PS 320 - Classical Mechanics
Reynolds problems

1. **Rope**

   A uniform rope of length L and mass M rests on a table.

   (a) If you lift one end of the rope upward with a constant speed v, show that the vertical position of the rope's instantaneous center of mass moves upward with constant acceleration, and determine its value. That is, show that the vertical component of the acceleration of the center of mass is constant. NOTE: The shape of the portion of the rope left on the table is unimportant.

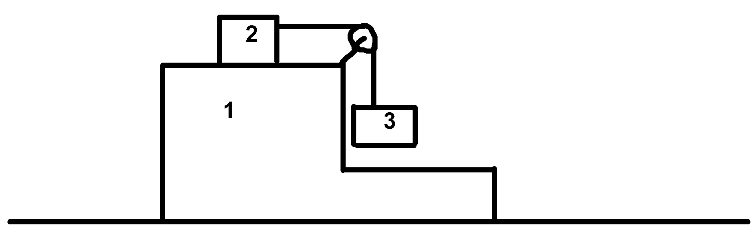
   (b) Next, suppose you hold the rope suspended in air, with its lower end just touching the table. If you now lower the rope with a constant speed v onto the table, show that the rope's center of mass has precisely the same upward, constant acceleration as in part (a).

2. **Rotating spring and bob**

   A bob of mass m is attached to a spring having spring constant k, and the system is whirled around in a horizontal circle at angular velocity ω. In the initial (relaxed) condition the distance of the center of mass of the bob from the axis of rotation is r_0. Find the steady-state distance, r, of the bob from the axis of rotation as a function of the imposed value of ω. Interpret the mathematical result by describing the physical effects predicted.

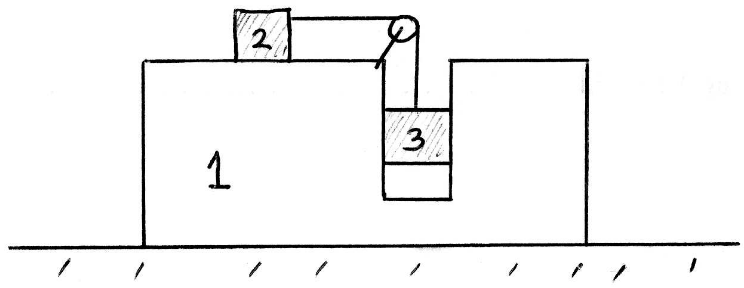
3. **Pedagogical machine I**

   Consider the "pedagogical machine" shown below. Mass 2 and mass 3 are connected by a rope that is hung over a pulley, and mass 1 is free to move horizontally on the table. All the surfaces are frictionless, and mass 3 does not touch mass 1. Find the initial acceleration of mass 1.



4. **Pedagogical machine II**

   Consider the "pedagogical machine" shown below. Mass 2 and mass 3 are connected by a rope that is hung over a pulley, and mass 1 is free to move horizontally on the table. All the surfaces are frictionless, and mass 3 is confined to a frictionless channel in mass 1. Find the acceleration of mass 1.



5. **Traffic**

   A motorist is approaching a green traffic light with speed v_0 when the light turns to amber.

   (a) If his reaction time is τ, during which he makes his decision to stop and applies his foot to the brake, and if his maximum braking deceleration is a, what is the minimum distance s_{min} from the intersection at the moment the light turns to amber in which he can bring his car to a stop?

   (b) If the amber light remains on for a time t before turning to red, what is the maximum distance s_{max} from the intersection at the moment the light turns to amber such that he can continue into the intersection at speed v_0 without running the red light?

   (c) Show that if his initial speed v_0 is greater than

   $$v_{0,\text{max}} = 2a(t - \tau),$$

   there will be a range of distances from the intersection such that he can neither stop in time nor continue through without running the red light.

   (d) Make some reasonable estimates of τ, t, and a, and calculate $v_{0,\text{max}}$ in miles per hour. If $v_0 = \frac{2}{3}v_{0,\text{max}}$, calculate s_{min} and s_{max}.

6. **Attractive potential well I**

   A particle of mass m has speed $v = \alpha/x$, where x is its displacement. Find the force $F(x)$ responsible for this motion, and the potential energy function $U(x)$ responsible for this force.

7. **Two masses, one swinging**

   The equations of motion for two masses, one swinging, are given by Morin on page 15, Eqs. (1.16)

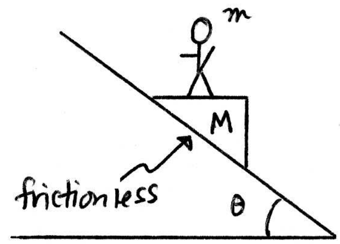
   $$2\ddot{r} = r\dot{\theta}^2 - g(1 - \cos\theta),$$

   $$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} - \frac{g\sin\theta}{r}.$$

   Linearize these equations for small θ and small ϵ, where $r = r_0 + \epsilon$ (r_0 is a constant). Retain only linear terms, i.e., only those to the first power in any combination of θ and ϵ and their time derivatives. What type of system does this describe?

8. **Weight on a sliding scale**

A person of mass m stands on a platform (of mass M) that is sliding down a frictionless inclined plane, as in the figure. What is the weight that would be recorded by a scale on the platform? That is, calculate the normal force between the platform and the person. Of course, there must be static friction between the platform and the person in order for the person to remain stationary with respect to the platform. HINT: Draw free body diagrams for the person and the platform, and solve for the acceleration and the various normal forces and friction forces.



9. **Water droplet I**

Solve the equation for the velocity of a raindrop

$$\frac{dv}{dt} = g - kv^2$$

that results from assuming that it collects water at a rate proportional to both its mass and velocity ($\dot{m} = kmv$). This nonlinear, ordinary differential equation is known as a Riccati equation, after the man who developed a method to solve equations of this type. His method is to transform it into a linear equation, solve this linear equation, and finally make the inverse transformation. Use the following transformation

$$z(t) = \frac{1}{v(t) - \sqrt{g/k}}$$

to obtain a differential equation for $z(t)$. It is a linear equation, so you can solve it. Then apply the inverse transformation

$$v(t) = \frac{1}{z(t)} + \sqrt{g/k}$$

to obtain $v(t)$.

10. **Water droplet II**

A water droplet falling in the atmosphere is spherical. Assume that as a spherical water droplet passes through a cloud, it acquires mass at a rate proportional to kA where k is a positive constant and A is its cross-sectional area. Consider a droplet of initial radius r_0 that enters a cloud with velocity v_0. Assume no resistive force. Show (a) that the radius increases linearly with time, and (b) that if r_0 is negligibly small then the speed increases linearly with time within the cloud.

11. **Attractive potential well II**

    Consider a particle moving in the region $x > 0$ under the influence of the potential energy function

    $$U(x) = U_0 \left(\frac{a}{x} + \frac{x}{a} \right),$$

    where U_0 and a are positive constants. Plot the potential (*without* your graphing calculator!), find the equilibrium points, and determine whether they are maxima or minima.

12. **Three superballs**

    Solve the superball problem, but this time use *three* balls. If the masses satisfy the inequality

    $$m_1 \gg m_2 \gg m_3,$$

    and if the heaviest ball (m_1) is on the bottom with the lightest ball (m_3) on top, find (approximately) the height that m_3 bounces to. Assume that the balls have negligible radii, and that the initial height is h. State *clearly* all the approximations that you use.

    Extra credit: Can you find the height that the top ball bounces to if there are N superballs? Assume that the masses satisfy the inequality as above.

13. **Rocket**

    A rocket starts from rest in free space by emitting mass. At what fraction of the initial mass is the momentum a maximum?

14. **Chain and scale**

    A chain of mass M and length L is suspended vertically with its lowest end touching a scale. The chain is released and falls onto the scale. What is the reading of the scale (i.e., the force that the scale must exert upward) when a length of chain, x, has fallen (i.e., as a function of time)? (Hint: Consider the change in momentum when a length of chain $\mathrm{d}x$ falls on the scale in a time $\mathrm{d}t$, and don't forget about the portion of the chain that is already lying on the scale. The maximum reading of the scale is $3Mg$.)

15. **Drag on an angled, flat surface**

    Show that the drag force experienced by a flat surface of cross-sectional area A moving through air of density ρ with speed V is given by

    $$F_d = (1 + \cos 2\theta) A \rho V^2,$$

    where the surface is inclined at an angle θ with respect to the direction of motion, i.e., $\theta = 0$ denotes a surface perpendicular to the direction of motion. Assume that the gas particles are initially motionless, and that the collisions between the gas particles and the sphere are perfectly elastic. Note: For each collision there will be a component of $\Delta \mathbf{p}$ perpendicular to the direction of motion, but these will all cancel for a symmetric object. We are only interested in the component of $\Delta \mathbf{p}$ *parallel* to the direction of motion because that is what causes drag.

16. **Drag on a sphere**

    Show that the drag force experienced by a sphere of cross-sectional area A moving through air of density ρ with speed V is given by

    $$F_d = A\rho V^2.$$

    Assume that the gas particles are initially motionless, and that the collisions between the gas particles and the sphere are perfectly elastic. To show this, I suggest using the result that the drag force on an angled flat surface is $F_d = (1 + 2\cos 2\theta)A\rho V^2$, where θ is the angle with respect to the direction of motion, and then breaking up the front surface of the sphere into concentric rings that have the same angle of attack (relative to the direction of motion). Finally, integrating over all concentric rings will result in the total drag force. Note: For each collision there will be a component of $\Delta\mathbf{p}$ perpendicular to the direction of motion, but these will all cancel for a symmetric object. We are only interested in the component of $\Delta\mathbf{p}$ *parallel* to the direction of motion because that is what causes drag.

17. **Neutron star**

    A neutron star is a collection of neutrons bound together by their mutual gravitation with a density comparable to that of an atomic nucleus (approximately 10^{12} g/cm^3). Assuming that the neutron star is a sphere, show that the maximum frequency with which it may rotate (if mass is not to fly off at the equator) is $f = (\rho G/3\pi)^{1/2}$, where ρ is the density. Calculate f for a density of 10^{12} g/cm^3. It is now accepted that pulsars, which emit regular bursts of radiation at repetition rates up to about 30/sec, are rotating neutron stars.

18. **A massive sphere**

    Consider a sphere of mass M where the gravitational field vector $\mathbf{g}$ is *independent* of the the radial position r within the sphere (of course it falls of like r^{-2} outside the sphere). Find the function describing the mass density $\rho(r)$ as a function of radial position. Note: There is spherical symmetry so that ρ does not depend on either of the other two spherical coordinates, θ and ϕ. (answer: $\rho(r) = M/(2\pi R^2 r)$)

19. **Sheet and sphere**

    A uniform solid sphere of mass M and radius R is fixed a distance h above a thin infinite sheet of mass density σ (mass per unit area). With what force does the sphere attract the sheet? (answer: $2\pi GM\sigma$)

20. **Self-gravitating cloud**

    Show that the gravitational self energy (i.e., the energy of assembly piecewise from infinity) of a uniform sphere of mass M and radius R is

    $$U = -\frac{3}{5}\frac{GM^2}{R}.$$

HINT: Calculate the work done to bring a mass dm in from infinity and spread it uniformly over the surface of the mass m that is already assembled. Then integrate over the entire mass M.

21. **The proto-sun and Kelvin**

The gravitational self-energy is the amount of energy released when a cloud of gas and dust contracts (gravitationally) into a protostar. This gravitational energy is ultimately converted into light, and is what allows protostars to shine before the density and temperature at their center are large enough to trigger nuclear reactions. In the mid-nineteenth century, nuclear reactions were not known, and Lord Kelvin (William Thomson) calculated the lifetime of the sun in the following manner.

(a) Given the Sun's current luminosity (rate of energy release in the form of radiation), calculate the rate at which its radius should decrease (assume that no mass is lost). HINT: Find the relation between dU/dt and dR/dt from your result in the "Self-gravitating cloud" problem. Would it be possible for us (with current technologies) to observe such a radius change? You might have to look up some numbers in the library for this part.

(b) Assume that the Sun in the past has been shining with its current luminosity. How long can it have been shining if all that energy had been released during its gravitational collapse from "infinity" (i.e., a very large interstellar cloud). (answer: about 20 million years)

Note: The above estimates are incorrect, because there is a theorem of classical mechanics known as the "virial theorem," which says that half of any gravitational energy released must go into heating the cloud, and the other half can be radiated away. However, the above estimates were good enough to show that the "Kelvin" time scale is much shorter than the geologic time scale, and, since the Earth cannot be older than the Sun, there must be another energy supply for the Sun besides gravitational energy.

"The sun must, therefore, either have beencreated as an active source of heat at some time of not immeasurable antiquity, by an over-ruling decree; or the heat which he has already radiated away, and that which he still possesses, must have been acquired by a natural process, following permanently established laws. Without pronouncing the former supposition to be essentially incredible, we may safely say that it is in the highest degree improbabli, if we can show the latter to be not contradictory to known physical laws. And we do show this and more, by merelypointing to certain actions, going on before us at present, which, if sufficiently abundant at some past time, must have given the sun heat enough to account for all we know of his past radiation and present temperature."

— Lord Kelvin, 1862

22. **Tunnel through the Earth I**

A particle is dropped into a hole drilled straight through the center of the Earth. Neglecting rotational effects and friction, show that the particle's motion is simple harmonic if you assume the Earth is spherical and has a uniform density. Show that the period of oscillation is about 84 minutes.

23. **Tunnel through the Earth II**

    Let's try the same thing as with the previous tunnel problem, but this time we do not require that the tunnel passes through the center of the Earth (although we do require that the tunnel is straight). That is, the tunnel is a straight chord from one spot on the surface of the Earth to another. Assuming that the tunnel's surface is frictionless, show that the period of oscillation is the same as when the tunnel passes through the center of the Earth. If you designed a transportation system with this idea, you could travel from any point on the Earth's surface to any other in only 42 minutes!

24. **Shape of reference geoid**

    Calculate the shape (r as a function of θ) of the Earth (the reference geoid) in the following manner. Start with the expression for the gravitational potential

    $$\Phi(r,\theta) = -\frac{GM}{r} + \frac{GMJ_2a^2}{r^3}P_2(\cos\theta) - \frac{1}{2}r^2\omega^2\sin^2\theta,$$

    where a is the Earth's equatorial radius and ω is its angular velocity of rotation. Choose the potential surface of interest to be that which passes through the equatorial radius

    $$\Phi_0 = \Phi(a, \pi/2).$$

    Then, setting $\Phi = \Phi_0$, and assuming $r = a(1+\epsilon)$, where $|\epsilon| \ll 1$, solve for $\epsilon(\theta)$. Use your knowledge of Taylor series expansions to retain only the largest terms. Then, compare this shape to that of a true ellipse (i.e., determine $\epsilon(\theta)$ for an ellipse and plot both functions on the same graph).

25. **Chain and peg**

    A flexible chain of length L is hung over a peg with one end slightly longer than the other. Assuming that the chain slides off with no friction, derive and solve the differential equation of motion to show that

    $$y(t) = y_0 \cosh\left(t\sqrt{\frac{2g}{L}}\right),$$

    where $2y$ is the difference in length of the two ends, and $y = y_0$ when $t = 0$. (The above solution only is valid, of course, for $0 < y < L/2$.)

26. **Water droplet III**

    We have modeled a falling raindrop gaining mass in two different ways. Let's take a third model, one that I think is more realistic. In this model, the time rate of change of the mass is proportional to both the cross-sectional area A and the velocity v (but not the mass directly)

    $$\dot{m} = kAv,$$

where k is a positive constant. This makes more sense because more water vapor will be swept up by a larger cross-sectional area, and, of course, the faster the drop moves the faster it will sweep up water vapor. This assumes that the density of the raindrop and the water vapor is constant.

Derive the dynamical equation for this raindrop, as well as any auxiliary differential equations that might be needed for a *complete* solution. (Make sure that all constants and variables are defined.) Although you do not have to solve it, discuss how you would go about solving it.

27. **Tunnel through the Earth III**

In Problem 18 we considered a spherically symmetric distribution of mass M whose gravitational field is

$$\vec{g} = g\hat{r},$$

where $g = -MG/R^2$ is a negative constant. (Recall that this means the mass density is not constant.) Consider a particle of mass m in a straight frictionless tunnel drilled through a chord of this sphere (i.e., it does not pass through the center, but its closest approach to the center is a distance a, similar to Problem 23). (a) What is the component of the gravitational force on the particle in the direction of its motion? (b) What is the potential energy of the particle as a function of its position in the tunnel? (c) Sketch this potential energy, and describe (qualitatively) the motion of the particle.