Flux Limiters And TVD Criteria

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- **Total Variation Diminishing Criteria**
- Summary of Flux Limiters

Total-Variation

For a quantity Q specified on a numerical grid, the “total variation” of the solution at a given time is given by:

$$TV(Q) = \sum_{i=1}^{N-1} |Q_{i+1} - Q_i|$$

If the system is not to develop nonphysical maxima or minima (indicating oscillations, for example), this must remain constant or decreasing over each time step:

$$\boxed{TV(Q^{n+1}) \leq TV(Q^n)}$$

[We follow the descriptions of *LeVeque*, 2002; *Durran*, 2010; both provide excellent introductions.]

“Monotonicity” of a Method

Methods which satisfy TVD criteria also preserve monotonicity of solutions. A scheme is **monotone** if:

Given: $Q_i^n \geq Q_{i+1}^n$

Then: $Q_i^{n+1} \geq Q_{i+1}^{n+1}$

Pure second order methods cannot be monotone; indeed, any linear monotone scheme is at best first-order accurate.

We seek to use **flux limiters** to control the inclusion of higher-order terms, creating schemes with greater than first order accuracy, while preserving TVD

TVD as a Goal for FVMs

TVD criteria is independent of a scheme's stability, and is not a sufficient condition for a "satisfactory" solution.

Indeed, useful higher order methods do not satisfy TVD alone, and many TVD low-order methods are too diffusive for practical application. (Hence interest in flux limiters...)

"High resolution" TVD methods will be of greatest utility for systems with initially discontinuous or steep solutions, or those apt to produce shocks or steep solutions.

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Flux Limited Methods

We will focus first on Flux-Limited flux form finite difference methods, using fluxes defined by combinations of low (F_L) and high (F_H) order flux approximations.

$$\boxed{F_{i-1/2} = F^L_{i-1/2} + \phi_{i-1/2}(F^H_{i-1/2} - F^L_{i-1/2})}$$

Here, the "flux limiter" is given by *phi*.

Flux limiters seek to maintain TVD behavior of the solution while achieving greater than first order accuracy.

Flux Limited Methods

$$\boxed{F_{i-1/2} = F^L_{i-1/2} + \phi_{i-1/2}(F^H_{i-1/2} - F^L_{i-1/2})}$$

We can explore a few limiting cases:

$\phi_{i-1/2} = 0$ *Defaults to low-order method.*

$\phi_{i-1/2} = 1$ *Defaults to high-order method.*

For discussion, we will assume that F^H is defined by a **Lax-Wendroff** flux, F^L is defined by an **Upwind** flux (i.e., our example code online).

Calculations for Flux Limiter

The flux limiter function will be dependent on the ratio of solution slopes across an interface between two cells:

$$\theta^n_{i+1/2} = \frac{Q^n_i - Q^n_{i-1}}{Q^n_{i+1} - Q^n_i}$$

Note that this expression is valid only for a positive upwind direction (c>0), where if c<0 the ratio of slopes becomes:

$$\theta^n_{i+1/2} = \frac{Q^n_{i+2} - Q^n_{i+1}}{Q^n_{i+1} - Q^n_i}$$

The flux limiter value *phi* is a function of *theta*: $\phi(\theta^n_{i+1/2})$

Flux Limited Methods

$$\boxed{F_{i-1/2} = F^L_{i-1/2} + \phi_{i-1/2}(F^H_{i-1/2} - F^L_{i-1/2})}$$

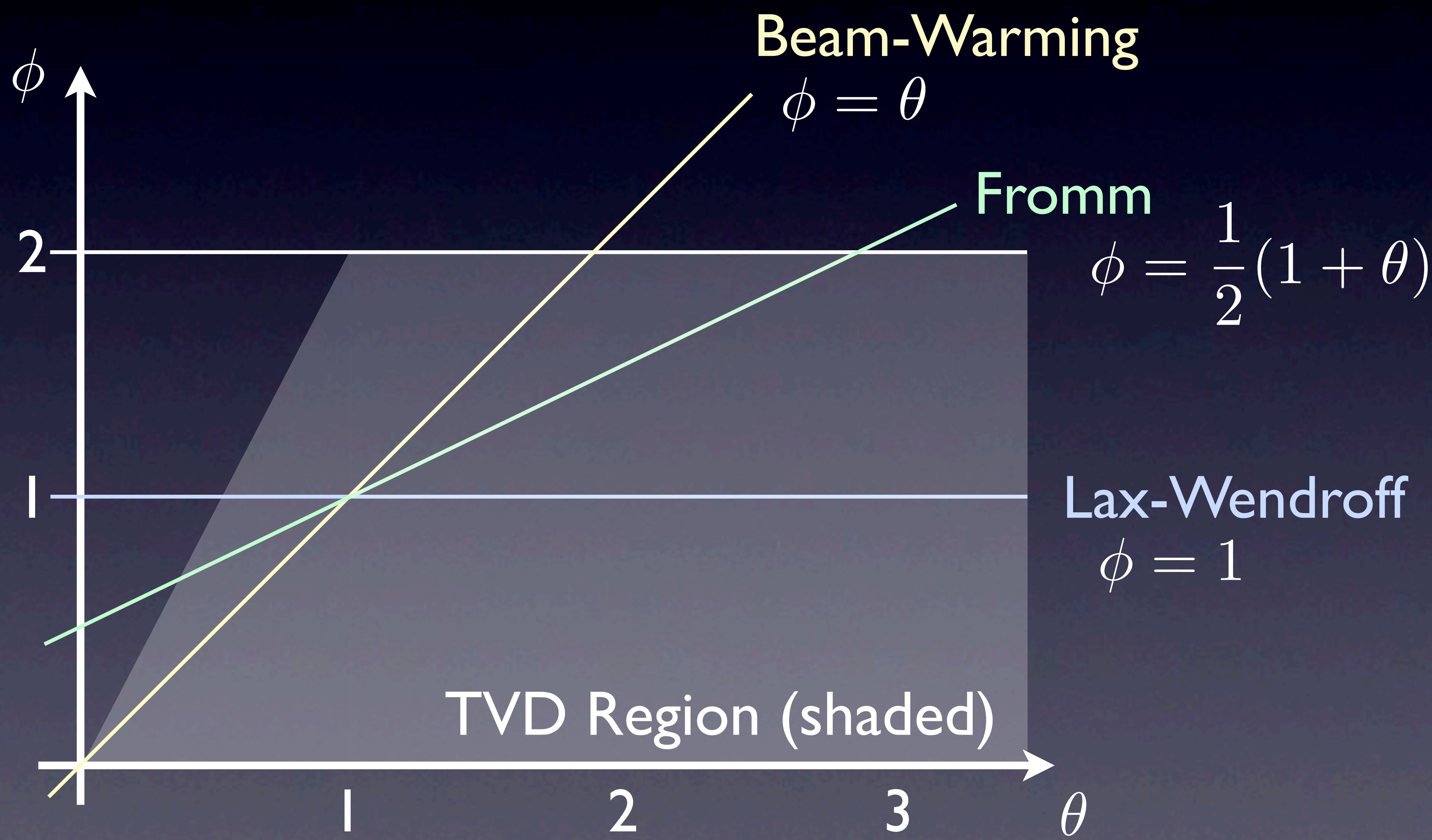
We can explore a few more limiting cases:

$$\phi_{i-1/2} = \theta$$ *Beam-Warming Method*

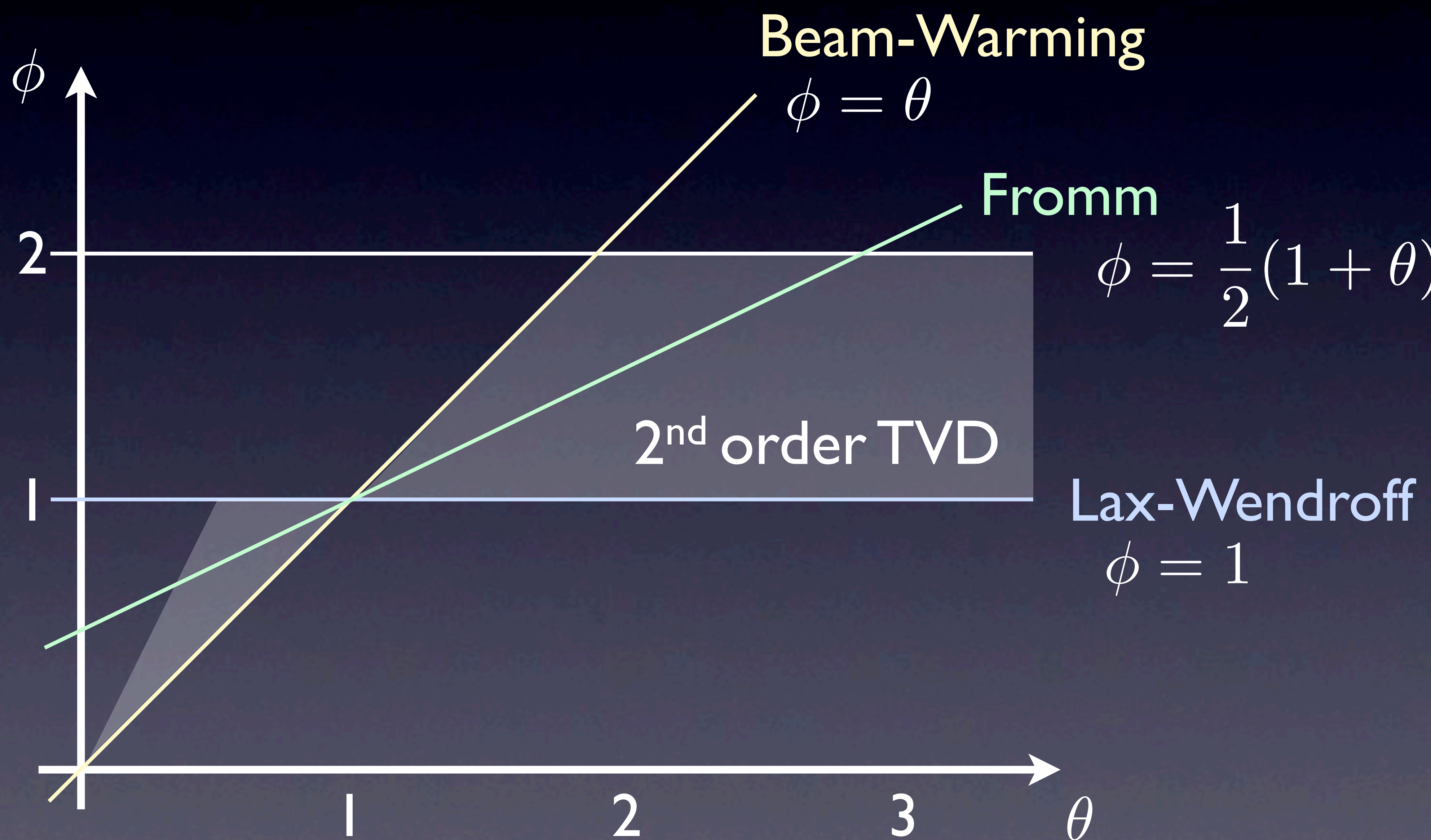
$$\phi_{i-1/2} = \frac{1}{2}(1+\theta)$$ *Fromm Method*

Beam-warming takes upwind differences, Fromm takes centered differences (asymmetrically), Lax-Wendroff takes downwind differences (symmetrically).

Plotting Phi vs. Theta

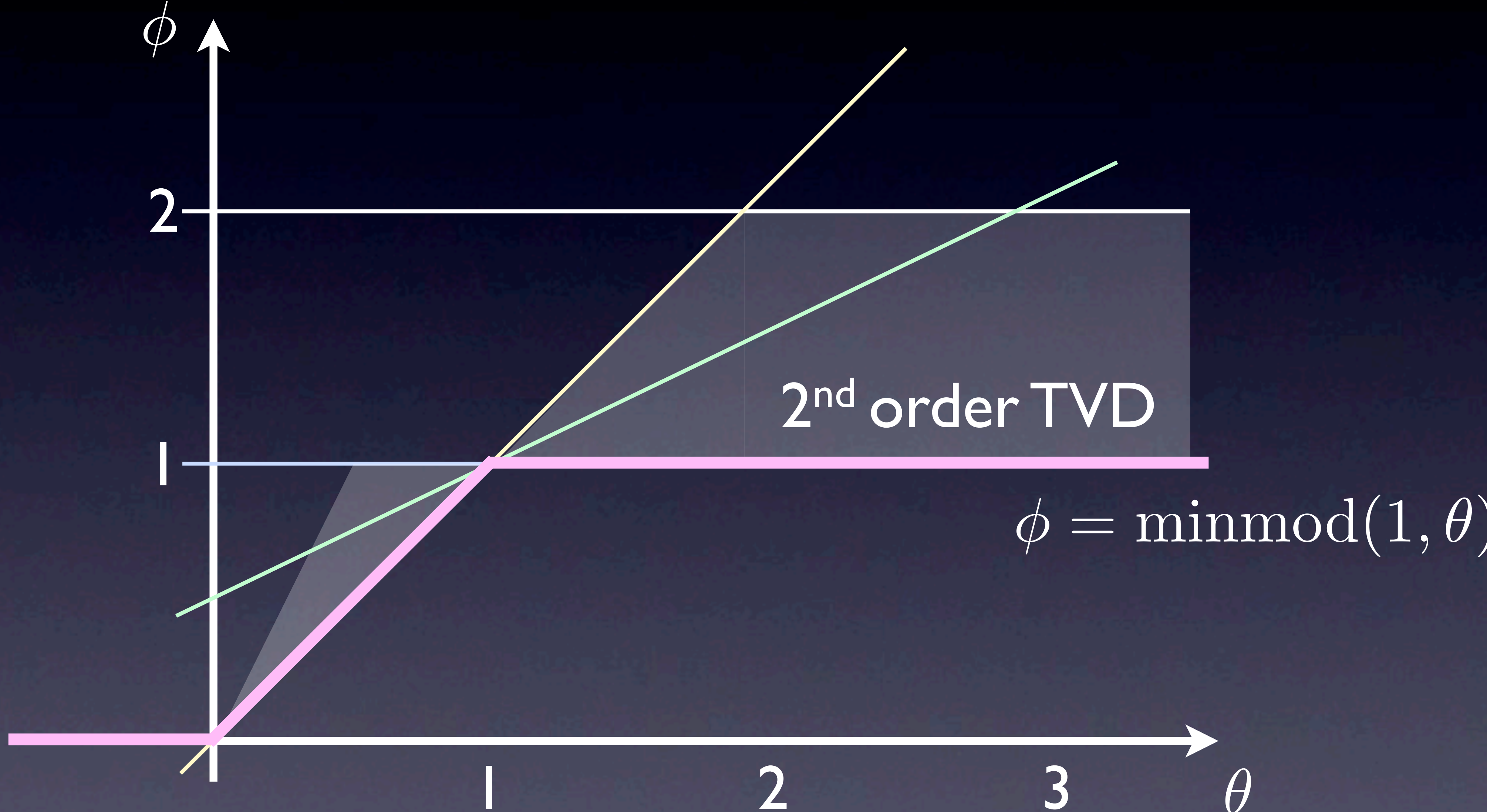


Plotting Phi vs. Theta



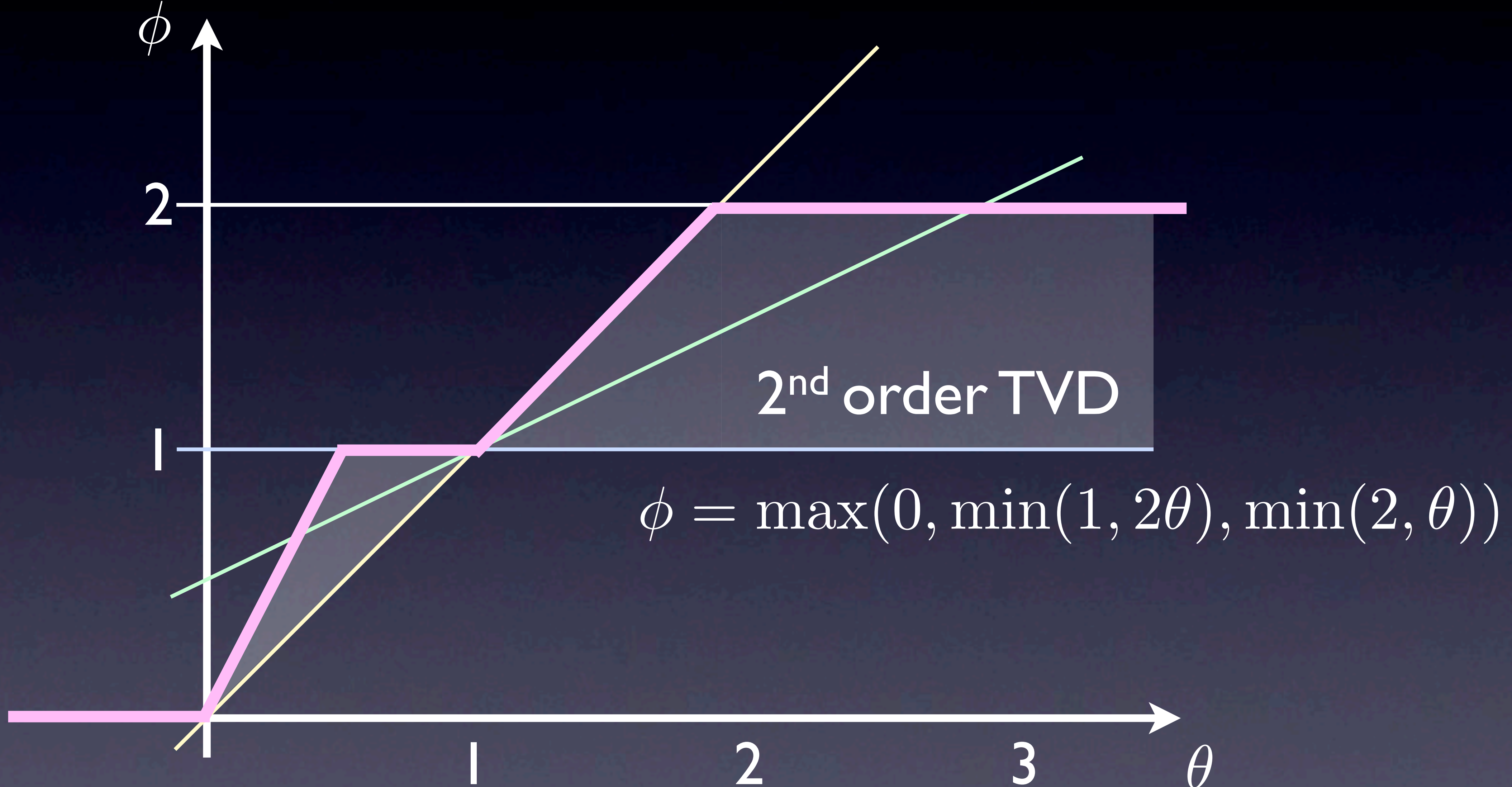
Only the region bounded by these methods is 2^{nd} order!

Minmod Flux Limiter



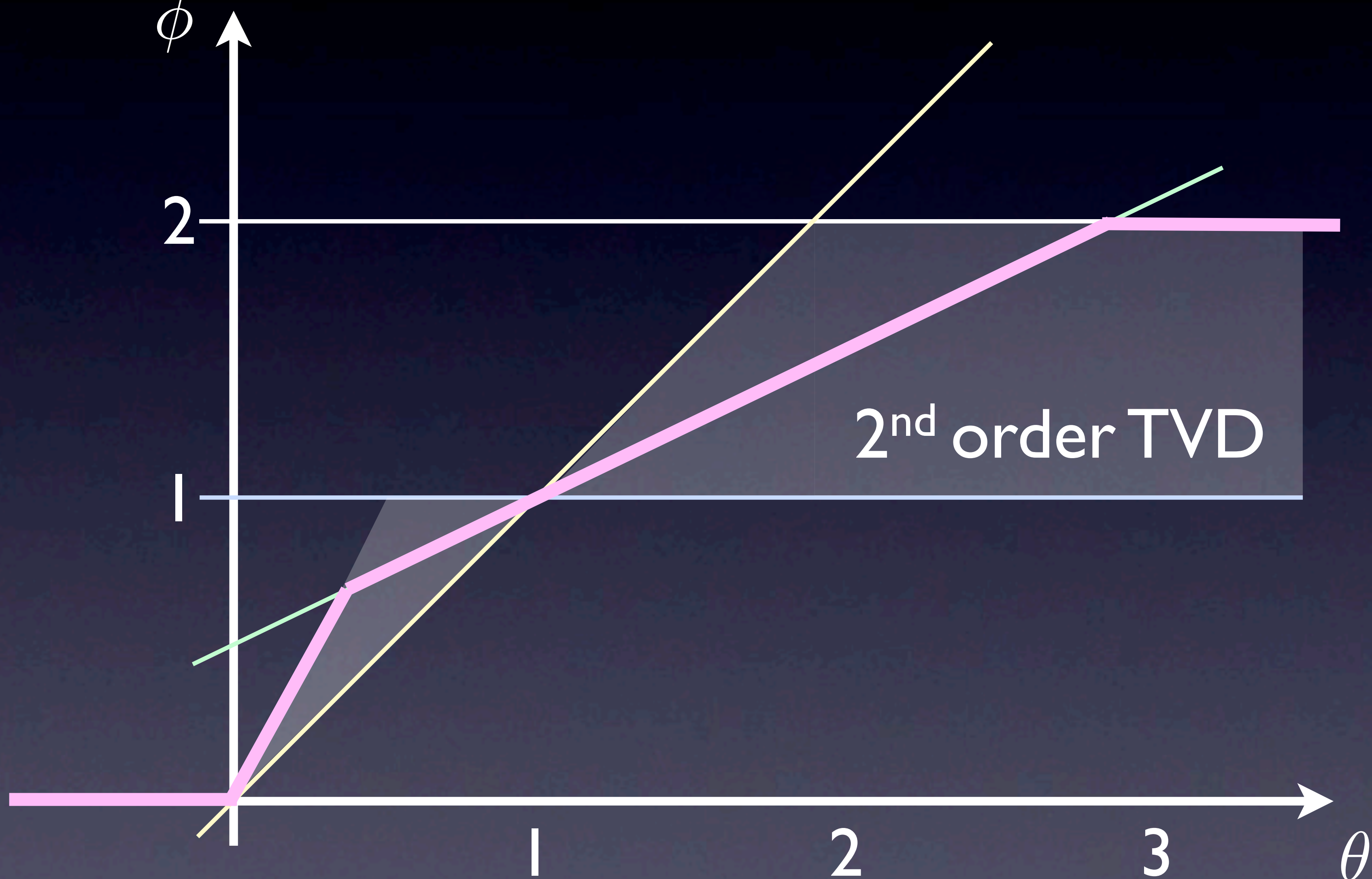
Errs on the “safe” side

Superbee Flux Limiter



Takes the riskier path.

Monotized Center Flux Limiter



Splits the difference.