Problem 1

For this problem we will determine the energy from in and outside the sphere using the following expression:

$$W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}dV \tag{1}$$

So, the first step is to find the $\mathbf{D}$ and $\mathbf{E}$ fields inside the sphere. This can be done through Guass's Law. Setting it up and evaluating the surface integral gives the following:

$$4\pi R^2\mathbf{D} = \frac{4\pi}{3}R^3\rho\hat{a}_R \tag{2}$$

Which reduces to:

$$\mathbf{D} = \frac{R\rho}{3}\hat{a}_R \tag{3}$$

The $\mathbf{E}$ field is easy to obtain from here, and the integral now looks like the following after switching to spherical coordinates and evaluating $d\phi$ and $d\theta$:

$$W = \frac{1}{2}\int_0^b \frac{1}{\epsilon_o}\left(\frac{\rho R}{3}\right)^2 4\pi R^2 dR \tag{4}$$

Now after evaluating the integral we get:

$$W_i = \frac{2\pi b^5\rho^2}{45\epsilon_o} \tag{5}$$

We will use the same formula for the outside of the sphere but now we have a different $\mathbf{D}$ and $\mathbf{E}$ field.

$$\mathbf{D} = \frac{b^3\rho}{3r^2}\hat{a}_R \tag{6}$$

Now the integral looks as follows:

$$W = \frac{1}{2}\int_b^\infty \frac{1}{\epsilon_o}\left(\frac{b^3\rho}{3r^2}\right)^2 4\pi R^2 dR \tag{7}$$

Evaluating the integral gives the following:

$$W_o = \frac{2\pi b^5\rho^2}{9\epsilon_o} \tag{8}$$

Problem 2

For this question we will use the fact that for their energy to be equal then there ratio must be one. The ratio looks like the following:

$$\frac{W_1}{W_2} = \frac{(1/2)\epsilon_r\epsilon_o E^2 xyd}{(1/2)\epsilon_o E^2(l-x)yd} = \frac{\epsilon_r x}{l-x} = 1 \tag{9}$$

From that we can solve for x and get:

$$x = \frac{l}{\epsilon_r + 1} \tag{10}$$

Problem 3

For this problem we will be solving the Laplace equation for between the plates. The boundary conditions for the problem are the following.

$$V(0) = 0; V(d) = V_o \tag{11}$$

Conveniently, this is a one dimensional problem so the Laplace equation reduces to:

$$\frac{d^2V}{dy^2} = 0 \tag{12}$$

This can be solved through integration, but since there are 2 different permittivities it needs to be solved independently for each region. Then, use the continuity of the **D** field and the potential through the interface to connect the 2 solutions. The following solutions and conditions are as follows:

$$V_b = c_1 y + c_2 \tag{13}$$

$$V_t = c_3 y + c_4 \tag{14}$$

$$V_b(0) = 0; V_t(d) = V_o \tag{15}$$

$$V_b(d/2) = V_t(d/2) \tag{16}$$

$$\mathbf{D}_b(d/2) = \mathbf{D}_t(d/2); D = -\epsilon \nabla V \tag{17}$$

Now, there is a bit of algebra required to find all of the constants. They are:

$$c_1 = \frac{4V_0}{(3+\epsilon_r)d} \tag{18}$$

$$c_2 = 0 \tag{19}$$

$$c_3 = \frac{4V_0\epsilon_r}{(3+\epsilon_r)d} \tag{20}$$

$$c_4 = \frac{3V_0(1-\epsilon_r)}{3+\epsilon_r} \tag{21}$$

You can now plug these constants in to their appropriate equations and determine V and **E**. The next part of this question is to find the surface charge densities, this can be done through evaluating the **D** field at 0 and d.

$$\rho_b = \mathbf{D}_b(0) \cdot \hat{n} = -\frac{4V_0\epsilon_r\epsilon_0}{(3+\epsilon_r)d} \tag{22}$$

$$\rho_t = \mathbf{D}_t(d) \cdot \hat{n} = \frac{4V_0\epsilon_r\epsilon_0}{(3+\epsilon_r)d} \tag{23}$$

Now for the last part, the potential function for a capacitor with no dielectric is the following:

$$V(y) = \frac{V_0}{d} y \tag{24}$$

Problem 4

For this problem, refer to the figure in Cheng p.166. From this figure we can set up the following electric field from the image line charge and the actual line charge:

$$\mathbf{E} = \frac{\rho_l[(x-d)\hat{a}_x + y\hat{a}_y]}{2\pi\epsilon_0[(x-d)^2 y^2]} - \frac{\rho_l[(x+d)\hat{a}_x + y\hat{a}_y]}{2\pi\epsilon_0[(x+d)^2 y^2]} \tag{25}$$

Then to find the surface charge we take only the x **D**-field and set x=0. This results in the following:

$$\rho_s = \frac{\rho_l d}{\pi(d^2 + y^2)} \tag{26}$$

Now we should recover an image line charge if we integrate over all of y. Integrating yields:

$$\frac{-\rho_l d}{\pi} \frac{\arctan(\frac{y}{d})}{d} \tag{27}$$

Evaluating the integral from negative infinity to positive infinity causes the arctan to go to π giving $-\rho_l$.

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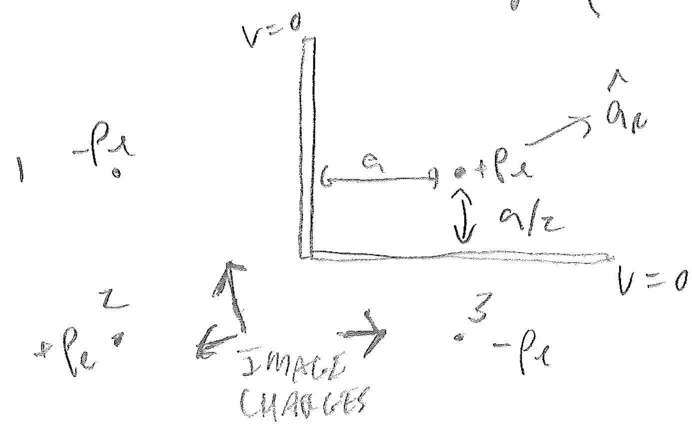
Find force on line charge ρ_ℓ



Force must be per unit length $F_\ell = \frac{F}{\ell} = \rho_\ell E$

$$\bar{E} = \bar{E}_1 + \bar{E}_2 + \bar{E}_3$$

$$\bar{E}_1 = \frac{-\rho_\ell}{2\pi\epsilon_0 2a}\hat{a}_x \qquad \bar{E}_3 = \frac{-\rho_\ell}{2\pi\epsilon_0 a}\hat{a}_y$$

$$\bar{E}_2 = \frac{+\rho_\ell \hat{a}_R}{2\pi\epsilon_0\sqrt{4a^2+a^2}} = \frac{\rho_\ell \hat{a}_R}{2\pi\epsilon_0\sqrt{5a^2}}$$

$$\hat{a}_R = \hat{a}_x \frac{a}{\sqrt{a^2+a^2/4}} + \hat{a}_y \frac{a/2}{\sqrt{a^2+a^2/4}} =$$

$$= \hat{a}_x \frac{2}{\sqrt{5}} + \hat{a}_y \frac{1}{\sqrt{5}}$$

$$\bar{E} = \left(\frac{-\rho_\ell}{4\pi\epsilon_0 a} + \frac{\rho_\ell 2}{2\pi\epsilon_0\sqrt{5a^2}\sqrt{5}}\right)\hat{a}_x +$$

$$\left(\frac{-\rho_\ell}{2\pi\epsilon_0 a} + \frac{\rho_\ell}{2\pi\epsilon_0\sqrt{5a^2}\sqrt{5}}\right)\hat{a}_y =$$

$$\boxed{\bar{E} = \frac{\rho_\ell}{\pi\epsilon_0}\left(\frac{1}{5a} - \frac{1}{4a}\right)\hat{a}_x + \frac{\rho_\ell}{\pi\epsilon_0}\left(\frac{1}{10a} - \frac{1}{2a}\right)\hat{a}_y}$$

$$\boxed{\bar{F}_\ell = \rho_\ell \bar{E}} \qquad \bar{F}_\ell = \frac{\rho_\ell^2}{\pi\epsilon_0}\left(\frac{-1}{20a}\hat{a}_x - \frac{2}{5a}\hat{a}_y\right)$$

P.4-6 Poisson's eq. $\nabla^2 V = -\frac{A}{\epsilon r} \longrightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = -\frac{A}{\epsilon r}$.

Solution: $V = -\frac{A}{\epsilon} r + c_1 \ln r + c_2$.

B.C.: $\begin{cases} \text{At } r=a, & V_0 = -\frac{A}{\epsilon} a + c_1 \ln a + c_2. \\ \text{At } r=b, & 0 = -\frac{A}{\epsilon} b + c_1 \ln b + c_2. \end{cases}$

$c_1 = \frac{\frac{A}{\epsilon}(b-a) - V_0}{\ln(b/a)}$,

$c_2 = \frac{V_0 \ln b + \frac{A}{\epsilon}(a \ln b - b \ln a)}{\ln(b/a)}$.