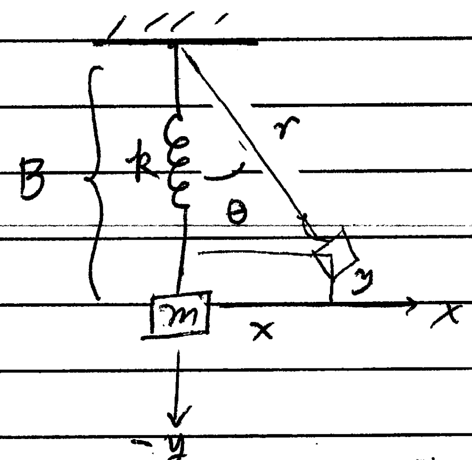
Spring – Pendulum

A spring with constant k is hung vertically and allowed to swing back and forth. It's equilibrium length is b, but when it hangs, the gravitational field extends it further:



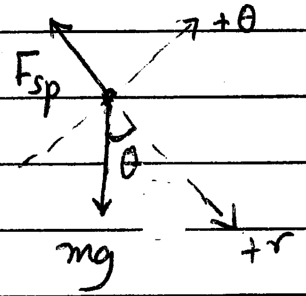
The new equilibrium length is B, found by equating the spring force with the gravitational force

$$k(B-b) = mg$$

or $$\boxed{B = b + \frac{mg}{k}}$$

It is about this new equilibrium position that the mass oscillates.

Method I polar coordinates: the general position of the mass is most conveniently described by (r,θ) where the origin is the support point of the spring/pendulum. A free-body diagram is



Decomposing the forces in polar coordinates give

$$\Sigma F_r = mg\cos\theta - F_{sp}$$
$$\Sigma F_\theta = -mg\sin\theta$$

where $F_{sp} = k(r-b)$

Newton's 2nd Law in polar coordinates is

$$\Sigma F_r = ma_r = m(\ddot{r} - r\dot{\theta}^2)$$
$$\Sigma F_\theta = ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

So, our two coupled, second-order ODEs for $r(t)$ and $\theta(t)$ are

$$\ddot{r} = g\cos\theta - \omega_0^2(r-b) + r\dot{\theta}^2$$
$$r\ddot{\theta} = -g\sin\theta - 2\dot{r}\dot{\theta}$$

where $\omega_0^2 = \frac{k}{m}$

These are highly nonlinear, but linearizing decouples these dynamic eqns.

let $r = B + \epsilon \qquad \epsilon \ll B \qquad (\dot{r} = \dot{\epsilon},\ \ddot{r} = \ddot{\epsilon})$

$\theta = \theta \qquad \theta \ll 1$

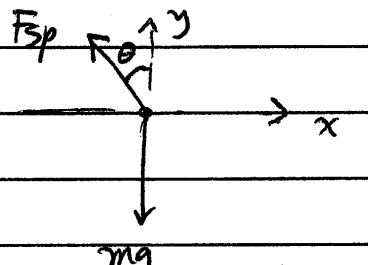
Keeping only terms first order in any combination of $\epsilon, \theta, \dot{\epsilon}, \dot{\theta}, \ddot{\epsilon}, \ddot{\theta}$ give

$$\ddot{\epsilon} \doteq g - \omega_0^2(\epsilon + B - b) = -\omega_0^2\epsilon$$
$$B\ddot{\theta} \doteq -g\theta$$

or: $\boxed{\begin{array}{l} \ddot{\epsilon} + \omega_0^2\epsilon = 0 \\ \ddot{\theta} + \Omega^2\theta = 0 \end{array}} \qquad \Omega^2 = \frac{g}{B}$

We have two simple harmonic oscillators for radial and angular motion.

<u>Method II</u> rectangular coordinates: The position of m can be denoted (x,y), where the origin is the (changing) equilibrium position. The forces must be decomposed in these x,y coordinates.



$$\sum F_x = -F_{sp}\sin\theta = m\ddot{x}$$

$$\sum F_y = +F_{sp}\cos\theta - mg = m\ddot{y}$$

It is still true that $F_{sp} = k(r-b)$, but we need to express r and θ in terms of (x,y). Three expressions can be deduced from the original figure at the top of page 1:

$$\sin\theta = \frac{x}{r} \qquad \cos\theta = \frac{B-y}{r} \qquad r^2 = x^2 + (B-y)^2$$

The equations of motion, in terms of x and y, become

$$\ddot{x} = -\omega_0^2 x \,\frac{\sqrt{x^2+(B-y)^2} - b}{\sqrt{x^2+(B-y)^2}}$$

$$\ddot{y} = \omega_0^2 (B-y)\,\frac{\sqrt{x^2+(B-y)^2} - b}{\sqrt{x^2+(B-y)^2}} - g$$

These, of course, are extremely ugly equations, simply because (x,y) are not the natural coordinates for the problem. Linearization will simplify things

Let $x \ll B$ and $y \ll B$

So x, y are already first order (similar to θ in Method I)

expanding the square roots:

$$x^2 + (B-y)^2 = x^2 + B^2 - 2By + y^2$$
$$= B^2 + (-2By + x^2 + y^2)$$
$$= B^2\left[1 + \left(-2\frac{y}{B} + \frac{x^2+y^2}{B^2}\right)\right]$$

Since we will retain only those terms linear in x, y, the squared terms can be eliminated. Although in general they need to be retained. The fractions become

$$\frac{\sqrt{x^2+(B-y)^2} - b}{\sqrt{x^2+(B-y)^2}} \cong 1 - \frac{b}{B}\left[1 - \frac{2y}{B}\right]^{-1/2} \approx 1 - \frac{b}{B}\left(1 + \frac{y}{B}\right)$$
$$= 1 - \frac{b}{B} - \frac{by}{B^2}$$

The equations of motion are

$$\ddot{x} \cong -\omega_0^2 x \left(1 - \frac{b}{B} - \frac{by}{B^2}\right) \approx -\frac{g}{B} x$$

where I've used the fact that $1 - \frac{b}{B} = \frac{mg/k}{b + mg/k}$ and I've thrown away the xy term.

$$\ddot{y} \cong \frac{k}{m}(B-y)\left(1 - \frac{b}{B} - \frac{by}{B^2}\right) - g$$

keeping all terms except the y^2 term, we have (after much simplification)

$$\boxed{\ddot{y} \approx -\frac{k}{m} y}$$ The y motion is a spring oscillator.

and the x motion $\boxed{\ddot{x} = -\frac{g}{B} x}$ is pendulum motion.

Again – 2 un coupled simple harmonic oscillators

NOTE: If we wished to retain higher order terms and study the nonlinear oscillations, it is clear that polar coordinates are more suitable.