Classical Mechanics (Twelve Days of) Christmas Problems

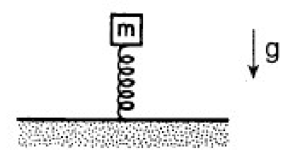
Dec 26 A particle of mass m is constrained to move in the x direction, and it is subject to the time-dependent force $F_x(t) = ae^{-bt}$. Obtain the position x and velocity v_x as functions of time, given that the initial velocity is $v_x(0) = v_0$ and the initial position is $x(0) = x_0$. Sketch both x and v_x versus t.

Dec 27 An object moving in one dimension is slowed by a drag force that is a function of velocity, $F(v)$. Observations show that its velocity decreases in the following way

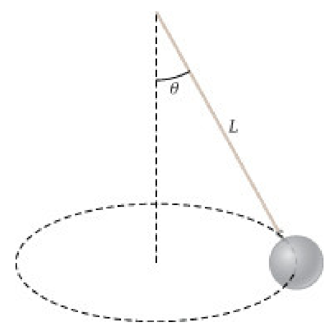
$$v = c^2(t - t_s)^2,$$

where c is a constant and t_s is the time at which the object stops. Find the force F as a function of v.

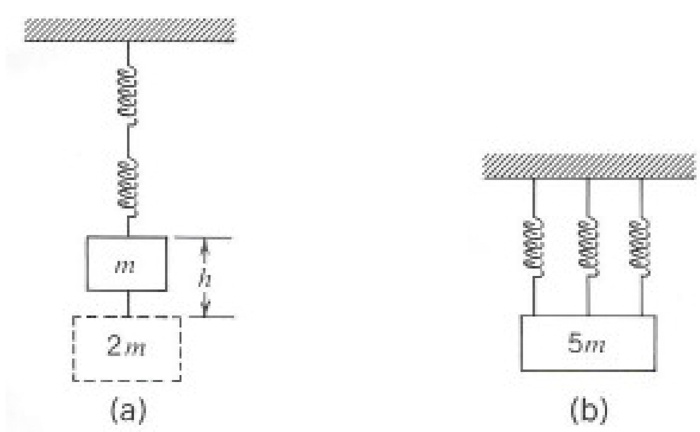
Dec 28 A massless spring of equilibrium length ℓ_0 and spring constant k has a mass m attached to one end. The system is set on a table with the mass vertically above the spring. (a) What is the new equilibrium height of the mass above the table? (b) The spring is compressed a distance c below the new equilibrium point and released. Find the motion of the mass assuming that the free end of the spring remains in contact with the table. (c) Find the critical compression distance for which the spring will break contact with the table.



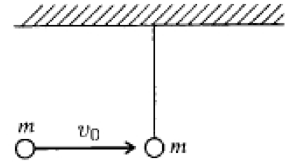
Dec 29 Consider the "conical pendulum" in the figure, where a mass m is attached to a string of length L and it is swung in a horizontal circle such that the angle the string makes with the vertical is θ. (a) Calculate the tension in the string. (b) Obtain the angular velocity ω as a function of the angle θ. (c) Show that in the limit $\theta \to 0$, ω reduces to the well-known planar pendulum angular frequency, $\sqrt{g/\ell}$.



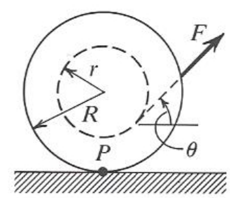
Dec 30 When the mass is doubled in Figure (a), the end of the two-spring system descends an additional distance h. What is the frequency of oscillation for the arrangement in Figure (b)? All individual springs shown are identical.



Dec 31 A mass m moving horizontally with velocity v_0 strikes a pendulum bob of the same mass m. (a) If the two masses stick together, find the maximum height reached by the pendulum bob. (b) If the masses scatter elastically along the line of the initial motion,m find the resulting maximum height.



Jan 1 A yo-yo rests on a table and the free end of its string is gently pulled at an angle θ to the horizontal. If $\theta \approx \pi/2$, then the yo-yo will accelerate to the left, but if $\theta \approx 0$, then the yo-yo will accelerate to the right. (a) Find the critical angle θ_c such that the yo-yo remains stationary, even though it is free to roll. (b) Find the acceleration of the yo-yo as a function of the angle θ.



Jan 2 An "n-sphere" is the set of points in $(n+1)$-dimensional Euclidean space that are at a fixed distance a from the origin. You know that the volume enclosed by a 2-sphere (the ordinary sphere) is $4\pi a^3/3$, and the "volume" enclosed by a 1-sphere (the ordinary circle) is πa^2. Calculate the volume of the 3-sphere by evaluating the integral

$$V_3 = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} \int_{-\sqrt{a^2-x^2-y^2-z^2}}^{\sqrt{a^2-x^2-y^2-z^2}} du\ dz\ dy\ dx.$$

I suggest that you start with the simpler problems of confirming (via direct integration) the area of a circle

$$V_1 = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\ dx = \pi a^2$$

and the volume of a sphere

$$V_2 = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz\ dy\ dx = \frac{4}{3}\pi a^3.$$

HINT: In the V_2 integral, switch to polar coordinates after the first (trivial) integration over z, and in the V_3 integral, switch to spherical coordinates after the first (trivial) integration over u.

Jan 3 Two boats leave from opposite banks of a river at the same time and travel at constant but different speeds. They pass each other 700 yards from one bank and continue to the other side of the river, where they turn around. On their return trip the boats pass again—this time 400 yards from the opposite bank. How wide is the river?

Jan 4 Einstein's general relativity is a modification of Newton's theory of gravitation. For a satellite of mass m orbiting a central object of mass M, the lowest order correction (due to the weak gravitational field) can be expressed as a modification of Newton's Law of Universal Gravitation

$$F = \frac{GMm}{r^2}\left(1 + 6\frac{v^2}{c^2}\right).$$

Show that if the satellite's orbital period under the pure Newtonian force GMm/r^2 is T_0, then the modified period T is approximately

$$T \approx T_0\left(1 - \frac{12\pi^2 r^2}{c^2 T_0^2}\right).$$

HINT: Treat the relativistic correction as a small fractional increase in G and use the value of v corresponding to the Newtonian orbit. ...(to be continued)...

Jan 5 ...(continued)... Show that, in each revolution, a planet in a circular orbit would travel through an angle greater than under the Newtonian force by an amount $24\pi^3 r^2/c^2 T_0^2$.

Also, show that this angle is expressible as $6\pi GM/c^2 r$ where M is the mass of the Sun. Apply this result to the planet Mercury, and verify that the accumulated advance in angle amounts to 43 seconds of arc per century. This is the calculation that Einstein made with his new theory that explained the longstanding observations of Mercury.

Jan 6 A moving particle of mass M collides perfectly elastically with a stationary particle of mass $m < M$. Show that the maximum possible angle through which the incident particle can be deflected is $\arcsin(m/M)$.

Bonus Problem! If all the starlight falling on the state of New York could be directed into Yankee stadium, would there be enough light for a night game?