28. **Pendulum phase portrait**

Draw the phase portrait for the pendulum (supported by an inextensible rod)

$$\ddot{\theta} + \omega^2 \sin\theta = 0.$$

Indicate the stable equilibrium points as well as the unstable equilibrium points. Indicate which paths represent oscillatory motion and which represent the pendulum executing circular motion. Can you derive the equation of the phase paths?

29. **Cubic potential**

Construct the phase portrait for the potential energy function $U(x) = -(\lambda/3)x^3$.

30. **Logistic model**

Consider the "logistic" model for the population of a single species, say rabbits. If the rabbits have a birth rate that is larger than the death rate then their population increases without bound. However, realistically we might assume that as their population grows too large, they start competing with each other for the available food. One such model that "explains" this is the logistic model:

$$\frac{dx}{dt} = ax - ex^2$$

where $-ex^2$ represents the decrease in population, and $x(t)$ is the population of the rabbits. (Both a and e are positive.)

(a) Find the equilibrium values of x.

(b) Since this is a Riccati equation, we can solve it using the variable-transformation technique that we used before. Transform the dependent variable from x to y via the following transformation: $x = (1/y) + (a/e)$. Solve for $y(t)$, and then perform the inverse transformation to solve for $x(t)$.

(c) Sketch $x(t)$ for two different initial conditions: $x(0) = 0.1(a/e)$ and $x(0) = 10(a/e)$.

(d) Analyze the equation and the solutions to say something qualitative about the dynamics of the rabbit population.

31. **Predator-prey analysis**

(a) Let $f(x)$ and $g(y)$ have local minima at $x = a$ and $y = b$ respectively. Show that $f(x) + g(y)$ has a minimum at (a, b). (Think of $f(x) + g(y)$ as a function of two variables $z(x, y)$ that represents the height or potential energy as a function of position.) Deduce that there exists a neighborhood of (a, b) in which all solutions of the family of equations

$$f(x) + g(y) = \text{constant}$$

represent closed curves surrounding (a, b).

(b) For the predator-prey problem discussed in class (wolves and rabbits), show using the above result that all solutions are periodic (assuming $x > 0$ and $y > 0$).

The system we considered was

$$\frac{dx}{dt} = ax - cxy$$
$$\frac{dy}{dt} = -by + dxy.$$

32. **Energy in oscillation**

    Consider a simple harmonic oscillator. (a) Calculate the *time* averages of the kinetic and potential energies over one cycle, and show that these quantities are equal. (b) Calculate the *spatial* averages of the kinetic and potential energies over one cycle. Discuss both results, and explain why they make sense.

33. **The Spring-Pendulum I**

    A spring-pendulum consists of a mass m suspended by a massless spring with unextended length b and spring constant k, and the mass is free to move in both the vertical and horizontal directions. Write the equations of motions (Newton's second law) in two ways. (a) Use Cartesian coordinates (x and y) where the origin is the equilibrium position of the mass. You should obtain two second order differential equations, one for x and the other for y. Then, linearize the equations for small x and small y (compared to what?), and qualitatively characterize the system. (b) Use polar coordinates (r and θ) where the origin is the point of support of the pendulum. Now your ODEs will be for r and θ as functions of time. Again, linearize the equations and compare and contrast with the results from part (a).

34. **Bobbing in liquid**

    An object of uniform cross-sectional area A and mass density ρ floats in a liquid of density ρ_0 and at equilibrium displaces a volume V. Show that the period of small oscillations about the equilibrium position is give by

    $$T = 2\pi\sqrt{V/gA}$$

    where g is the acceleration due to gravity.

35. **Damped oscillator I**

    If the amplitude of a damped oscillator decreases to $1/e$ of its initial value after n periods, show that the frequency of the oscillator must be approximately $[1-(8\pi^2n^2)^{-1}]$ times the frequency of the corresponding undamped oscillator. HINT: Assume that the damping is small and use a Taylor series expansion. The damping parameter must be small compared to what?

36. **Damped oscillator II**

    Express the displacement $x(t)$ and the velocity $\dot{x}(t)$ for the overdamped oscillator in terms of hyperbolic functions.

37. **Quality Factor**

    Show that, if a driven oscillator is only lightly damped and driven near resonance, the Q of the system is approximately

    $$Q \approx 2\pi \times \left(\frac{\text{Total energy}}{\text{Energy lost during one period}} \right).$$

    HINT: calculate the work done by the friction force.

38. **Straight Line**

    Using the calculus of variations, show that the shortest distance between two points in a plane is a straight line.

39. **Refraction of Light**

    Consider a medium in which the index of refraction is a function of altitude y (for example, the atmosphere). If light attempts to traverse the atmosphere, it will refract, and we are interested in finding the path $y(x)$ that light takes. Using calculus of variations and Fermat's principle of least time, derive the differential equation that describes the path of the light. HINT: the second form of Euler's equation is useful.

40. **Pendulum on a train**

    Find the frequency of small oscillations of a simple pendulum hanging from the ceiling of a railroad car that has a constant acceleration a in the positive x direction. HINT: be sure to choose your generalized coordinates relative to an inertial frame of reference. Explain why the sign of the acceleration a does not affect the frequency ω (i.e., you should get the same answer if the train were acclerating in the negative x direction).

41. **Sphere in a Cylinder**

    A uniform solid sphere of radius ρ and mass m is constrained to roll without slipping on the lower half of the inner surface of a hollow cylinder of inside radius R. Determine the Lagrangian, the equation of constraint, and Lagrange's equation of motion. Find the frequency of small oscillations about the bottom. (Don't forget the elementary physics of rotational motion!)

42. **Double Pendulum**

    A double pendulum consists of two simple pendula, with one pendulum suspended from the bob of the other. The lengths of the two rods are ℓ_1 and ℓ_2 and the masses of the bobs are m_1 and m_2 (the rods are massless). Find Lagrange's equations of motion for the system (note that there will be two [coupled] equations, one for each generalized coordinate). Do not assume small angles. Do assume that they are both confined to move in the same plane.

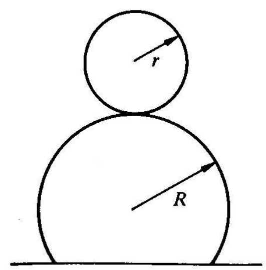
    EXTRA CREDIT: If $\ell_1 = \ell_2$ and $m_1 = m_2$ and the displacements from equilibrium are small (i.e., assume small angles), solve the two (coupled) equations of motion for the two normal modes of the system.

43. **The Spring-Pendulum II**

A spring-pendulum consists of a mass m suspended by a massless spring with unextended length b and spring constant k. (a) Find Lagrange's equations of motion (do not assume small angles). (b) Taylor expand the equations of motion around the equilibrium position (i.e., keep only the linear terms) and determine the frequencies of small oscillations. You should obtain two *uncoupled* linear oscillators.

44. **The hoop on a cylinder**

A uniform hoop of mass m and radius r rolls without slipping on a fixed cylinder of radius R as shown in the figure. The only external force is that of gravity. If the hoop starts rolling from rest on top of the cylinder, find, by the method of Lagrange undetermined multipliers, the point at which the hoop falls off the cylinder. (You know that moment of inertia of the hoop is mr^2, but it is instructive to solve the problem for a disk of arbitrary moment of inertia I.)



45. **Lagrangian I**

The Lagrangian for a particular physical system can be written as

$$L' = \frac{m}{2}\left(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2\right) - \frac{k}{2}\left(ax^2 + 2bxy + cy^2\right),$$

where a, b, and c are arbitrary constants but subject to the condition that $b^2 - ac \neq 0$. What are the equations of motion? Examine particularly the two cases $a = 0 = c$ and $b = 0$, $c = -a$. What is the physical system described by the above Lagrangian?

46. **Lagrangian II**

A particle of mass m moves in one dimension such that it has the Lagrangian

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2 V(x) - V^2(x),$$

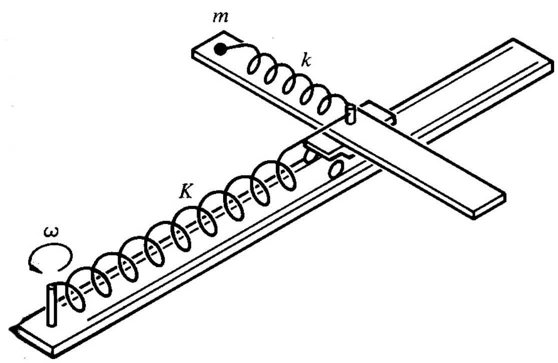
where V is some arbitrary differentiable function of x. Find the equation of motion for $x(t)$ and describe the physical nature of the system on the basis of this equation.

47. **Double carriage/spring**

A carriage runs along rails on a rigid beam. The carriage is attached to one end of a spring, equilibrium length r_0 and force constant K, whose other end is fixed on the beam. On the carriage there is another set of rails perpendicular to the first along which a particle of mass m moves, held by a spring fixed on the beam, of force constant k and zero equilibrium length. Beam, rails, springs and carriage are assumed to have zero mass. The whole system is forced to move in a plane about the point of attachment of the first spring, with a constant angular speed ω. The length of the second spring is at all times considered small compared to r_0.

(a) What is the energy of the system? Is it conserved?

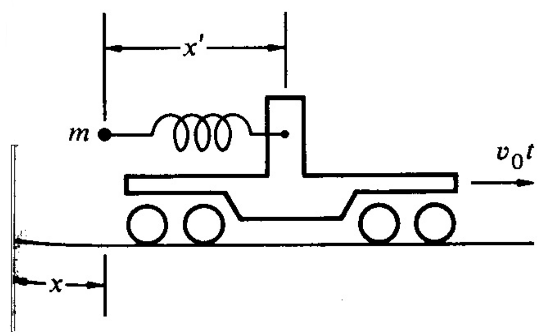
(b) Using generalized coordinates in the laboratory frame, what is the Jacobi integral (i.e., H) for the system? Is it conserved?



48. **Spring on a cart**

Consider a mass m attached to a spring of constant k, the other end of which is fixed on a massless cart that is being moved uniformly by an external force with speed v_0. Determine the equation of motion of the mass.

Investigate the behavior of the Hamiltonian function for the case where x is the generalized coordinate, and the case where x' is the generalized coordinate. Determine, for each of these cases, whether H is constant, and separately whether H is conserved.



49. An underdamped harmonic oscillator is subject to an applied force

$$f(t) = A\, e^{-at} \cos(\omega t).$$

Find a particular solution by expressing f as the real part of a complex exponential function, and looking for a solution for x having the same exponential time dependence. HINT: This means rewriting the equation of motion as

$$Re\left[\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \mathcal{F}(t)\right],$$

where $f = Re(\mathcal{F})$. Now you can solve the complex equation for a complex x, but at the end take the real part. CHECK: In the limit $a \to 0$, you should obtain the standard result for a harmonically driven oscillator.

50. Find the particular solution to the underdamped harmonic oscillator if the driving force is proportional to $\cos^2 \omega t$.

51. A force $A \cos \omega t$ acts on a damped oscillator starting at time $t = 0$ (the external force is zero for $t < 0$). What must the initial values of x and v be in order that there be no transient?

52. In my posted solution to Reynolds # 33, I obtained the following equations of motion in polar coordinates,

$$\ddot{r} = g \cos\theta - \omega_0^2 (r - b) + r\dot{\theta}^2,$$
$$r\ddot{\theta} = -g \sin\theta - 2\dot{r}\dot{\theta}.$$

Show that these are identical to Morin's equations (6.12) and (6.13) with the appropriate change of variables.

53. Show that the geodesic on the surface of a right circular cylinder is a helix. A *geodesic* is the shortest path between two points in a curved space.

54. Consider a disk of mass M rolling without slipping down a stationary plane inclined an angle α with respect to the horizontal. Choose reasonable coordinates (say, the distance s down the plane and the angle θ orienting the disk). Obtain the Lagrangian and the equations of motion. What is the constraint equation? What is the force of constraint?

55. A particle of mass m is subject to a force

$$F = -kx + kx^3/a^2,$$

where k and a are constants. (a) Find $V(x)$ and draw the phase portrait. Discuss the kinds of motion that can occur. (b) Show that if $E = \frac{1}{4}ka^2$ the integral of the energy equation* can be evaluated simply. Find $x(t)$ for this case, choosing x_0 and t_0 in any convenient way. Show that your result agrees with the qualitative discussion in part (a) for this particular energy.

*Energy conservation

$$\frac{1}{2}mv^2 + V(x) = E,$$

can be solved for $v = dx/dt$ and then integrated by separation of variables

$$t - t_0 = \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx'}{\sqrt{E - V(x')}}.$$