EP711:
Equilibrium and Stability

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Hydrostatic Equilibrium

For an atmosphere with slowly-varying or constant *T*, the density and pressure decay exponentially with a scale height *H*:

$$\frac{dp}{dz} = -\rho g = -\frac{pg}{RT} = -p/H$$

Integrating, we find that (assuming constant scale height, $H=RT/g$, i.e., isothermal) solutions for the pressure (and density) vary from reference altitude z_o to z:

$$p(z) = p_o \exp\left(-\int_{z_o}^{z} \frac{dz'}{H}\right) = p_o e^{-(z-z_o)/H}$$

Adiabatic Processes and the Potential Temperature

Without exchanging energy/matter with the background, adiabatic changes in temperature and pressure are governed by the following expression, where changes from one initial state at pressure *p* and temperature *T* (*p*,*T*) to another state at reference pressure p_o and temperature θ (p_o,θ) must satisfy:

$$\theta p_o^{-(\gamma-1)/\gamma} = T p^{-(\gamma-1)/\gamma}$$

Potential temperature is thus defined as the temperature that a parcel of air would have if compressed or expanded *adiabatically* from a pressure *p* to a reference pressure p_o.

$$\theta = T \left(\frac{p_o}{p} \right)^{(\gamma-1)/\gamma}$$

Temperature Lapse Rate

We define the environmental temperature lapse rate as minus the derivative of temperature with altitude:

$$\Gamma = -\frac{dT}{dz}$$

The *adiabatic lapse rate* describes the change in temperature of an air parcel as it is displaced vertically (without exchange of heat) in a stratified atmosphere:

$$\Gamma_d = -\frac{dT'}{dz'} = \frac{g}{c_p}$$

We will use this to assess the stability of the atmosphere.

Temperature Lapse Rate

Where did this come from?

$$\Gamma_d = -\frac{dT'}{dz'} = \frac{g}{c_p}$$

Hydrostatic equilibrium combined with First Law of Thermodynamics:

$$dp = -\rho g dz \quad \Rightarrow \quad \frac{dp}{p} = -\frac{dz}{H} \quad \Rightarrow \quad d\ln p = -\frac{dz}{H}$$

$$c_p d\ln T - R d\ln p = 0 \quad \Rightarrow \quad c_p dT' + g dz' = 0$$

Hydrostatic Stability

Assuming a *dry* atmosphere,

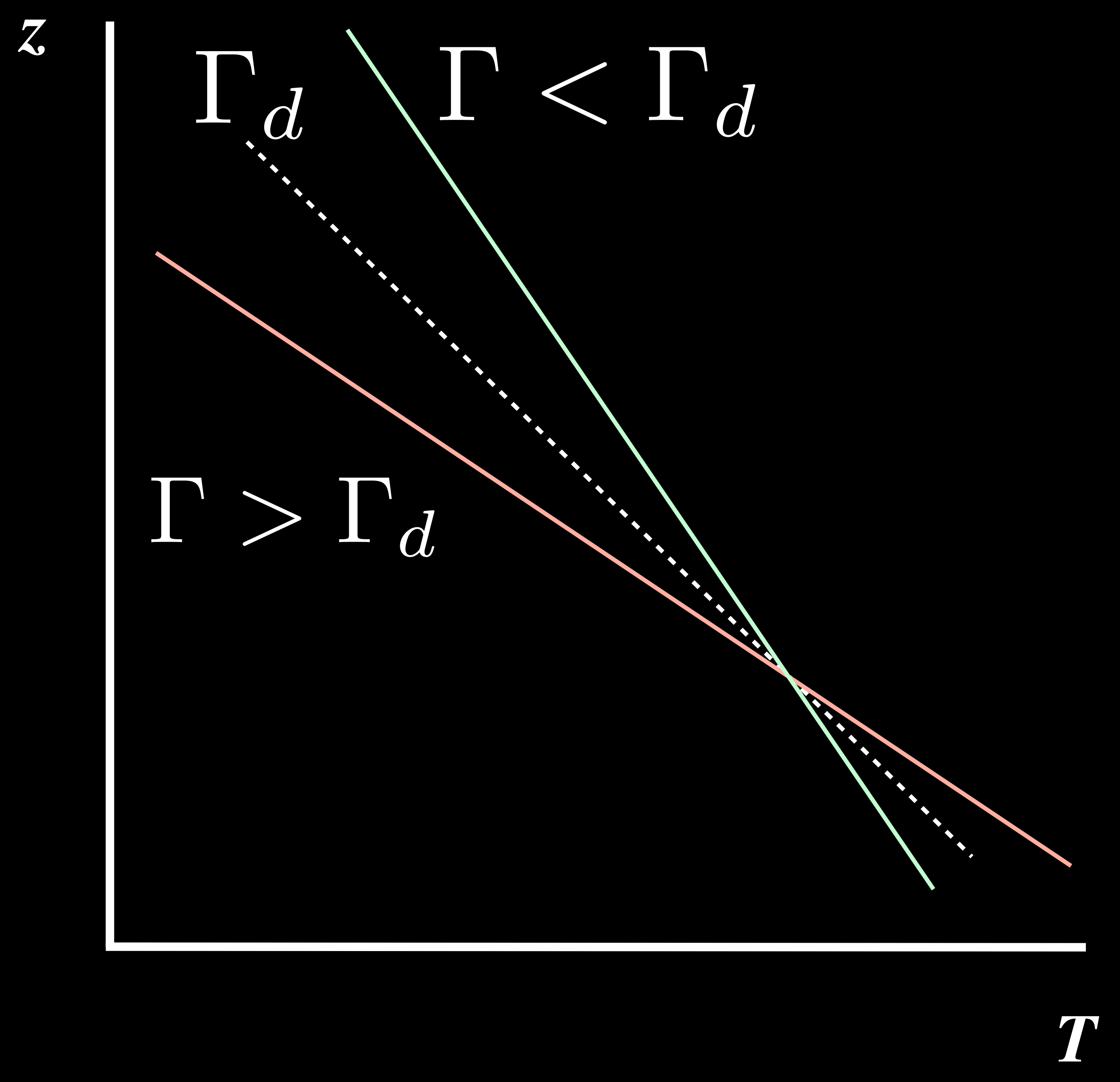
The atmosphere is stable when $\Gamma < \Gamma_d$

The atmosphere is neutrally stable when $\Gamma = \Gamma_d$

The atmosphere is unstable when $\Gamma > \Gamma_d$

Where the temperature increases in the atmosphere (*temperature inversion*), stability is very high – For example, the lower thermosphere!

Hydrostatic Stability



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For example, the lower thermosphere!

Potential Temperature

Recall the potential temperature:

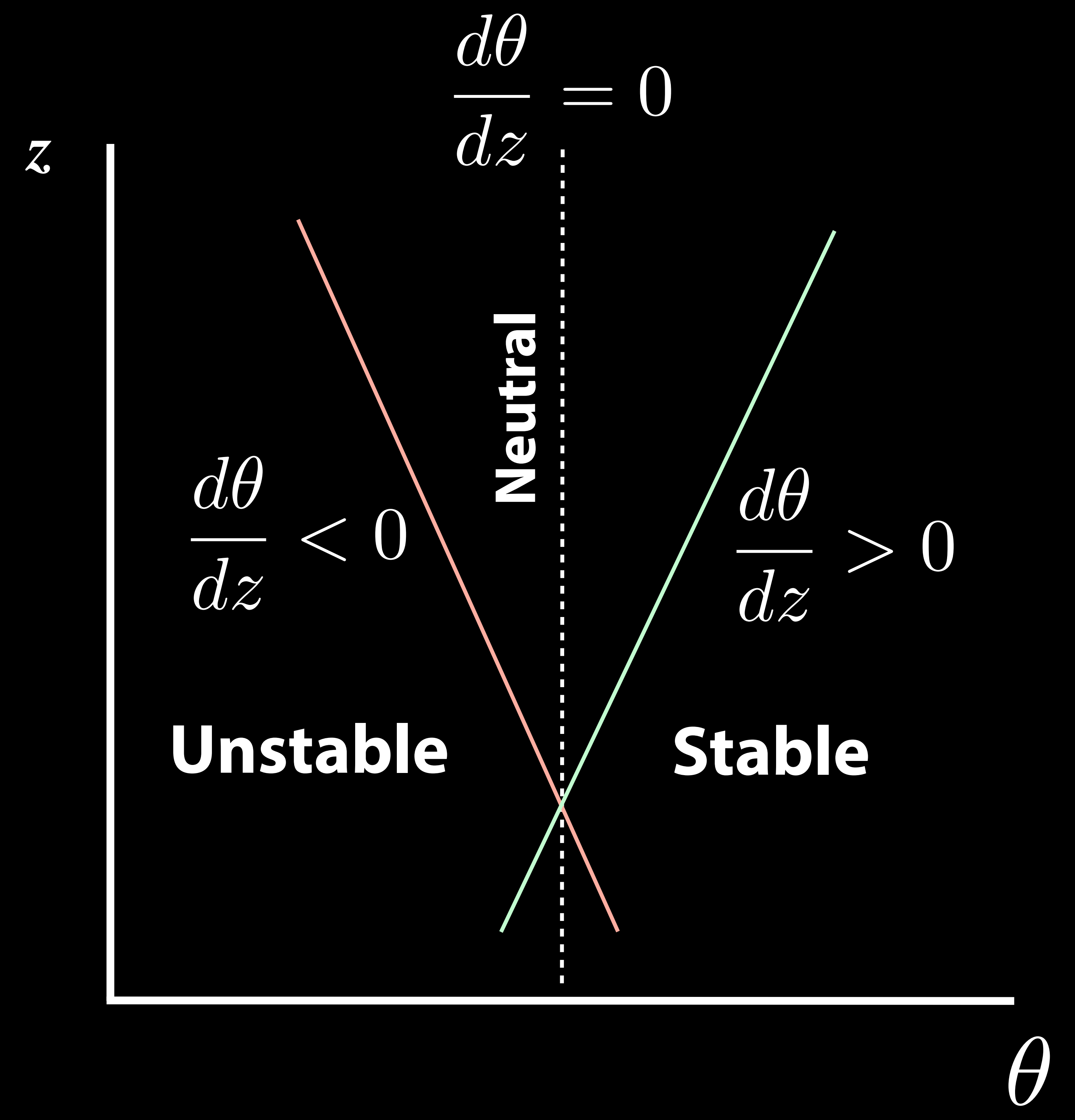
$$\theta = T\left(\frac{p_o}{p}\right)^{(\gamma-1)/\gamma}$$

The change in potential temperature can also be related to the lapse rate:

$$\frac{d\theta}{dz} = \frac{\theta}{T}(\Gamma_d - \Gamma)$$

So, we can expect that when the lapse rate equals the dry adiabatic lapse rate, that the atmosphere will be neutrally stable, and that the potential temperature will be constant.

Potential Temperature



We can also relate this to the Brunt frequency: $N^2 = \frac{g}{\theta}\frac{d\theta}{dz}$

$$N^2 = \frac{g}{T}\left(\Gamma_d - \Gamma\right) = \frac{g}{T}\left(\frac{\partial T}{\partial z} + \frac{g}{c_p}\right)$$

Brunt Väisälä Frequency

The atmosphere is stable when $N^2 > 0$

The atmosphere is neutrally stable when $N^2 = 0$

The atmosphere is unstable when $N^2 < 0$

Gravity waves are especially apt to propagate in regions with high (stable) Brunt frequencies – For example, the lower thermosphere! [e.g., *Walterscheid et al.*, 2001]

Dynamic Stability

Strong shears (vertical variations of wind) lead to reduced dynamic stability. This can be quantified by the *gradient Richardson number*:

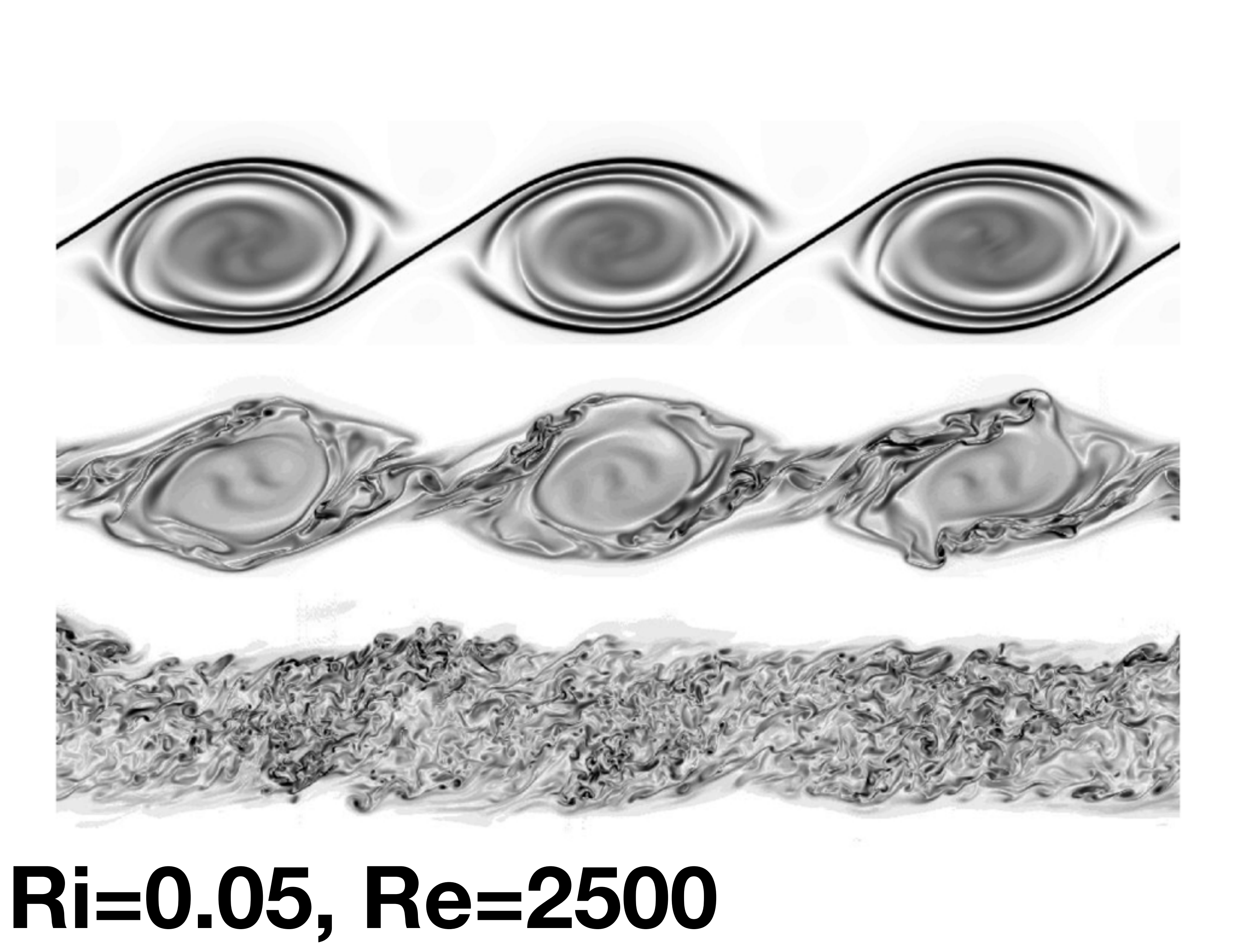
$$R_i = \frac{N^2}{(\partial u/\partial z)^2}$$

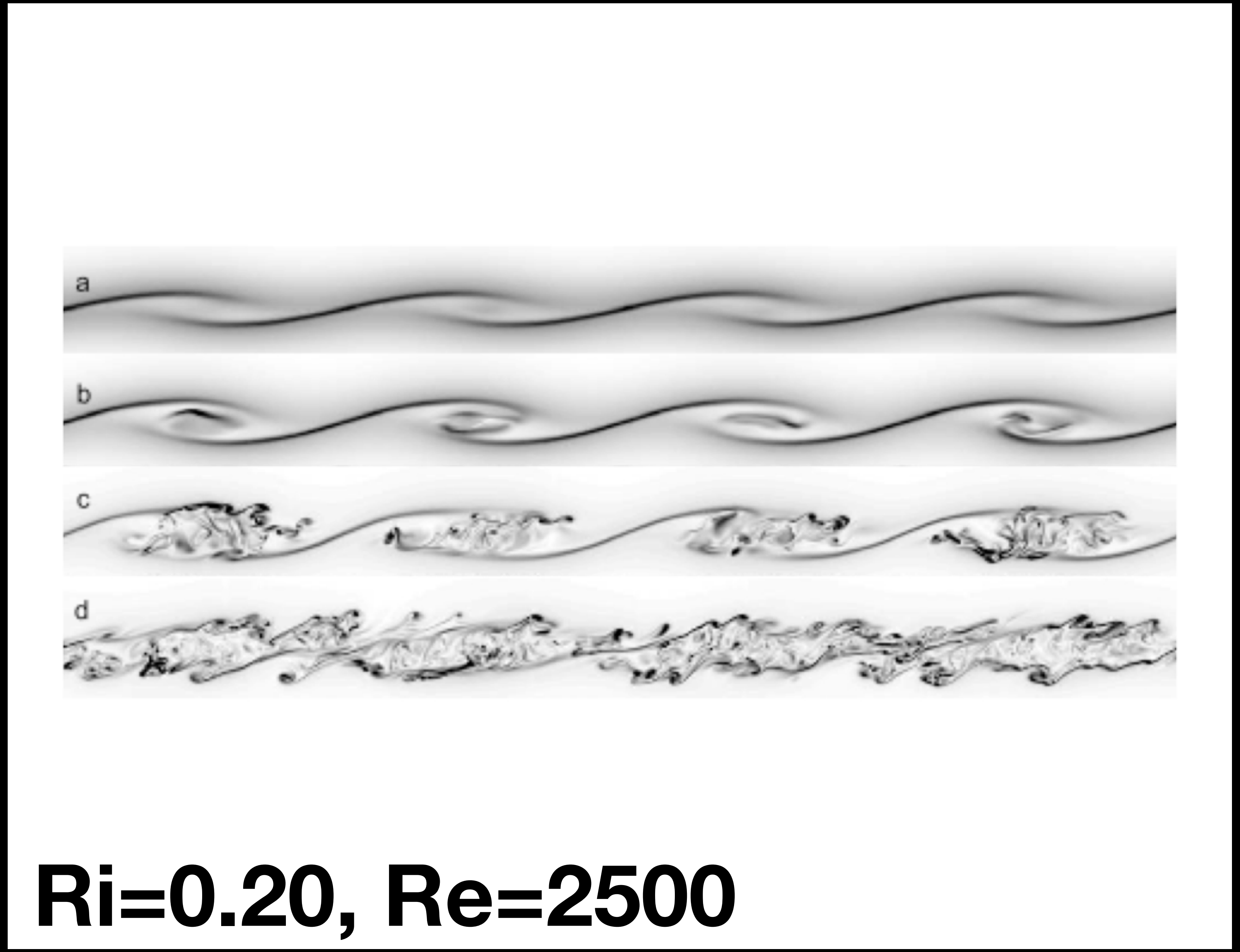
relating stability of stratification to strengths of the shear.

$R_i < 1/4$ is a *necessary* condition for dynamic instability, which may lead to the formation of familiar Kelvin-Helmholtz instabilities, and eventually turbulence.

$R_i < 0$ implies possibility of buoyancy-driven instability.

Dynamic Stability





KH Instability Examples [*Fritts et al.*]

KH instabilities triggered by
incident gravity wave at
shear layer with Ri=0.11.