EP711:
Navier-Stokes Systems for Atmospheric Dynamics

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Molecular Viscosity

Viscosity in a gas arises through collisions rather than internal friction, such that it tends to increase with temperature. For "air", the dynamic viscosity is given by:

$$\eta_a \simeq 1.8325 \times 10^{-5} \left(\frac{416.16}{T+120}\right)\left(\frac{T}{296.16}\right)^{1.5}$$

it has units (Pa*s=kg/(m*s)).

The *kinematic* viscosity is given by: $\nu = \dfrac{\eta}{\rho}$

Kinematic viscosity describes diffusivity of momentum, with units (m^2/s), and is inversely proportional to density (increases with decreasing density, and hence altitude).

Mean Free Path

Exponential decrease in density with altitude (and increase in kinematic viscosity) indicates exponential increase in mean free path and collision times with altitude. Note:

$$l = \frac{k_B T}{\sqrt{2}\pi d^2 p}$$

Which suggests a mean free path of ~1 km at ~250 km altitude. This is likely longer than reality, as it assumes hard sphere scattering. Note that *d* is diameter, *p* is pressure.

Mean Collision Time

Time between collisions increases exponentially with altitude.
Note average *velocity* of particles (~500-1000m/s):

$$v = \sqrt{\frac{k_B T}{m}}$$

This suggests a mean time between collisions of: $t = \dfrac{l}{v}$

When collisions occur over time and spatial scales << the scales of interest, "continuum" can be assumed.

Effects of Viscosity on Momentum Conservation:

The conservative Navier-Stokes Momentum Equation is:

$$\frac{\partial(\rho\vec{u})}{\partial t} + \nabla \cdot (\rho\vec{u}\vec{u} + p\delta_{ij} - \underset{\substack{\uparrow \\ \textbf{Viscous Stress Tensor}}}{\tau_{ij}}) = 0$$

The viscous stress tensor is then given by:

$$\tau_{ij} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k} \right) + \zeta\delta_{ij}\frac{\partial v_k}{\partial x_k}$$

where “eta” is the dynamic viscosity and “zeta” is referred to as either the second or bulk viscosity (caution with definitions and terms!), assumed to be zero for monatomic gases [O], and generally neglected for atmospheric dynamics.

Effects of Viscosity on Momentum Conservation:

Fortunately, for most atmospheric applications, the dynamic viscosity does not vary significantly with altitude, and a simpler form of the viscous force term may be used:

$$\frac{\partial(\rho\vec{u})}{\partial t} + \nabla\cdot(\rho\vec{u}\vec{u}) = -\nabla p + \rho\vec{g} + \eta\nabla^2\vec{u} + \frac{\eta}{3}\nabla(\nabla\cdot\vec{u})$$

Where the atmospheric dynamics can be modeled as incompressible (nondivergent velocity field), we can simplify further to only include the Laplacian term.

$$\frac{\partial(\rho\vec{u})}{\partial t} + \nabla\cdot(\rho\vec{u}\vec{u}) = -\nabla p + \rho\vec{g} + \eta\nabla^2\vec{u}$$

Thermal Conduction / Conductivity:

Likewise, can define a thermal conductivity "kappa" for the atmosphere, which is related to the molecular viscosity via the Prandtl number:

$$Pr = \frac{C_p \eta}{\kappa}$$

where a typical Pr ≈ 0.7 for the atmosphere.

Note that we can also define a thermal diffusivity, equivalent to the molecular diffusivity (kinematic viscosity):

$$\alpha = \frac{\kappa}{(\rho C_p)}$$

Hence, $Pr = \dfrac{\nu}{\alpha}$

Thermal Conduction / Conductivity:

Thermal conduction in a fluid is given by a heat flux of internal energy:

$$\vec{q} = -\kappa \nabla T$$

A conservation law for energy can be written as:

$$\frac{\partial E}{\partial t} + \nabla \cdot ((E + p)\vec{u} - \vec{u} \cdot \tau_{ij} - k\nabla T) = 0$$

Let us derive (on the board) expressions for conservation of internal and kinetic energy – Of practical use for developing numerical schemes!

Thermal Conduction / Conductivity:

Effectively, the addition of thermal conductivity provides diffusion of temperature, e.g., the heat equation:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

By comparison, for momentum diffusion, velocity is "diffused" in a similar manner:

$$\frac{\partial \vec{u}}{\partial t} = \nu \nabla^2 \vec{u}$$

Note that *thermal diffusivity* and *momentum diffusivity* (kinematic viscosity) both have the same units [m^2/s].