Time-Splitting Methods; Two-Dimensional Problems

Jonathan B. Snively
Embry-Riddle Aeronautical University
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Advection Equation

The general continuity equation (and conservative advection equation) is given by:

$$\frac{\partial Q}{\partial t} + \nabla \cdot (Q \vec{v}) = 0$$

For non-divergent flow where $\nabla \cdot \vec{v} = 0$, the simple advection equation may be used:

$$\frac{dQ}{dt} = \left[ \frac{\partial Q}{\partial t} + \vec{v} \cdot \nabla Q \right] = 0$$

Or, in one dimension:

$$\frac{\partial Q}{\partial t} + u \frac{\partial Q}{\partial x} = 0$$
Diffusion and Heat Equations

The diffusion equation describes the diffusive flux of some quantity over time. Physically, it is a result of Fick’s law:

\[ \vec{F} = -\kappa \nabla u \]

This flux of the quantity being diffused is proportional to the gradient of the quantity as it varies in space. Therefore, after substitution into a continuity equation:

\[ \frac{\partial u}{\partial t} = -\nabla \cdot (-\kappa \nabla u) = \nabla \cdot (\kappa \nabla u) \]

If \( \kappa \) does not vary spatially, the heat equation may be used:

\[ \frac{\partial u}{\partial t} = \kappa \nabla^2 u \]

\[ \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0 \]
Advection-Diffusion Equation

Many simple physical systems involve both advective transport terms (also referred to as “convective” terms) and diffusion terms. They do not behave as specifically “hyperbolic” or “parabolic”.

\[
\frac{\partial Q}{\partial t} - \nabla \cdot (-\kappa \nabla Q + \vec{v}Q) = 0
\]

In 1D, and assuming \( \kappa \) and \( u \) do not vary, this becomes:

\[
\frac{\partial Q}{\partial t} + u \frac{\partial Q}{\partial x} = \kappa \frac{\partial^2 Q}{\partial x^2}
\]

(Note that \( u, v, w \) are East-West, North-South, and Up-Down velocities, respectively.)
Time-Splitting Methods

The system can be split into hyperbolic (advection) and parabolic (diffusion) equations. Why? Each equation may then be solved in an optimal manner.

Step One, Solve:
\[
\frac{\partial Q}{\partial t} + u \frac{\partial Q}{\partial x} = 0
\]

Step Two, Solve:
\[
\frac{\partial Q}{\partial t} = -\kappa \frac{\partial^2 Q}{\partial x^2}
\]
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Advection Equation in 2D

The general continuity equation (and conservative advection equation is given by:

$$\frac{\partial Q}{\partial t} + \nabla \cdot (Q\vec{v}) = 0$$

For non-divergent flow where $\nabla \cdot \vec{v} = 0$, the simple advection equation may be used:

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \vec{v} \cdot \nabla Q = 0$$

Or, in two dimensions:

$$\frac{\partial Q}{\partial t} + v_x \frac{\partial Q}{\partial x} + v_y \frac{\partial Q}{\partial y} = 0$$
Heat Equations

Starting with the diffusion equation:

\[ \frac{\partial u}{\partial t} = -\nabla \cdot (-\kappa \nabla u) = \nabla \cdot (\kappa \nabla u) \]

If \( \kappa \) does not vary spatially, the heat equation is given:

\[ \frac{\partial u}{\partial t} = \kappa \nabla^2 u \]

Or, in two dimensions:

\[ \frac{\partial Q}{\partial t} - \kappa \frac{\partial^2 Q}{\partial x^2} - \kappa \frac{\partial^2 Q}{\partial y^2} = 0 \]
2D Finite Difference Methods

Indices are given by: $n, i, j, k$

Where $t = n^*\Delta t$, $x = i^*\Delta x$, $y = j^*\Delta y$, $z = k^*\Delta z$

Spatial and Temporal Step Sizes are given by $\Delta t, \Delta x, \Delta y, \Delta z$. 
Lax-Friedrichs Method in 2D

For Advection:

\[
\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = \frac{1}{4} \left( u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n \right) - \frac{\Delta t}{2\Delta x} v_x \left( u_{i+1,j}^n - u_{i-1,j}^n \right) - \frac{\Delta t}{2\Delta y} v_y \left( u_{i,j+1}^n - u_{i,j-1}^n \right)
\]

For equal grid-spacing:

\[
\Delta t \leq \frac{\Delta x (= \Delta y)}{\sqrt{v_x^2 + v_y^2 \sqrt{2}}}
\]
Lax-Friedrichs Method in 2D

For Fluxes:

\[ u_{i,j}^{n+1} = \frac{1}{4} (u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n) \]

\[ -\frac{\Delta t}{2\Delta x} (F_{i+1,j}^n - F_{i-1,j}^n) \]

\[ -\frac{\Delta t}{2\Delta y} (F_{i,j+1}^n - F_{i,j-1}^n) \]

General stability constraint for N-dimensional problems of equal grid-spacing:

\[ \Delta t \leq \frac{\Delta \text{space}}{|\vec{v}| \sqrt{N}} \]
Explicit Forward Euler in 2D

\[ u_{i,j}^{n+1} = u_{i,j}^n + \frac{\kappa \Delta t}{(\Delta x)^2} (u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n) \]
\[ + \frac{\kappa \Delta t}{(\Delta y)^2} (u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n) \]

Determining stability is left as an exercise to the reader!
Locally One Dimensional (LOD) Methods

For parabolic equations, where implicit solutions (such as Crank-Nicolson) are often used, it is beneficial to treat each dimension separately. This is a spatial equivalent time-splitting (fractional step) methods. Note that the following expression is similar to the previous explicit expression:

\[ u_{i,j}^* = u_{i,j}^n + \frac{\kappa \Delta t}{(\Delta x)^2} (u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n) \]

\[ u_{i,j}^{n+1} = u_{i,j}^* + \frac{\kappa \Delta t}{(\Delta y)^2} (u_{i,j+1}^* - 2u_{i,j}^* + u_{i,j-1}^*) \]

Although obvious for the explicit method, the same approach can be applied to Crank-Nicolson...