Appendix H

Reduced Mass

Consider two particles of mass m_1 and m_2 located at positions $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ respectively, as shown in Fig. H.1. If, in addition to the forces that they exert on each other, there is an external force $\vec{\mathbf{F}}_{\text{ext}}$ that is exerted on each of them, then the equations of motion for each of the particles are

$$\vec{\mathbf{F}}_{\text{ext}} + \vec{\mathbf{F}}_{21} = m_1 \frac{d^2}{dt^2}\vec{\mathbf{r}}_1, \tag{H.1}$$

$$\vec{\mathbf{F}}_{\text{ext}} + \vec{\mathbf{F}}_{12} = m_2 \frac{d^2}{dt^2}\vec{\mathbf{r}}_2, \tag{H.2}$$

where $\vec{\mathbf{F}}_{12}$ is the force exerted by m_1 on m_2, and $\vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}$ (by virtue of Newton's Third Law). The two (vector) dependent variables in this description that are to be solved for as functions of time are $\vec{\mathbf{r}}_1(t)$ and $\vec{\mathbf{r}}_2(t)$. This "two-body problem" can be reduced to an equivalent "one-body problem" by making the following change of variables

$$\vec{\mathbf{R}} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{m_1 + m_2}, \tag{H.3}$$

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2,$$

where $\vec{\mathbf{R}}$ is the position of the center of mass of the system and $\vec{\mathbf{r}}$ is the relative position of the particles. It doesn't matter whether you use the original set of dependent variables,

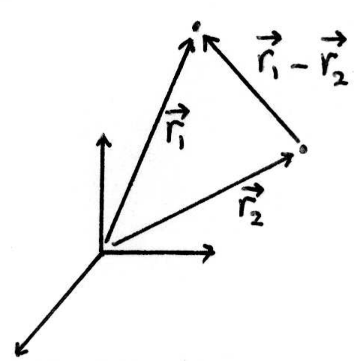

Figure H.1: Geometry for two particles moving in one coordinate system. If there is no external forces, then only the relative position vector $\vec{\mathbf{r}}$ is needed.

$\vec{\mathbf{r}}_1(t)$ and $\vec{\mathbf{r}}_2(t)$, or the new set, $\vec{\mathbf{R}}(t)$ and $\vec{\mathbf{r}}(t)$. Therefore, in order to recast Eqs. (H.1) and (H.2) we need to invert the transformation in Eq. (H.3) and express $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ in terms of $\vec{\mathbf{R}}$ and $\vec{\mathbf{r}}$. Doing this I obtain

$$\vec{\mathbf{r}}_1 = \frac{(m_1 + m_2)\vec{\mathbf{R}} + m_2\vec{\mathbf{r}}}{m_1 + m_2}, \tag{H.4}$$

$$\vec{\mathbf{r}}_2 = \frac{(m_1 + m_2)\vec{\mathbf{R}} - m_1\vec{\mathbf{r}}}{m_1 + m_2}.$$

Plugging these into the original equations of motion, (H.1) and (H.2), I obtain two new equations of motion, for $\vec{\mathbf{R}}$ and $\vec{\mathbf{r}}$. First, adding (H.1) and (H.2) gives

$$2\vec{\mathbf{F}}_{\text{ext}} = (m_1 + m_2)\frac{d^2}{dt^2}\vec{\mathbf{R}}, \tag{H.5}$$

and then subtracting (H.2) from (H.1) results in

$$\begin{aligned} \vec{\mathbf{F}}_{12} &= \left(\frac{m_1 - m_2}{2}\right)\frac{d^2}{dt^2}\vec{\mathbf{R}} + \frac{m_1 m_2}{m_1 + m_2}\frac{d^2}{dt^2}\vec{\mathbf{r}} \\ &= \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\vec{\mathbf{F}}_{\text{ext}} + \frac{m_1 m_2}{m_1 + m_2}\frac{d^2}{dt^2}\vec{\mathbf{r}}. \end{aligned} \tag{H.6}$$

Eq. (H.5) is simply the equation of motion for the entire system—the total force, $2\vec{\mathbf{F}}_{\text{ext}}$, is equal to the total mass, $(m_1 + m_2)$, times the acceleration of the center of mass. The first term in Eq. (H.6) is zero if $\vec{\mathbf{F}}_{\text{ext}} = 0$, so that in the absence of external forces the internal dynamics are determined by

$$\vec{\mathbf{F}} = \mu\frac{d^2}{dt^2}\vec{\mathbf{r}}, \tag{H.7}$$

where $\vec{\mathbf{F}} = \vec{\mathbf{F}}_{12}$ is the internal force between the two particles, and

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{H.8}$$

is called the "reduced mass." Equation (H.7) is identical to that for *one* particle of mass μ moving in a central force $\vec{\mathbf{F}}$. Another, more suggestive, way of expressing the reduced mass is

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}.$$

The net result of this transformation has been to reduce a two-body problem to a one-body problem. For many situations, $m_1 \gg m_2$, and the approximation $\mu \approx m_2$ is valid. Such is the case, for example, for a low-mass planet orbiting a high-mass star.