

# Molecular Viscosity, Thermal Conduction, Direct Numerical Simulation

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# Contents

- **Molecular Viscosity**
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# Molecular Viscosity

Viscosity in a gas arises through collisions rather than internal friction, such that it tends to increase with temperature. For “air”, the dynamic viscosity is given by:

$$\eta_a \simeq 1.8325 \times 10^{-5} \left( \frac{416.16}{T + 120} \right) \left( \frac{T}{296.16} \right)^{1.5}$$

The *kinematic* viscosity is given by:

$$\nu = \frac{\eta}{\rho}$$

Kinematic viscosity describes diffusivity of momentum, and is inversely proportional to density (increases with decreasing density, and hence altitude).

# Effects of Viscosity

The Momentum Equation, excluding “bulk” viscosity, in vector form (vs. tensor form) [Gossard and Hooke, 1975]:

$$\frac{\partial(\rho\vec{u})}{\partial t} + \nabla \cdot (\rho\vec{u}\vec{u}) = -\nabla p + \rho g + \underbrace{(\nabla \cdot \eta \nabla)\vec{u} + (\nabla \eta \times \nabla) \times \vec{u} + \frac{1}{3}\nabla(\eta \nabla \cdot \vec{u})}_{\text{Viscous Force Density}}$$

Fortunately, for most atmospheric applications, the dynamic viscosity does not vary significantly with altitude, and a simpler form of the viscous force term may be used:

$$\frac{\partial(\rho\vec{u})}{\partial t} + \nabla \cdot (\rho\vec{u}\vec{u}) = -\nabla p + \rho g + \eta \nabla^2 \vec{u} + \frac{1}{3}\nabla(\eta \nabla \cdot \vec{u})$$

# Effects of Viscosity: Incompressible Case

For the compressible case with varying dynamic viscosity the “viscous force density” is given by:

$$(\nabla \cdot \eta \nabla) \vec{u} + (\nabla \eta \times \nabla) \times \vec{u} + \frac{1}{3} \nabla (\eta \nabla \cdot \vec{u})$$

For the incompressible case, we can drop the  $\text{div}(\mathbf{v})$  term:

$$(\nabla \cdot \eta \nabla) \vec{u} + (\nabla \eta \times \nabla) \times \vec{u}$$

For the constant-viscosity case:

$$\boxed{\eta \nabla^2 \vec{u}}$$

**(not bad!)**

# Example: 1D Acoustics

## Recall from your project assignment:

**1) Using a time-split method, incorporate viscosity** in the system describing linear acoustic wave propagation in one dimension. As a hint, consider that the momentum equation:

$$\rho \frac{\partial \vec{V}}{\partial t} = -\nabla p + \mu \nabla^2 v + \frac{1}{3} \mu \nabla (\nabla \cdot \vec{v})$$

For an incompressible fluid, the last term would become zero (non-divergent), however for acoustic waves it cannot be neglected. In 1D, the momentum equation becomes:

$$\frac{\partial u}{\partial t} + \frac{1}{\rho_0} \frac{\partial p}{\partial x} = \frac{4}{3} \frac{\mu}{\rho_0} \frac{\partial^2 u}{\partial x^2}$$

We can define a “kinematic viscosity”  $\nu$  in terms of “dynamic viscosity”  $\mu$ , where  $\nu = \mu / \rho_0$ .

In the “CLAW711” example, the viscosity input in `setprob.data` is given in terms of *kinematic* viscosity at ground level (which is everywhere, if `grav=0`). So, it would be perhaps easiest to specify kinematic viscosity as input to your solution! |

Here, since the problem is given in 1D, we can combine the two viscous terms into a single second-order term.

# Contents

- Molecular Viscosity
- **Thermal Conduction**
- Direct Numerical Simulation

# Thermal Conductivity

Likewise, can define a thermal conductivity “kappa” for the atmosphere, which is related to the molecular viscosity via the Prandtl number:

$$Pr = \frac{C_p \eta}{\kappa}$$

where a typical  $Pr \approx 0.7$  for the atmosphere.

Note that we can also define a thermal diffusivity, equivalent to the molecular diffusivity (kinematic viscosity):

$$\alpha = \frac{\kappa}{(\rho C_p)} \quad \text{Hence,} \quad Pr = \frac{\nu}{\alpha}$$

# Thermal Conductivity

Effectively, the addition of thermal conductivity provides diffusion of temperature, e.g., the heat equation:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

By comparison, for momentum diffusion, velocity is “diffused” in a similar manner:

$$\frac{\partial \vec{u}}{\partial t} = \nu \nabla^2 \vec{u}$$

Note that *thermal diffusivity* and *momentum diffusivity* (kinematic viscosity) both have the same units [m<sup>2</sup>/s].

# Contents

- Molecular Viscosity
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# Direct Numerical Simulation

*Whether or not turbulence is achieved depends on the number of  $\Delta X$ 's that can be jammed into one's computer. The question of directly simulating turbulent flows is largely one of economics, clever programming, and access to a big machine.*

*[Fox and Lilly, 1972]*

**The Goal:** To achieve sufficiently high resolution such that all turbulent processes above the “Kolmogorov Scale” are properly simulated.

*This is rarely achievable for any practical problem.*

# Direct Numerical Simulation

At very small spatial scales, turbulence is dissipated rapidly into heat by molecular viscosity. The Kolmogorov Length Scale is given by:

$$\eta = \left( \frac{\nu^3}{\epsilon} \right)^{1/4} \quad \text{NOTE: My apologies for reusing "eta"!$$

Where  $\nu$  is the kinematic viscosity and  $\epsilon$  is the kinetic energy dissipation rate [*Kundu and Cohen, 2002*]:

$$\epsilon \sim \frac{u'^3}{l}$$

*← RMS velocity of eddy motions*

*← Length scale of eddy motions*

# Direct Numerical Simulation

The turbulent Reynolds Number describes the relative effects of viscosity on the flow field, i.e., it compares the eddy scales of interest to the effects of viscosity.

$$Re = \frac{u'l}{\nu}$$

To achieve DNS (capturing all relevant scales) the number of grid points in one direction is on the order of  $N=Re^{3/4}$

Note that such simulations are necessarily 3-dimensional!

**Example:** DNS at a Reynold's number of 10000 requires 1000 grid cells in each dimension.