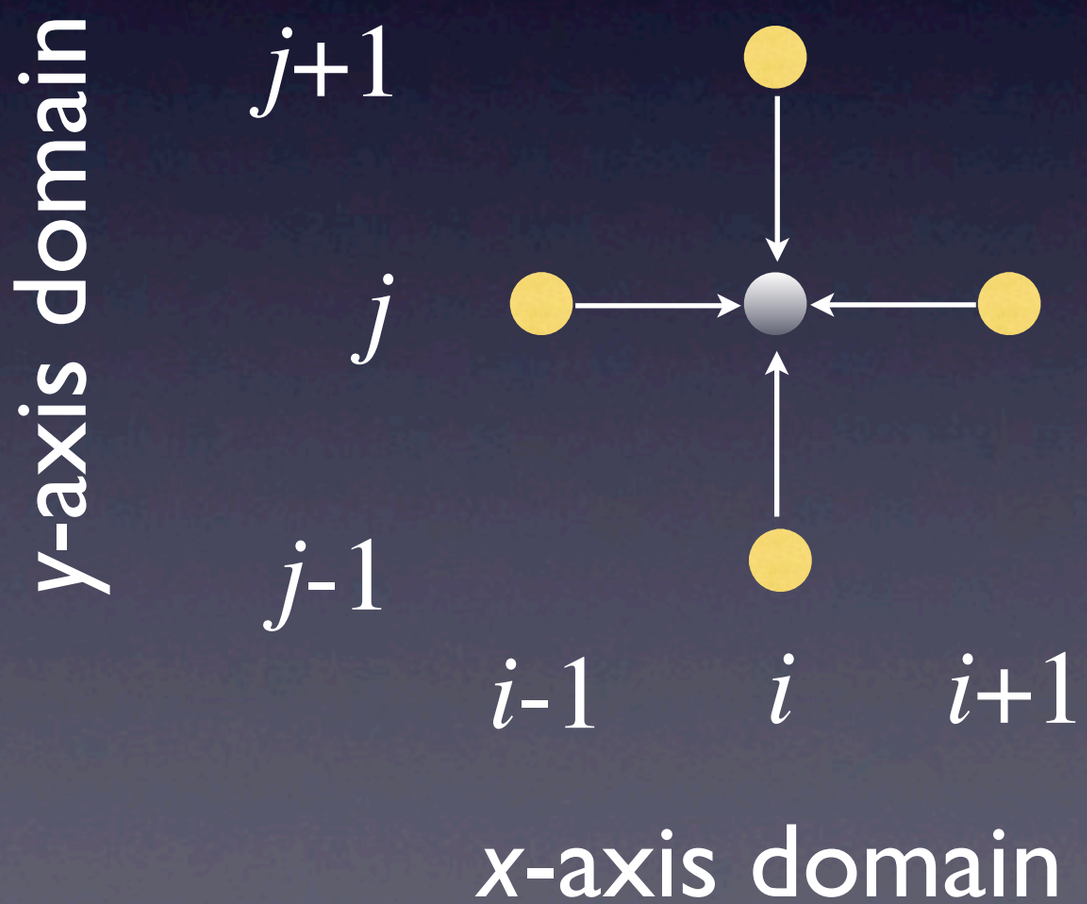
Where $t=n*\Delta t$, $x=i*\Delta x$, $y=j*\Delta y$, $z=k*\Delta z$

Spatial and Temporal **Step Sizes** are given by Δt,Δx,Δy,Δz.

Lax-Friedrichs Method in 2D

For Advection:

$$u_{i,j}^{n+1} = \frac{1}{4}(u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n) - \frac{\Delta t}{2\Delta x} v_x (u_{i+1,j}^n - u_{i-1,j}^n) - \frac{\Delta t}{2\Delta y} v_y (u_{i,j+1}^n - u_{i,j-1}^n)$$



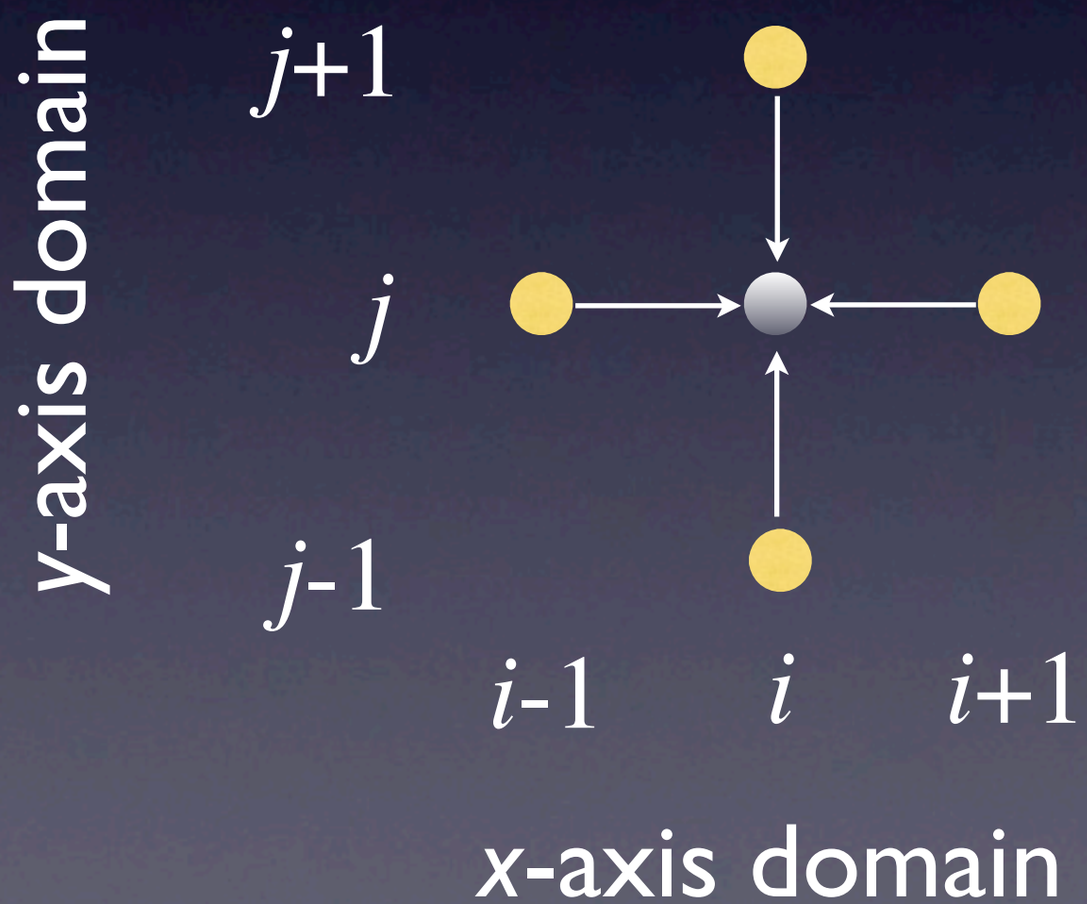
For equal grid-spacing:

$$\boxed{\Delta t \leq \frac{\Delta x (= \Delta y)}{\sqrt{v_x^2 + v_y^2}\sqrt{2}}}$$

Lax-Friedrichs Method in 2D

For Fluxes:

$$u_{i,j}^{n+1} = \frac{1}{4}(u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n) - \frac{\Delta t}{2\Delta x}(F_{i+1,j}^n - F_{i-1,j}^n) - \frac{\Delta t}{2\Delta y}(F_{i,j+1}^n - F_{i,j-1}^n)$$



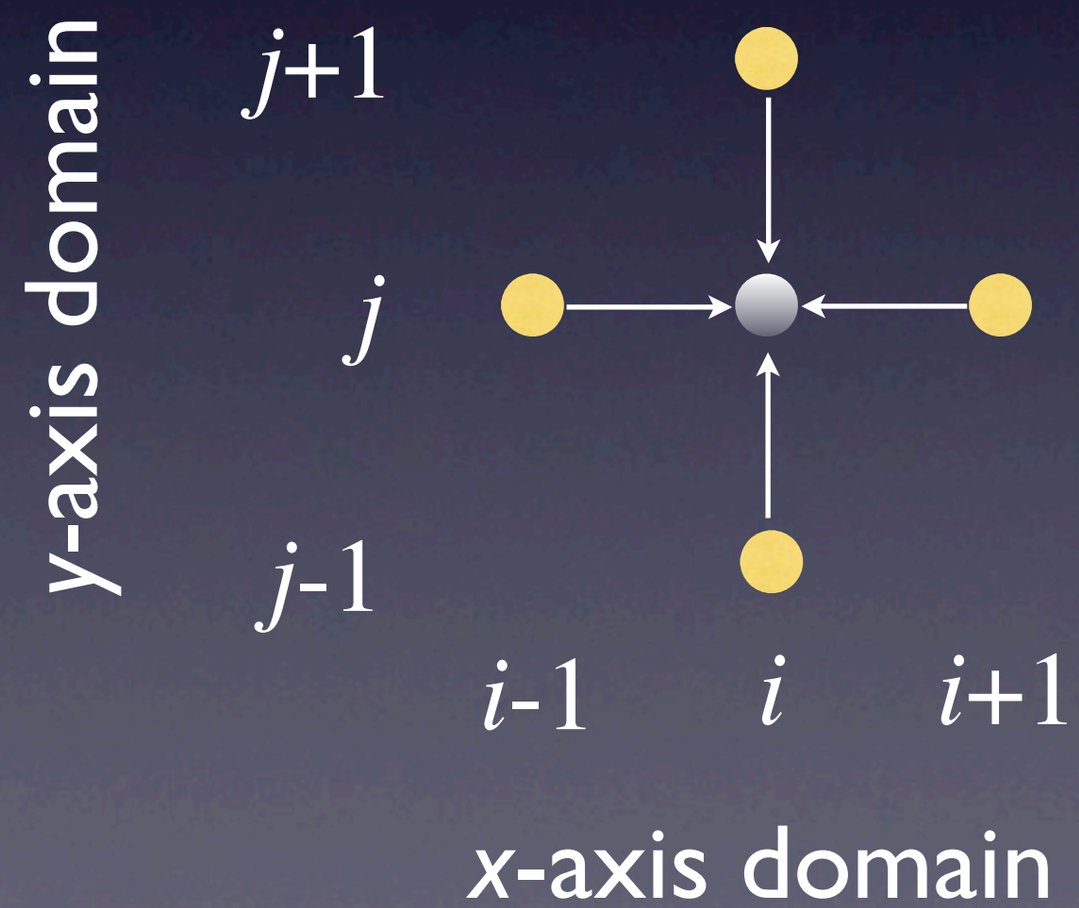
General stability constraint for N-dimensional problems of equal grid-spacing:

$$\Delta t \leq \frac{\Delta \text{space}}{|\vec{v}|\sqrt{N}}$$

Explicit Forward Euler in 2D

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \frac{\kappa \Delta t}{(\Delta x)^2}(u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n})$$
$$+ \frac{\kappa \Delta t}{(\Delta y)^2}(u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n})$$

Determining stability is left as an exercise to the reader!



Locally One Dimensional (LOD) Methods

For parabolic equations, where implicit solutions (such as Crank-Nicolson) are often used, it is beneficial to treat each dimension separately. This is a spatial equivalent time-splitting (fractional step) methods. Note that the following expression is similar to the previous explicit expression:

$$u^*_{i,j} = u^n_{i,j} + \frac{\kappa \Delta t}{(\Delta x)^2}(u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j})$$

$$u^{n+1}_{i,j} = u^*_{i,j} + \frac{\kappa \Delta t}{(\Delta y)^2}(u^*_{i,j+1} - 2u^*_{i,j} + u^*_{i,j-1})$$

Although obvious for the explicit method, the same approach can be applied to Crank-Nicolson...