EP440: Engineering Electromagnetics, Exam #3

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Solve Problem 1, then choose 3 others. Please put a box around all final answers! If attempting more than 4, write "Do Not Grade" on the problem that you do not want to have graded. (Sorry, there is no "best of" option.)

For each problem, show as much work as possible on the pages provided (and the backs of the pages if necessary). You may find the following tables useful.

$$
\begin{aligned}
k_e &= \frac{1}{4\pi\epsilon_o} \simeq 9\times 10^9 \quad [\mathrm{N\cdot m^2/C^2}]\\
\epsilon_o &= 8.854\times 10^{-12} \quad [\mathrm{C^2/N\cdot m^2}] \text{ or } [\mathrm{F/m}]\\
\mu_o &= 4\pi\times 10^{-7} \quad [\mathrm{N/A^2}] \text{ or } [\mathrm{H/m}]\\
c &= 2.99792458\times 10^8 \simeq 3\times 10^8 \quad [\mathrm{m/s}]\\
m_e &= 9.109\times 10^{-31} \quad [\mathrm{kg}]\\
m_p &= 1.673\times 10^{-27} \quad [\mathrm{kg}]\\
q_e &= 1.602\times 10^{-19} \quad [\mathrm{C}]
\end{aligned}
$$

$$
\begin{aligned}
\nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}\\
\nabla\times\mathbf{H} &= \mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}\\
\nabla\cdot\mathbf{D} &= \rho\\
\nabla\cdot\mathbf{B} &= 0
\end{aligned}
$$

$$\mathbf{F} = q(\mathbf{v}\times\mathbf{B}+\mathbf{E})$$

$$\mathbf{J}=\sigma\mathbf{E},\ \mathbf{D}=\epsilon\mathbf{E},\ \mathbf{B}=\mu\mathbf{H}$$

$$\mathbf{D}=\epsilon_o\mathbf{E}+\mathbf{P} \qquad \mathbf{B}=\mu_o(\mathbf{H}+\mathbf{M})$$

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{J}$$

$$
\begin{aligned}
\int\frac{dx}{x} &= lnx\\
\int e^{ax}dx &= \frac{1}{a}e^{ax}\\
\int\frac{dx}{\sqrt{a^2-x^2}} &= \arcsin\frac{x}{a}\\
\int\frac{dx}{\sqrt{x^2+a^2}} &= \ln\left(x+\sqrt{x^2+a^2}\right)\\
\int\frac{xdx}{\sqrt{x^2+a^2}} &= \sqrt{x^2+a^2}\\
\int\frac{dx}{x^2+a^2} &= \frac{1}{a}\arctan\frac{x}{a}\\
\int\frac{dx}{(x^2+a^2)^{3/2}} &= \frac{1}{a^2}\frac{x}{\sqrt{x^2+a^2}}\\
\int\frac{xdx}{(x^2+a^2)^{3/2}} &= -\frac{1}{\sqrt{x^2+a^2}}
\end{aligned}
$$

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=\mathbf{B}\cdot\mathbf{C}\times\mathbf{A}=\mathbf{C}\cdot\mathbf{A}\times\mathbf{B}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

$$\nabla(\psi V)=\psi\nabla V+V\nabla\psi$$

$$\nabla\cdot(\psi\mathbf{A})=\psi\nabla\cdot\mathbf{A}+\mathbf{A}\cdot\nabla\psi$$

$$\nabla\times(\psi\mathbf{A})=\psi\nabla\times\mathbf{A}+\nabla\psi\times\mathbf{A}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\cdot\nabla V=\nabla^2 V$$

$$\nabla\times\nabla\times\mathbf{A}=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$$

$$\nabla\times\nabla V=0$$

$$\nabla\cdot(\nabla\times\mathbf{A})=0$$

$$\int_v\nabla\cdot\mathbf{A}\,dv=\oint_s\mathbf{A}\cdot d\mathbf{s} \qquad \text{(Divergence theorem)}$$

$$\int_s\nabla\times\mathbf{A}\cdot d\mathbf{s}=\oint_C\mathbf{A}\cdot d\boldsymbol{\ell} \qquad \text{(Stokes's theorem)}$$

$$E_{1t}=E_{2t} \qquad \mathbf{a}_{n2}\cdot(\mathbf{D}_1-\mathbf{D}_2)=\rho_s$$

$$B_{1n}=B_{2n} \qquad \mathbf{a}_{n2}\times(\mathbf{H}_1-\mathbf{H}_2)=\mathbf{J}_s$$

$$U_L=\frac{1}{2}LI^2 \qquad U_C=\frac{1}{2}CV^2$$

$$w_m=\frac{1}{2}\mathbf{H}\cdot\mathbf{B}=\frac{B^2}{2\mu}=\frac{1}{2}\mu H^2$$

$$w_e=\frac{1}{2}\mathbf{D}\cdot\mathbf{E}=\frac{D^2}{2\epsilon}=\frac{1}{2}\epsilon E^2$$

$$e^{j\phi} = \cos(\phi) + j\sin(\phi) \qquad e^{j\pi/2} = j \qquad e^{-j\pi/2} = -j \qquad j^2 = -1 \qquad \sqrt{j} = \left(\sqrt{2}/2 + j\sqrt{2}/2\right)$$

Cartesian Coordinates (x, y, z)

$$\nabla V = \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \mathbf{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Cylindrical Coordinates (r, ϕ, z)

$$\nabla V = \mathbf{a}_r \frac{\partial V}{\partial r} + \mathbf{a}_\phi \frac{\partial V}{r\partial \phi} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{\partial A_\phi}{r\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\phi r & \mathbf{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \mathbf{a}_r\left(\frac{\partial A_z}{r\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{a}_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \mathbf{a}_z \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Spherical Coordinates (R, θ, ϕ)

$$\nabla V = \mathbf{a}_R \frac{\partial V}{\partial R} + \mathbf{a}_\theta \frac{\partial V}{R\partial \theta} + \mathbf{a}_\phi \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \mathbf{a}_R & \mathbf{a}_\theta R & \mathbf{a}_\phi R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix} = \mathbf{a}_R \frac{1}{R\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right] + \mathbf{a}_\theta \frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}(RA_\phi)\right] + \mathbf{a}_\phi \frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right]$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

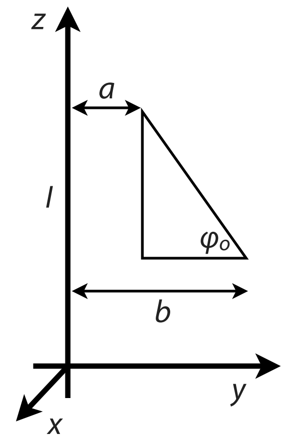
1. Let's explore Maxwell's equations for time-harmonic fields:

a. *Assuming solutions* ~ e^{jwt}, write the time-harmonic form of Maxwell's equations in terms of vector field phasors **E** and **H**, including conduction current **J**=σ**E**.

b. *Using these time-harmonic form equations*, please derive a vector Helmholtz wave equation for the **E** field in a **conducting** medium (ε, μ, $\sigma \neq 0$).
Please box (i) your Helmholtz wave equation, and (ii) an expression for complex wavenumber k_c, in terms of (iii) an expression for complex permittivity ε_c.

2. Find the mutual inductance between the infinitely long, straight, current-carrying wire and a triangular loop of wire as shown below. Both are located in the *y*-*z* plane in free space. Express your result in terms of a, b, and φ_o.

As a hint, tan $\varphi_o =$ ...something...something **cough-cough**.



3. Demonstrate the concept of "Displacement Current" for the case of a parallel plate capacitor (permittivity *ε*, plate area *s*, plate separation distance *d*).

<u>To achieve this, you will need to:</u>

a. Show that the displacement current density is given by: $\mathbf{J}_d = \dfrac{\partial \mathbf{D}}{\partial t}$

Please approach this using Maxwell's equations and conservation of charge.

b. Verify that total displacement current i_d within the capacitor is equal to the conduction current i_c in the wires when the capacitor is connected to a sinusoidal voltage source $v(t) = V_0 \sin(\omega t)$.

4. The magnetic field of an f=100 MHz uniform plane wave in free space is given by the following expression (in time-harmonic phasor form):

$$\mathbf{H} = H_0(\mathbf{a}_x - \mathbf{a}_y)e^{-j\beta z}$$

Determine the following, carefully boxing each result:

a. The wave direction of propagation and an exact value for β.

b. *Real* expressions for time-varying electric **E**(z,t) and magnetic **H**(z,t) fields in terms of H_o.

c. A description of the wave's polarization, including a diagram of **E** and **H** field vectors relative to the wave's direction of propagation.

5. Seawater has electrical properties given approximately by $\mu=\mu_0$, $\varepsilon=72\varepsilon_0$, and $\sigma=4$ [S/m].

Determine the following, carefully boxing each result:

a. Find the ratio of the amplitudes of conduction and displacement currents $|\mathbf{J}_c|/|\mathbf{J}_d|$ given a frequency of f=24.0 kHz. Use the approximate expression for $\varepsilon_o=10^{-9}/(36\pi)$ to avoid need for a calculator.

b. Taking a hint from this result, find an *approximate* expression for skin depth $\delta=1/\alpha$ in terms of frequency f, starting with the complex propagation constant (*show work!*):

$$\gamma = jk_c = j\omega\sqrt{\mu\epsilon}\left(1 - j\frac{\sigma}{\omega\epsilon}\right)^{1/2} = \alpha + j\beta$$

c. For underwater communications with submarines, frequencies ~ 24kHz are often used. Suggest a reason for this (including a sentence or two), comparing quantitatively with more conventional radio communications at f~24MHz.