Problem 1

For this problem we will start off with the Ampere's Law in terms of the E field and H fields.

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \tag{1}$$

We then take the curl of the equation above and use a vector identity and fact that the curl of E is defined by Faraday's Law to get the following:

$$\nabla \nabla \cdot \mathbf{H} - \nabla^2 \mathbf{H} = -(\sigma + \epsilon \frac{\partial}{\partial t}) \mu \frac{\partial \mathbf{H}}{\partial t} \tag{2}$$

Now the first term of the last equation goes to zero and we can distribute the right hand side to get the final result:

$$\nabla^2 \mathbf{H} = \sigma \mu \frac{\partial \mathbf{H}}{\partial t} + \epsilon \mu \frac{\partial^2 \mathbf{H}}{\partial t^2} \tag{3}$$

In order to get this result for the E-field is the same procedure and you get the following:

$$\nabla^2 \mathbf{E} = \sigma \mu \frac{\partial \mathbf{E}}{\partial t} + \epsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{4}$$

Problem 2

This one is pretty straight forward.

Problem 3

Waves can be represented by complex exponents knows as phasors. In phasor form, the electric and magnetic fields for a wave are:

$$\mathbf{E}(\mathbf{r}, t) = e^{-j(\mathbf{k}\cdot\mathbf{r} - \omega t)} \mathbf{E}_0 \qquad \mathbf{H}(\mathbf{r}, t) = e^{-j(\mathbf{k}\cdot\mathbf{r} - \omega t)} \mathbf{H}_0$$

Substituting these forms of the electric and magnetic field into the source free Maxwells equations gives the phasor forms for the equations.
Gauss' Laws:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 & \nabla \cdot \mathbf{H} &= 0 \\ \left(\nabla e^{-j(\mathbf{k}\cdot\mathbf{r} - \omega t)}\right) \cdot \mathbf{E}_0 &= 0 & \left(\nabla e^{-j(\mathbf{k}\cdot\mathbf{r} - \omega t)}\right) \cdot \mathbf{H}_0 &= 0 \\ -i\mathbf{k} \cdot \left(e^{-j(\mathbf{k}\cdot\mathbf{r} - \omega t)} \mathbf{E}_0\right) &= 0 & -j\mathbf{k}\left(\cdot e^{-j(\mathbf{k}\cdot\mathbf{r} - \omega t)} \mathbf{H}_0\right) &= 0 \\ \boxed{\mathbf{k} \cdot \mathbf{E} = 0} & & \boxed{\mathbf{k} \cdot \mathbf{H} = 0} & \end{aligned}$$

Faraday's Law:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t} \mathbf{H}$$

$$\left(\nabla e^{-j(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right)\times\mathbf{E}_0 = -\mu\frac{\partial}{\partial t}e^{-j(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{H}_0$$

$$-i\mathbf{k}\times\left(e^{-j(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{E}_0\right) = -j\mu\omega e^{-j(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{H}_0$$

$$\boxed{\mathbf{k}\times\mathbf{E} = \omega\mu\mathbf{H}}$$

Ampere's Law:

$$\nabla\times\mathbf{H} = \epsilon\frac{\partial}{\partial t}\mathbf{E}$$

$$\left(\nabla e^{-j(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right)\times\mathbf{H}_0 = \mu\frac{\partial}{\partial t}e^{-j(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{E}_0$$

$$-i\mathbf{k}\times\left(e^{-j(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{H}_0\right) = i\epsilon\omega e^{-j(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{E}_0$$

$$\boxed{\mathbf{k}\times\mathbf{H} = -\omega\epsilon\mathbf{E}}$$

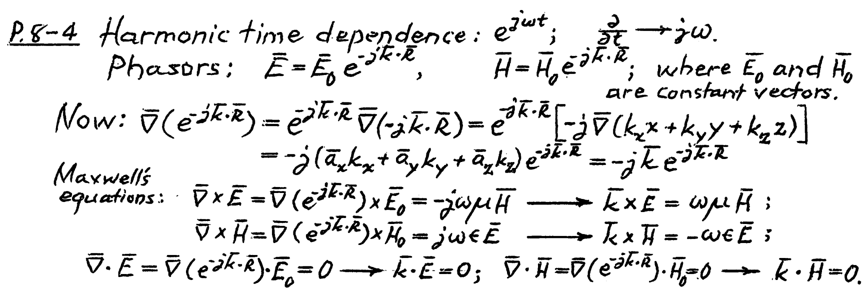
P.8-4 Harmonic time dependence: $e^{j\omega t}$; $\frac{\partial}{\partial t}\to j\omega$.
Phasors: $\bar{E}=\bar{E}_0 e^{-j\bar{k}\cdot\bar{R}}$, $\bar{H}=\bar{H}_0 e^{-j\bar{k}\cdot\bar{R}}$; where $\bar{E}_0$ and $\bar{H}_0$ are constant vectors.
Now: $\bar{\nabla}(e^{-j\bar{k}\cdot\bar{R}}) = e^{-j\bar{k}\cdot\bar{R}}\bar{\nabla}(-j\bar{k}\cdot\bar{R}) = e^{-j\bar{k}\cdot\bar{R}}\left[-j\bar{\nabla}(k_x x+k_y y+k_z z)\right]$
$= -j(\bar{a}_x k_x+\bar{a}_y k_y+\bar{a}_z k_z)e^{-j\bar{k}\cdot\bar{R}} = -j\bar{k}e^{-j\bar{k}\cdot\bar{R}}$
Maxwell's equations: $\bar{\nabla}\times\bar{E}=\bar{\nabla}(e^{-j\bar{k}\cdot\bar{R}})\times\bar{E}_0=-j\omega\mu\bar{H} \longrightarrow \bar{k}\times\bar{E}=\omega\mu\bar{H}$;
$\bar{\nabla}\times\bar{H}=\bar{\nabla}(e^{-j\bar{k}\cdot\bar{R}})\times\bar{H}_0=j\omega\epsilon\bar{E} \longrightarrow \bar{k}\times\bar{H}=-\omega\epsilon\bar{E}$;
$\bar{\nabla}\cdot\bar{E}=\bar{\nabla}(e^{-j\bar{k}\cdot\bar{R}})\cdot\bar{E}_0=0 \longrightarrow \bar{k}\cdot\bar{E}=0$; $\bar{\nabla}\cdot\bar{H}=\bar{\nabla}(e^{-j\bar{k}\cdot\bar{R}})\cdot\bar{H}_0=0 \longrightarrow \bar{k}\cdot\bar{H}=0$.

Figure 1: Cheng Solution for P.8-4

Problem 4

Part a

The wavelength and the frequency can be easily calculated from the coefficients of the z and t terms of the wave respectively:

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{1/\sqrt{3}} \qquad f = \frac{\omega}{2\pi} = \frac{10^8}{2\pi}$$

$$\boxed{\lambda = 10.9\ [\mathrm{m}]} \qquad \boxed{f = 15.9\ [\mathrm{MHz}]}$$

Part b

The dielectric constant is determined from the wave speed:

$$u_g = \frac{c}{\sqrt{\epsilon_r}} = \frac{\omega}{k}$$

$$\epsilon_r = \left(\frac{kc}{\omega}\right)^2$$

Substituting the appropriate values gives:

$$\boxed{\epsilon_r = \left(\frac{(1/\sqrt{3})3 \cdot 10^8}{10^8}\right)^2 = 3}$$

Part c

I determine polarization by sketching the x and y components and determining the motion that way. As $\mathbf{a}_x$ is positive and decreasing, $\mathbf{a}_y$ is negative and increasing (in the negative direction). This can only happen with left handed polarization. Since the magnitudes of each component are different, the wave is then left handed elliptically polarized

Part d

In order to find the magnetic field we must first find the characteristic impedance:

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \frac{\eta_0}{\sqrt{\epsilon_r}} = \frac{120\pi}{\sqrt{3}} \ [\Omega].$$

We know that the magnetic field is orthogonal to both the direction of motion of the wave and to the electric field, hence its direction is given by the cross product of these two elements. We also know that it is decreased in magnitude from the electric field by a factor of $1/\eta$, and that the fields are given by the real component of the phasor form. Therefore the magnetic field intensity is:

$$\mathbf{H} = \mathfrak{Re}\left\{\frac{1}{\eta}\mathbf{a}_z \times \mathbf{E}\right\}$$

$$\boxed{\mathbf{H}(z,t) = \frac{\sqrt{3}}{120\pi}\left[\sin\left(10^8 t - \frac{z}{\sqrt{3}}\right)\mathbf{a}_x \ + \cos\left(10^8 t - \frac{z}{\sqrt{3}}\right)\mathbf{a}_y\right] [\mathrm{Am}^{-1}]}$$

P.8-6 Phasor: $\bar{E} = \bar{a}_x 2 e^{-jz/\sqrt{3}} + \bar{a}_y j e^{-jz/\sqrt{3}}$ (V/m).

a) $\omega = 10^8$ (rad/s) $\longrightarrow$ $f = 10^8/2\pi = 1.59\times 10^7$ (Hz),

$\beta = 1/\sqrt{3}$ (rad/m) $\longrightarrow$ $\lambda = 2\pi/\beta = 2\sqrt{3}\pi$ (m).

b) $u = \frac{c}{\sqrt{\epsilon_r}} = \frac{\omega}{\beta} \longrightarrow \epsilon_r = \left(\frac{\beta c}{\omega}\right)^2 = 3,$

c) Left-hand elliptically polarized.

d) $\eta = \sqrt{\frac{\mu}{\epsilon}} = \frac{120\pi}{\sqrt{\epsilon_r}} = \frac{120\pi}{\sqrt{3}}$ (Ω),

$\bar{H} = \frac{1}{\eta}\bar{a}_z \times \bar{E} = \frac{\sqrt{3}}{120\pi}\left(\bar{a}_y 2 e^{-jz/\sqrt{3}} - \bar{a}_x j e^{-jz/\sqrt{3}}\right),$

$\bar{H}(z,t) = \frac{\sqrt{3}}{120\pi}\left[\bar{a}_x \sin(10^8 t - z/\sqrt{3}) + \bar{a}_y \cos(10^8 t - z/\sqrt{3})\right]$ (A/m).

Figure 2: Cheng Solution for P.8-6

Problem 5

Part a.

This material is not in the "good conductor" region, so we have to invoke the full force of the Helmholtz Equation to obtain attenuation and phase constants, the intrinsic impedance, the phase velocity, wavelength, and skin depth.

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)} - 1\right]^{1/2}$$

$$\boxed{\alpha = 84\ [\mathrm{Npm}^{-1}]}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)} + 1\right]^{1/2}$$

$$\boxed{\beta = 300\pi[\mathrm{radm}^{-1}]}$$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon - \frac{\sigma}{\omega}i}} = \frac{\sqrt{\mu}}{\left(\epsilon^2 + \left(\frac{\sigma}{\omega}\right)^2\right)^{1/4}} e^{\frac{i}{2}\tan^{-1}\left(\frac{\sigma}{\epsilon\omega}\right)}$$

$$\boxed{\eta_c = 41.8e^{0.0283\pi j}}$$

$$u_p = \frac{\omega}{\beta} = 33.3\cdot 10^6\ [\mathrm{m/s}] \qquad \lambda = \frac{2\pi}{\beta} = 0.67[\mathrm{cm}] \qquad \delta = \frac{1}{\alpha} = 1.19\ [\mathrm{cm}]$$

Part b.

The amplitude of the wave is given by $0.1e^{-\alpha y}$ A/m. The distance required for the field to decrease to 0.01 A/m is:

$$0.1e^{-\alpha y} = 0.01$$

$$\boxed{y = \frac{1}{\alpha}\ln(10) = 2.74 \text{ [cm]}}$$

Part c.

From the calculated values above we can write the magnetic field intesnity as a function of position and time:

$$\mathbf{H}(y,t) = 0.1e^{-\alpha y}\sin\left(10^{10}\pi t - \beta y - \pi/3\right)$$

At $y = 0.5$ [m], the magnetic field intensity is:

$$\mathbf{H}(0.5,t) = 0.1e^{-42}\sin\left(10^{10}\pi t - 150\pi - \pi/3\right)$$

Since 150π is a multiple of 2π, it does not change the sin function, and reducing all the other terms gives:

$$\boxed{\mathbf{H}(0.5,t) = 5.75\cdot 10^{-20}\sin\left(10^{10}\pi t - \pi/3\right)}$$

The electric field can be found using cross products and the characteristic impedance similarly to a previous problem, but using the imaginary in place of the real part due to the wave being represented by a sin function:

$$\mathbf{E}(0.5,t) = \mathfrak{Im}\left\{\eta_c\mathbf{H}(0.5,t)\times\mathbf{a}_y\right\}$$

$$\boxed{\mathbf{E}(0.5,t) = 2.41\cdot 10^{-18}\sin\left(10^{10}\pi t - \pi/3 + 0.0283\pi\right)\,[\text{Vm}^{-1}]}$$