COMMENTARY ON

Queen Dido, Soap Bubbles, and a Blind Mathematician

A VARIETY of natural phenomena exhibit what is called the minimum principle. The principle is displayed where the amount of energy expended in performing a given action is the least required for its execution, where the path of a particle or wave in moving from one point to another is the shortest possible, where a motion is completed in the shortest possible time, and so on. A famous example of this economy of physical behavior was discovered by Heron of Alexandria. He found that the equality of the angles of incidence and reflection formed by a light ray meeting a plane mirror assures the shortest possible path of the ray in moving from its source to a reflected point by way of the mirror.[1] Sixteen hundred years later Fermat showed that the minimum principle also defined the law of the refraction of light.[2] Other important instances arise in mechanics (e.g., a flexible chain suspended freely from its two ends "assumes a form in which its potential energy is a minimum"), electrodynamics, relativity and quantum theory.

The minimum property and its inverse twin, the maximum property, find expression in certain simple statements of geometry (suggested by practical experience); such as that a straight line is the shortest distance between two points in the plane, or, that of all closed curves of equal length the circle encloses the largest area. Many of these "self-evident" truths were also known to the ancients. The Phoenician princess Dido obtained from a native North African chief a grant of as much land as she could enclose with an ox-hide. A clever girl, she cut the ox-hide into long thin strips, tied the ends together, and staked out a large and valuable territory on which she built Carthage.[3] Horatio—he who made his reputation defending the bridge—was rewarded by a gift of as much land as he could plough round in a day, another illustration of an isoperimetrical problem. The second law of thermodynamics provides a more modern

[1] For a proof of this theorem, regarded as the "germ of the theory of geometrical optics," see Courant and Robbins, *What Is Mathematics?* New York, 1941, pp. 329–332. Also, for a general discussion of the development of the minimal principle of physics, see Wolfgang Yourgrau and Stanley Mandelstam, *Variational Principles in Dynamics and Quantum Theory*, New York, 1955.

[2] Fermat's general principle states that where light travels from any point M to any point M′ in another medium, i.e., in all cases of refraction through prisms, lenses, etc., "the ray pursues that path which requires least time." See Thomas Preston, *The Theory of Light*, Fifth Edition, London, 1928, pp. 102–103.

[3] Lord Kelvin, *Popular Lectures and Essays*, London, 1894, Vol. 2, pp. 571–572.

(and a more discouraging) example of the maximum principle: the entropy (disorder) of the universe tends toward a maximum.

The search for maximum and minimum properties played an important part in the development of modern science. Fermat's discoveries in optics, James Bernoulli's work on the path of quickest descent, were among the labors that led to the conviction that physical laws "are most adequately expressed in terms of a minimum principle that provides a natural access to a more or less complete solution of particular problems."[4] Nor can one disregard the extent to which animistic tendencies and mysticism motivated the search for a unifying principle, even among the outstanding contributors to a rational system of mechanics—Euler, Lagrange, Hamilton, Gauss, to name a few.[5]

Pierre de Maupertuis (1698–1759), the French mathematician and astronomer who enunciated the Principle of Least Action, was convinced that this comprehensive law of conservation demonstrated "God's intention to regulate physical phenomena by a general principle of highest perfection."[6] Other scientists and philosophers shared this view and even went so far as to suggest that a single rule of economy, a rule both grand and simple, would embrace all the phenomena of nature. Mathematicians, fortunately, were able to bring some clarity and order to the problem of least and most, thus preventing physics from abandoning the real world and wandering into a swamp of ideals and noble conjectures.

The calculus of variations is the branch of mathematics which deals with the type of problem we have been considering. Euler was the first to give the subject systematic treatment, though the name itself followed the notation introduced a little later by Lagrange. The method of this calculus was "to find the change caused in an expression containing any number of variables when one lets all or any of the variables change." It deals not only with the maximum and minimum properties discernible in physical events, such as the behavior of light rays, the equilibrium of a mechanical system, the resistance encountered by a bullet moving through air or by a boat through water; it treats also of economic, engineering and operational research problems in which it is sought to maximize (e.g., production, profit) or minimize (e.g., cost, time) certain critical variables.

In the first of the selections which follow, Karl Menger, a noted Austrian mathematician, now professor of mathematics at Illinois Institute of Technology, presents a general survey of the main topics and methods of

[4] Courant and Robbins, *op. cit.*, p. 330.

[5] Philip E. B. Jourdain, *The Principle of Least Action*, Chicago, 1913, p. 1.

[6] Courant and Robbins, *op. cit.*, p. 383; Jourdain, *op. cit.*, p. 10 *et seq.* Maupertuis, it should be noted, was discriminating in his choice of evidence as to the existence of God. When it was suggested that proof of divine wisdom lay in the fact that there were folds in the skin of a rhinoceros—without which that unhappy creature would have been unable to move—he asked: "What would be said of a man who should deny a Providence because the shell of a tortoise has neither folds nor joints?"

the calculus of variations. Dr. Menger is an able expositor and in his brief article gives a very satisfactory popular sketch of a difficult subject. The second selection is taken from a series of lectures on soap bubbles, delivered to "juvenile and popular audiences" at the beginning of the century by the British physicist C. Vernon Boys.[7] Boys was a talented scientist, though not known to a wide public. He was highly skilled as an experimentalist and as a designer of delicate measuring apparatus. His most celebrated piece of work was weighing the earth, which is to say measuring the gravitation constant. In 1895, following a method used by the great eighteenth-century chemist and physicist Henry Cavendish, he measured the gravitational force exerted by two lead balls of 4½ inches diameter on two tiny gold balls (a quarter of an inch in diameter) suspended from a metal beam by fine quartz threads. Comparing the pull of the lead balls on the gold balls (as measured by the deflection of the beam) with the pull of the earth on the gold balls (i.e., their weight), he was able to establish a ratio of about 1 : 1,000,000,000 and thus to fix with remarkable accuracy the mass of the earth.[8] These experiments had to be conducted in a cellar, in the middle of the night to eliminate the vibration caused by passing traffic at ordinary times. Besides making the equipment for this experiment, he designed a camera, which bears his name, to photograph the speed of lightning flashes, and he measured by photography the speed of flying bullets. Except for a brief period as an assistant professor of physics at Imperial College, Boys held no academic post. His income was derived from being a "Gas Referee"—someone who prescribes the method of testing the quality of gas—and an expert witness in patent cases. He died in 1944 at the age of eighty-nine.[9]

The excerpt from Boys' delightful little book discusses some of the brilliant work on bubbles and liquid films by the blind Belgian physicist J. Plateau (1801–1883).[10] Plateau's results bring to notice a particularly fruitful aspect of the co-operation between mathematics and experimental research. That mathematics is a handmaiden of science is a commonplace; but it is less well understood that experiments stimulate mathe-

[7] C. V. Boys, *Soap-Bubbles: Their Colours and the Forces Which Mould Them*, New and Enlarged Edition, London, 1931.

[8] "He found that two point masses of 1 gramme each, 1 centimetre apart, would attract one another with a force of 6.6576×10^{-8} dynes, which makes the density of the Earth 5.5270 times that of water." Sir William Dampier, *A History of Science*, Fourth Edition, Cambridge, 1949, p. 178.

[9] Lord Rayleigh gives an interesting account of Boys' life in *Obituary Notices of Fellows of the Royal Society*, Vol. 4, No. 13, November 1944, pp. 771–788.

[10] J. Plateau, "Sur les Figures d'Équilibre d'une Masse Liquide Sans Pésanteur," Mémoires de l'Académie Royale de Belgique, Nouvelle Série, XXIII, 1849; also, by the same author, a comprehensive treatise, *Statique Expérimentale et Théoretique des Liquides*, Paris, 1873. More recent accounts of research on soap films are given in Courant and Robbins, *op. cit.*, pp. 385–397; R. Courant, "Soap Film Experiments with Minimal Surfaces." *American Mathematical Monthly*, XLVII (1940), pp. 167–174; and in D'Arcy Wentworth Thompson's classic, *On Growth and Form*, Second Edition, Cambridge, 1952.

matical imagination, aid in the formulation of concepts and shape the direction and emphasis of mathematical studies. One of the most remarkable features of the relationship is the successful use of physical models and experiments to solve problems arising in mathematics. In some cases a physical experiment is the only means of determining whether a solution to a specific problem exists; once the existence of a solution has been demonstrated, it may then be possible to complete the mathematical analysis, even to move beyond the conclusions furnished by the model—a sort of boot-strap procedure. It is interesting to point out that what counts in this action and reaction is as much the "physical way of thinking," the turning over in imagination of physical events, as the actual doing of the experiment. Thus, in the nineteenth century, "many of the fundamental theorems of function theory were discovered by Riemann [merely] by thinking of simple experiments concerning the flow of electricity in thin sheets"—without even approaching the laboratory.[11] The celebrated minimum problem associated with Plateau's name [12]—in its simplest version: to find the surface of smallest area with a given boundary—is connected with the solution of a system of partial differential equations. As described in the selection from Courant and Robbins, Plateau found a physical solution for "very general contours" by dipping wire frames of various shapes into liquids of low surface tension.[13] A film immediately spans the frame with a surface of least area—a fact you can discover for yourself by repeating Plateau's experiments with simple home-made equipment. The general mathematical solution of Plateau's problem was somewhat harder to derive. The human brain being less agile than a soap film, the problem was not solved until 1931.

[11] Courant and Robbins, *op. cit.*, pp. 385–386. The discussion above is largely based on the excellent treatment in the Courant book.

[12] The problem was actually first proposed by the famous French mathematician J. L. Lagrange (1736–1813) and can, if you like, be traced back to Dido.

[13] For biographical data on Courant and Robbins see p. 571.

For up and down and round, says he,
Goes all appointed things,
And losses on the roundabouts
Means profits on the swings.

—PATRICK R. CHALMERS

8 What Is Calculus of Variations and What Are Its Applications?

By KARL MENGER

THE calculus of variations belongs to those parts of mathematics whose details it is difficult to explain to a non-mathematician. It is possible, however, to explain its main problems and to sketch its principal methods for everybody.

The first human being to solve a problem of calculus of variations seems to have been Queen Dido of Carthage. When she was promised as much land as might lie within the boundaries of a bull's hide, she cut the hide into many thin strips, put them together into one long strip, the ends of which she united, and then she tried to secure as extensive a territory as possible within this boundary. History does not describe the form of the territory she chose, but if she was a good mathematician she covered the territory in the form of a circle, for to-day we know: Of all surfaces bounded by curves of a given length, the circle is the one of largest area. The branch of mathematics which establishes a rigorous proof of this statement is the calculus of variations.

Newton was the first mathematician to publish a result in this field. If a body moves in the air, it meets with a certain resistance, which depends on the shape of the body. The problem Newton studied was, what shape of body would guarantee the least possible resistance? Applications of this problem are obvious. The rifle bullet is designed in such shape as to meet with a minimum resistance in the air. Newton published a correct answer to a special case of this problem, namely, that the surface of the solid considered is obtained by revolving a curve around an axis. But he did not give the proof or the calculations that had led him to the answer. So Newton's solution had no great effect on the development of mathematics.

A new branch of mathematics started with another problem formulated and studied by the brothers Bernoulli in the seventeenth century. If a small body moves under the influence of gravity along a given curve from one point to another, then the time required naturally depends on the

form of the curve. Whether the body moves along a straight line (on an inclined plane) or along a circle makes a difference. Bernoulli's question was: which path takes the shortest time? One might think that the motion along the straight line is the quickest, but already Galileo had noticed that the time required along some curves is less than along a straight line. The brothers Bernoulli determined the form of the curve which takes the shortest possible time. It is a curve which was already well known in geometry for other interesting properties and had been called cycloid.

What is common to all these problems is this: A number is associated with each curve of a certain family of curves. In the first example (that of Queen Dido) the family consists of all closed curves with a given length, and the associated number is the area of the inclosed surface; in the second example (that of Newton) the number is the resistance which a body somehow associated with the curve meets in the air; in the third example (that of the brothers Bernoulli) the family of curves consists of all curves joining two given points, and the number associated with each curve is the time it takes a body to fall along this curve. The problem consists in finding the curve for which the associated number attains a maximum or a minimum—this is the largest or the smallest possible value; in Dido's example, the maximum area; in Newton's example, the minimum resistance; in Bernoulli's example, the shortest time.

Some problems concerning maxima and minima are studied in differential calculus, taught in college. They may be formulated in the following way: Given a single curve, where is its lowest and where is the highest point? or given a single surface, where are its peaks and where are its pits? With each point of the curve or the surface, there is associated a certain number, namely, the height of the point above a horizontal axis or a horizontal plane. We are looking for those points at which this height is greatest or least. In differential calculus we deal thus with maxima and minima of so-called functions of points, *i.e.*, of numbers associated with points; in calculus of variations, however, with maxima and minima of so-called functions of curves, that is, of numbers associated with curves or of numbers associated with still more complicated geometric entities, like surfaces.

A famous question concerning surfaces is the following problem, the so-called problem of Plateau: if a closed curve in our three-dimensional space is given, we can span into it many different surfaces, all of them bounded by the given curve, *e.g.*, if the given curve is a circle we can span into it a plane circular area or a hemisphere or other surfaces bounded by the circle. Each of these surfaces has an area. Which of all these surfaces has the smallest area? If the given curve lies in a plane, like a circle, then, obviously, the plane surface inscribed has the minimum area. If the given curve, however, does not lie in the plane, like a knotted curve in the three-dimensional space, then the problem of finding the surface of minimal area bounded by the curve is very complicated. The question, which was solved some years ago by T. Radó (Ohio State University) and in a still more general way by J. Douglas (Massachusetts Institute of Technology), has applications to physics, for if the curve is made of a thin wire and we try to span into it a thin soap film, then this film will assume just the form of the surface of minimum area.

We frequently find that nature acts in such a way as to minimize certain magnitudes. The soap film will take the shape of a surface of smallest area. Light always follows the shortest path, that is, the straight line, and, even when reflected or broken, follows a path which takes a minimum of time. In mechanical systems we find that the movements actually take place in a form which requires less effort in a certain sense than any other possible movement would use. There was a period, about 150 years ago, when physicists believed that the whole of physics might be deduced from certain minimizing principles, subject to calculus of variations, and these principles were interpreted as tendencies—so to say, economical tendencies of nature. Nature seems to follow the tendency of economizing certain magnitudes, of obtaining maximum effects with given means, or to spend minimal means for given effects.

In this century Einstein's general theory of relativity has as one of its basic hypotheses such a minimal principle: that in our space-time world, however complicated its geometry be, light rays and bodies upon which no force acts move along shortest lines.

If we speak of tendencies in nature or of economic principles of nature, then we do so in analogy to our human tendencies and economic principles. A producer most often will adopt a way of production which will require a minimum of cost, compared with other ways of equal results; or which, compared with other methods of equal cost, will promise a maximum return. It is obvious that for this reason the mathematical theory of economics is to a large extent application of calculus of variations. Such applications have been considered by G. C. Evans (University of California) and in particular by Charles F. Roos (New York City). A simple but interesting example, due to the economist H. Hotelling (Columbia University), is to find the most economic way of production in a mine. We may start with a great output and decrease the output later or we may increase the output in time or we may produce with a constant rate of output. Each way of production can be represented by a curve. If we have conjectures concerning the development of the price of the produced metal, then we may associate a number with each of these curves—the possible profit. The problem is to find the way of production which will probably yield the greatest profit.

In the mathematical theory of the maximum and minimum problems in calculus of variations, different methods are employed. The old classical method consists in finding criteria as to whether or not for a given curve the corresponding number assumes a maximum or minimum. In order to find such criteria a considered curve is a little varied, and it is from this method that the name "calculus of variations" for the whole branch of mathematics is derived. The first result of this method, which to-day is represented by G. A. Bliss (University of Chicago) and his school, was the equation of Euler-Lagrange, which states: A curve which minimizes or maximizes the corresponding number must in each of its points have a certain curvature which can be determined for each problem.

Another method consists in finding out quite in general whether or not a given problem is soluble at all. For example, we consider the two following extremely simple problems: two given points may be joined by all possible curves; which of them has the shortest length, and which of them has the greatest length? The first problem is soluble: The straight line segment joining the two points is the shortest line joining them. The second problem is not soluble: There is no longest curve joining two given points, for no matter how long a curve joining them may be, there is always one which is still longer. The length is a number associated with each curve which for no curve assumes a finite maximum.

This second method of calculus of variations was initiated by the German mathematician Hilbert at the beginning of the century. The Italian mathematician Tonelli found out twenty years ago that the deeper reason for the solubility of the minimum problem concerning the length, that is, for the existence of a shortest line between every two points, is the following property of the length: A curve between two fixed points being given, there are always other curves as near as you please to it, and yet much longer than the given curve (*e.g.*, some zigzag lines near the given curve). But there is no curve very near to the given curve and joining the same two points, which is much shorter than the given curve. This property of the length is called the semi-continuity of the length. Contributions to this Hilbert-Tonelli method are due to E. J. McShane (University of Virginia), L. M. Graves (University of Chicago) and to the author.

Another way of calculus of variations was started in this country. G. D. Birkhoff (Harvard University) was the first to consider so-called minimax problems dealing with "stationary" curves which are minimizing with respect to certain neighboring curves and at the same time maximizing with respect to other curves. While the minimum and maximum problems of calculus of variations correspond to the problem in the ordinary calculus of finding peaks and pits of a surface, the minimax problems correspond to the problem of finding the saddle points of the surface (the passes of a mountain). The simplest example of such a stationary curve is obtained in the following way: if we consider two points of the equator of the earth, then their shortest connection on the surface of the earth is the minor arc of the equator between them. There is, as we have seen, no longest curve joining the two points. But there is one curve on the surface of the earth which, though it is neither the shortest nor the longest one, plays a special rôle in some respects, namely, the major arc of the equator between the two points.

One of the greatest advances of calculus of variations in recent times has been the development of a complete and systematic theory of stationary curves due to Marston Morse (Institute of Advanced Study). The most simple example of this theory, which calculates the number of minimizing and maximizing curves as well as of stationary curves, is the following "geographical" theorem quoted by Morse: If we add the number of peaks and the number of pits on the surface of the earth, and subtract the number of passes, then the result will be the number 2, whatever the shape of the mountains may be (highlands excluded).

There are many technical details of calculus of variations which are hardly available to a non-mathematician. They are the type of theory which frequently leads to the belief that mathematical theories are remote from the urgent problems of the world and useless. Real mathematicians do not worry too much about these reproaches which are engendered by a lack of knowledge of the history of science. Mathematicians study their problems on account of their intrinsic interest, and develop their theories on account of their beauty. History shows that some of these mathematical theories which were developed without any chance of immediate use later on found very important applications. Certainly this is true in the case of calculus of variations: If the cars, the locomotives, the planes, etc., produced to-day are different in form from what they used to be fifteen years ago, then a good deal of this change is due to calculus of variations. For we use streamline form in order to decrease to the minimum possible the resistance of the air in driving. It is through physics that we learn the actual laws of this resistance. But if we wish to discover the form which guarantees the least resistance, then we need calculus of variations.

Lex perpetua naturae est ut agat minimo labore, mediis et modis simplicissimis, facillimis, certis et tutis: evitando, quam maxime fieri potest, incommoditates et prolixitates. —GIOVANNI BORELLI (*1608–1679*)

9 The Soap-bubble

By C. VERNON BOYS

IT can only be our familiarity with soap-bubbles from our earliest recollections, causing us to accept their existence as a matter of course, that prevents most of us from being seriously puzzled as to why they can be blown at all. And yet it is far more difficult to realize that such things ought to be possible than it is to understand anything that I have put before you as to their actions or their form. In the first place, when people realize that the surface of a liquid is tense, that it acts like a stretched skin, they may naturally think that a soap-bubble can be blown because in the case of soap-solution the "skin" is very strong. Now the fact is just the opposite. Pure water, with which a bubble cannot be blown in air and which will not even froth, has a "skin" or surface tension three times as strong as soap-solution, as tested in the usual way, *e.g.*, by the rise in a capillary tube. Even with a minute amount of soap present the surface tension falls off from about 3¼ grains to the linear inch to 1¼ grains, as calculated from experiments with bubbles by Plateau. The liquid rises but little more than one-third of the height in a capillary tube. The soap-film has two surfaces each with a strength of 1¼ grains to the inch and so pulls with a strength of 2½ grains to the inch. Many liquids will froth that will not blow bubbles. Lord Rayleigh has shown that a pure liquid will not froth though a mixture of two pure liquids, *e.g.*, water and alcohol, will. Whatever the property is which enables a liquid to froth must be well developed for it to allow bubbles to be blown. I have repeatedly spoken of the tension of a soap-film as if it were constant, and so it is very nearly, and yet, as Prof. Willard Gibbs pointed out, it cannot be exactly so. For, consider any large bubble or, for convenience, a plane vertical film stretched in a wire ring. If the tension of two grains and a half to the inch were really identically the same in all parts, the middle parts of the film being pulled upwards and downwards to an equal extent by the rest of the film above and below it would in effect not be pulled by them at all and like other unsupported things would fall, starting like a stone with the acceleration due to gravity. Now the middle part of such a film does nothing of the kind. It appears to be at rest, and if there is any downward movement it is too slow to be noticed. The upper part therefore of the film must be more tightly stretched than the lower part, the difference being the weight of the intervening film. If the ring is turned over to invert the film then the conditions are reversed, and yet the middle part does not fall. The bubble therefore has the remarkable property within small limits of adjusting its tension to the load. Willard Gibbs put forward the view that this was due to the surface material not being identical with the liquid within the thickness of the film. That the surface was contaminated by material which lowered its surface tension and which by stretching of the film became diluted, making the film stronger, or by contraction became concentrated, making the film weaker. His own words are so apt and so much better than mine that I shall quote from his *Thermodynamics*, p. 313: "For, in a thick film (as contrasted with a black film), the increase of tension with the extension, which is necessary for its stability with respect to extension, is connected with an excess of the soap (or of some of its components) at the surface as compared with the interior of the film."

This is analogous to the effect of oil on water. Lord Rayleigh has by a beautiful experiment supported the contamination theory, for he measured the surface tension of the surface of a soap-solution within the first hundredth of a second of its existence. He then found it to be the same

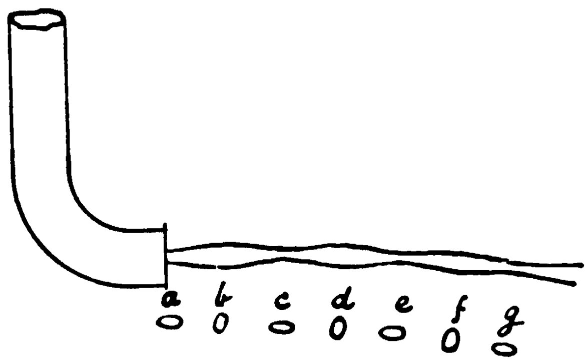

FIGURE 1

as that of water, for the surface contamination had no time to form. He allowed the liquid to issue from a small elliptical hole in a thin plate covering the end of a tube which came from a reservoir of the solution. When liquid issues from such a hole as at *a*, Figure 1, the cross section of the stream being elliptical, as shown below, tends in virtue of the surface tension to become circular, but when it gets circular the movement already set up in the section cannot be suddenly arrested and so the liquid continues its movement until it is elliptical in the other direction as at *b*, and this process is continued at a definite rate depending

upon the surface tension and density of the liquid and size of the jet. At the same time the liquid is issuing at a definite rate depending on the depth of the orifice below the free surface, and so when the conditions are well chosen the liquid travels from *a* to *c* while the ellipse goes through its complete evolution, and this is repeated several times. Now if the surface tension were less the evolution would take longer and the distance between the nodes *a c e g* would be greater. With the same liquid head the distance between the nodes is the same with pure water and with soap-solution, showing that their surface tensions at first are the same, but with alcohol, which has its own surface tension from the beginning, the distance between the nodes is greater, as the surface tension is lower in a higher proportion than that in which the density is lower. Professor Donnan has quite recently shown by direct experiment of surpassing delicacy, that there is a surface concentration of the kind and amount required by Gibbs's theory.

The following experiment also indicates the existence of a surface concentration. If a soap-bubble is blown on a horizontal ring so that the diameter of the ring is very little less than that of the bubble, and the wetted stopper from a bottle of ammonia is brought close to the upper side of the bubble, it will immediately shrink away from the stopper and slip through the ring as though annoyed by the smell of the ammonia. Or, if below, it will retire to the upper side of the ring if the stopper is held below it. What really happens is that the ammonia combines with some of the constituents of the soap which are concentrated on the surface, and so raises the tension of the film on one side of the ring; it therefore contracts and blows out the film on the other side which has not yet been influenced by the ammonia. That part of the film influenced by the ammonia also becomes thicker and the rest thinner, as shown by the colours, which are then far more brilliant and variegated.

Going back now to the soap-film we see then that whatever its shape the upper parts are somewhat more tightly stretched than the lower parts, and in the case of a vertical film the difference is equal to that which will support the intervening film. There is, however, a limit beyond which this process will not go; there is a limit to the size of a soap-bubble. I do not know what this is. I have blown spherical bubbles up to 2½ feet in diameter, and others no doubt have blown bubbles larger still. I have also taken a piece of thin string ten feet long and tied it into a loop after wetting it with soap solution and letting it untwist. Holding a finger of each hand in the loop and immersing it in soap-solution I have drawn it out and pulled it tight, forming a film in this way five feet long. On holding the loop vertical the film remained unbroken, showing that five feet is less than the limit with even a moderately thick bubble. With a thin bubble the limit should be greater still. Judging by the colour and using certain other information one concludes that the average thickness of the film must have been about thirty millionths of an inch and its weight about $\frac{9}{1000}$ of a grain per square inch. Taking a film an inch wide, the five feet or sixty inches would weigh close upon half a grain, that is about one-fifth of the total load that the film can carry, showing at least 20% capacity in the soap-film for adjusting its strength to necessity, which is far more than could have been expected.

I have also found that with the application of increased forces the bubble rapidly thins to a straw colour or white, so that the 20% increase of load is not exceeded, but a film of this colour might be thirty-three feet high, or a black film ten times as much.

The feeble tension of 2½ grains to the inch in a soap-film is quite enough in the case of the five-foot loop to make it require some exertion to keep it pulled so that the two threads are not much nearer in the middle than at the ends. In fact, this experiment provides the means by which the feeble tension of 2½ grains to the inch may be measured by means of a seven-pound weight. If for instance the threads are seventy inches long, equidistant at their ends, and are $\frac{1}{16}$ inch nearer in the middle than at their ends when placed horizontally and stretched with seven-pound weights, then the tension of the film works out at 2½ grains to the inch exactly. This is obtained as follows: The diameter of the circle of which the curved thread is a part is equal to half the length of the thread multiplied by itself and divided by the deviation of its middle point. The diameter of the circle then is $35 \times 35 \times 32$ inches. The tension of the thread, made equal to 7 lbs. or 7×7000 grains, is equal to the tension of the film in grains per inch multiplied by half the diameter of the circle of curvature of the thread. In other words the tension of the film in grains per inch is equal to the tension of the thread divided by half the diameter of the circle of curvature of the thread. The film tension then is equal to $\frac{7 \times 7000}{35 \times 35 \times 16}$ grains per inch. The fraction cancels at once to 2½ grains per inch. As the upper parts of a vertical film are stretched more tightly than the lower parts, the pair of threads will be drawn together in the middle, rather more than $\frac{1}{16}$ inch, as the pair of threads are gradually tilted from the horizontal towards a vertical position, and then as the film drains into the threads and become lighter the threads will separate slightly.

Large bubbles are short-lived not only because, if it were a matter of chance, a bubble a foot in diameter with a surface thirty-six times as great as that of one two inches in diameter would be thirty-six times as likely to break if all the film were equally tender, but because the upper parts being in a state of greater tension have less margin of safety than the lower parts. These large bubbles however by no means necessarily break at the top. When a large, free-floating bubble is seen in bright sunlight on a dark background it is almost possible to follow the process of breaking. What is really seen, however, is a shower of spray moving in the opposite direction to that in which the hole is first made, while the air which cannot be seen blows out in the opposite direction to that of the spray. By the time it is done, and it does not take long, the momentum given to the moving drops in one direction and to the moving air in the opposite direction are equal to one another, but as the air weighs far more than the water the spray is thrown the more rapidly.

The breaking of a bubble is itself an interesting study. Duprée long ago showed that the whole of the work done in extending a bubble is to be found in the velocity given to the spray as it breaks, and thence he deduced a speed of breaking of 105 feet a second or seventy-two miles an hour for a thin bubble, and Lord Rayleigh for a thicker bubble found a speed of forty-eight feet a second or thirty-three miles an hour. The speed of breaking of a soap-bubble is curious in that it does not get up speed and keep going faster all the time, as most mechanical things do, it starts full speed at once, and its speed only changes in accordance with its thickness, the thick parts breaking more slowly. It may be worth while to show how the speed is arrived at theoretically. Take in imagination a film between two parallel and wetted wires an inch apart and extend it by drawing a wetted edge of card, india-rubber or celluloid along the wires. Then the work done in extending it for a foot, say, not counting of course the friction of the moving edge, is with a tension of 2½ grains to the inch equal to $2\frac{1}{2} \times 12$ inch-grains, or if it is extended a yard to $2\frac{1}{2} \times 36$ inch-grains. Let us pull it out to such a length that the weight of the film drawn out is itself equal to 2½ grains, that is to the weight its own tension will just carry. By way of example let us consider a film not very thick or very thin, but of the well-defined apple-green colour. This, it can be calculated, is just under twenty millionths of an inch thick, and it weighs $\frac{5}{1000}$ or $\frac{1}{200}$ of a grain to the square inch. The length of this film one inch wide that will weigh 2½ grains is therefore 500 inches or forty-two feet nearly. The work done in stretching a film of this area is $2\frac{1}{2} \times 500$ inch-grains. The work contained in the flying spray must also be $2\frac{1}{2} \times 500$ inch-grains. Now the work contained in a thing moving at any speed is the same as that which would be needed to lift it to such a height that it would if falling without obstruction acquire that speed. The work done in lifting the 2½ grains through 500 inches is exactly the same as that done in pulling out the film 500 inches horizontally against its own tension, as the force and the distance are the same in both cases. The velocity of the spray, and therefore of the edge from which it is scattered is the same as that of a stone falling through a distance equal to the length of the film, the weight of which is equal to the tension at its end. Completing the figures a stone in this latitude falling forty-two feet acquires a velocity of fifty-two feet a second, which therefore is the speed of the breaking edge of the film. I conclude therefore, as the speed found in this example is intermediate between those found by Duprée and Lord Rayleigh, that I have chosen a film intermediate in thickness between those chosen by these philosophers. I would only add that a bubble has to be reduced to one quarter of its thickness to make it break twice as fast, then the corresponding length will be four times as great, and it requires a fall from four times the height to acquire twice the speed. A black film is about one thirty-sixth of the thickness of the apple-green film, it should therefore break six times as fast or 312 feet a second, or 212 miles an hour. The extra black film of half the thickness should break at the enormous speed of 300 miles an hour. These speeds would hardly be realized in practice as the viscosity of the liquid would reduce them.

Lord Rayleigh photographed the breaking soap-film by placing a ring on which it was stretched in an inclined position, and then dropping through it a shot wet with alcohol, and about a thousandth of a second later photographing it. For this purpose he arranged two electromagnets, one to drop the wet shot and the other to drop another shot at the same instant. The second shot was allowed a slightly longer fall, so as to take a thousandth of a second, or whatever interval he wanted, longer than the first; it then by passing between two knobs in the circuit of a charged Leyden jar let this off, and the electric spark provided the light and the sufficiently short exposure to give a good and sharp photograph. The sharp retreating edge is seen with minute droplets either just detached or leaving the film. Following Lord Rayleigh I photographed a breaking film by piercing it with a minute electric spark between two needles, one on either side of the film, and by means of a piano-wire spring like a mousetrap determining the existence of this spark, and then a ten-thousandth of a second or more later letting off the spark by the light of which it was taken. My electrical arrangements were akin to those by which I photographed bullets in their flight in one thirteen-millionth of a second, but my optical arrangements were similar to Lord Rayleigh's. One photograph of a vertical film which had become a great deal thinner in its upper part, is interesting as whereas all the lower part has a circular outline, the upper part breaks into a bay showing the speed of breaking upwards in the thin film to be much greater. In all the photographs the needle spark appears by its own light, so the point at which the break first occurs is seen as well as the circular retreating edge.

* * * * *

COMPOSITE BUBBLES

A single bubble floating in the air is spherical, and as we have seen this form is assumed because of all shapes that exist this one has the smallest surface in relation to its content, that is, there is so much air within, and the elastic soap-film, trying to become as small as possible, moulds the air to this shape. If the bubble were of any other shape the film could become of less surface still by becoming spherical. When however the bubble is not single, say two have been blown in real contact with one another, again the bubbles must together take such a form that the total surface of the two spherical segments and of the part common to both, which I shall call the interface, is the smallest possible surface which will contain the two quantities of air and keep them separate. As the soap-bubble provides such a simple and pleasing way of demonstrating the solution of this problem, which is really a mathematical problem, it will be worth while to devote a little time to its consideration. Let us suppose that the two bubbles which are joined by an interface are not equal, and that

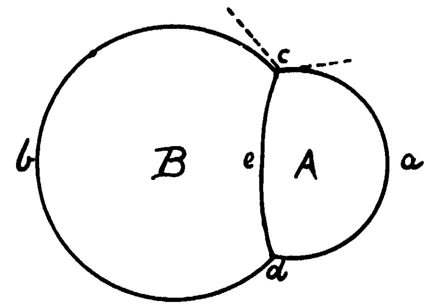

FIGURE 2

Figure 2 represents a section through the centres of both, *A* being the smaller and *B* the bigger bubble. In the first place we have seen that the pressure within a bubble is proportional to its curvature or to 1 divided by the radius of the bubble. The pressure in *A*, by which I mean the excess over atmospheric pressure, will therefore be greater than the pressure in *B* in the proportion in which the radius of *B* is greater than the radius of *A*, and the air can only be prevented from blowing through by the curvature of the interface. In fact this curvature balances the difference of pressure. Another way of saying the same thing is this: the curved and stretched film *dac* pushes the air in *A* to the left, and it takes the two less curved but equally stretched films *dbc* and *dec*, pushing to the right to balance the action of the more curved film *dac*. Or, most shortly of all, the curvature of *dac* is equal to the sum of the curvatures *dbc* and *dec*. Now consider the point *c* or *d* in the figure either of which represent a section of the circle where the two bubbles meet; at any point in this circle the three films meet and are all three pulling with the same force. They can only balance when the angles where they meet are equal or are each angles of 120°. Owing to the curvature of the lines, these angles do

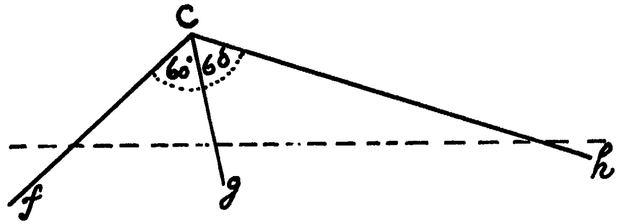

FIGURE 3

not look equal, but I have dotted in at *c* tangents to the three curves at the point *c*, and they clearly make equal angles with each other.

This equality of the angles is not an independent proposition to the last with regard to the curvatures; if either condition is fulfilled the other necessarily follows, as also does the one I opened with that the total surface must be the least possible. Plateau, the blind Belgian professor, discussed this, as he did everything that was known about the soap-bubble, in his book *Statique des Liquides*, published in Brussels, a book which is a worthy monument of the brilliant author. He there describes a simple geometrical construction by which any pair of bubbles and their interface may be drawn correctly.

From any point *C* draw three lines, *cf, cg, ch,* making angles of 60°, as shown in Figure 3. Then, on drawing any straight line across so as to cut the three lines such as the one shown dotted in the figure, the three points where it cuts the three lines will be the centres of three circles representing possible bubbles. The point where it cuts the middle line is the centre of the smaller bubble, and of the other two points, that which is nearer to *C* is the centre of the second bubble, and that which is further is the centre of the interface. Now, with the point of the compasses placed successively at each of these points, draw portions of circles passing through *C*, as shown in Figure 4, in which the construction lines of Figure 3 are shown dotted and the circular arcs are shown in full lines.

Having drawn a number of these on sheets of paper, with the curved lines very black so as to show well, lay a sheet of glass upon one, and having wetted the upper side with soap solution, blow a half bubble upon it, and then a second so as to join the first, and then with a very small pipe, or even a straw, with one end closed with sealing-wax opened afterwards

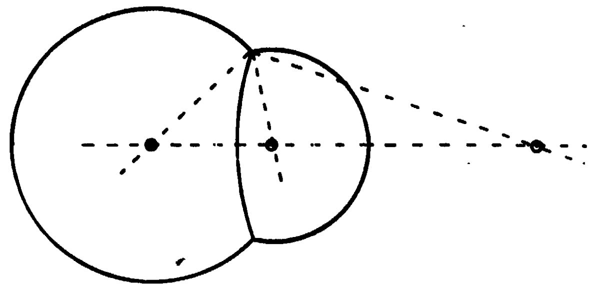

FIGURE 4

with a hot pin so as to allow air to pass very slowly, gently blow in or draw out air until the two bubbles are the same size as those in the drawing, moving the glass about so as to keep them over the figure. You will then find how the soap-bubble solves the problem automatically, and how the edges of the half bubbles exactly fit throughout their whole extent the drawings that you have made.

If the dotted line in Figure 3 cuts *cf* and *ch* at equal distances from *c*, then it will cut *cg* at half that distance from *c*, and we have the case of a bubble in contact with one of twice its diameter. In this case the interface has the same curvature as that of the larger bubble, but is reversed in direction, and each has half the curvature of the smaller bubble.

If the dotted line in Figure 3 cuts *cf* and *cg* at equal distances from *c*, it will be parallel to *ch* and will never cut it. The two bubbles will then be equal, and the interface will then have no curvature, or, in other words, it will be perfectly flat, and the line *cd*, Figure 2, which is its section, will be a straight line.

There are other cases where the same reciprocal laws apply as well as this one of the radii of curvature of joined soap bubbles. It may be written in a short form as follows: $\frac{1}{A} = \frac{1}{B} + \frac{1}{e}$, using the letters of Figure 2 to represent the lengths of the radii of the corresponding circles. For instance, a lens or mirror in optics has what is called a principal focus, *i.e.*, a distance, say *A*, from it at which the rays of the sun come to a focus and make it into a burning glass. If, instead of the sun, a candle flame is placed a little way off, at a distance, say *B*, greater than *A*, then the lens or mirror will produce an image of the flame at a distance, say *e*, such that $\frac{1}{A} = \frac{1}{B} + \frac{1}{e}$. Or, again, if the electrical resistance of a length of wire, say

A inches long, is so much, then the electrical resistance of two pieces of the same wire, B and e inches long, joined so that the current is divided between them, will be the same as that of A if $\frac{1}{A} = \frac{1}{B} + \frac{1}{e}$.

The soap-bubble then may be used to give a numerical solution of an optical and of an electrical problem.

Plateau gives one other geometrical illustration, the proof of which, however, is rather long and difficult, but which is so elegant that I cannot refrain from at least stating it. When three bubbles are in contact with one another, as shown in Figure 5, there are of course three interfaces meeting one another, as well as the three bubbles all at angles of 120°.

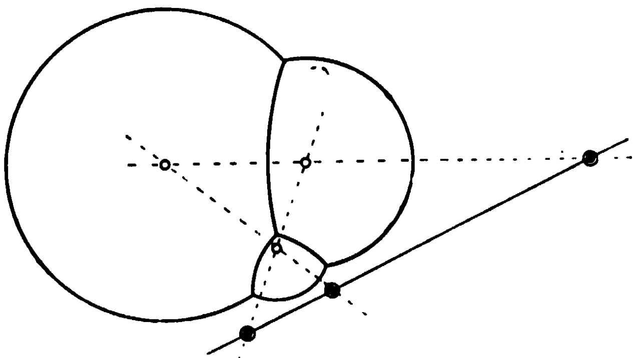

FIGURE 5

The centres of curvature also of the three bubbles and of the three interfaces, also necessarily lie in a plane, but what is not evident and yet is true is that the centres of curvature, marked by small double circles of the three interfaces, lie in a straight line. If any of you are adepts in geometry, whether Euclidean or analytical, this will be a nice problem for you to solve, as also that the surface of the three bubbles and of the three interfaces is the least possible that will confine and separate the three quantities of air. The proof that the three films drawn according to the construction of Figure 3 have the curvatures stated is much more easy, and I should recommend you to start on this first. If you want a clue, draw a line from the point where the dotted line cuts cg parallel to cf, and then consider what is before you.

I do not suppose there is anyone in this room who has not occasionally blown a common soap bubble, and while admiring the perfection of its form, and the marvelous brilliancy of its colours, wondered how it is that such a magnificent object can be so easily produced. I hope that none of you are yet tired of playing with bubbles, because, as I hope we shall see, there is more in a common bubble than those who have only played with them generally imagine. —SIR CHARLES VERNON BOYS (*Soap Bubbles*)

10 Plateau's Problem

By RICHARD COURANT
and HERBERT ROBBINS

EXPERIMENTAL SOLUTIONS OF MINIMUM PROBLEMS SOAP FILM EXPERIMENTS

INTRODUCTION

IT is usually very difficult, and sometimes impossible, to solve variational problems explicitly in terms of formulas or geometrical constructions involving known simple elements. Instead, one is often satisfied with merely proving the existence of a solution under certain conditions and afterwards investigating properties of the solution. In many cases, when such an existence proof turns out to be more or less difficult, it is stimulating to realize the mathematical conditions of the problem by corresponding physical devices, or rather, to consider the mathematical problem as an interpretation of a physical phenomenon. The existence of the physical phenomenon then represents the solution of the mathematical problem. Of course, this is only a plausibility consideration and not a mathematical proof, since the question still remains whether the mathematical interpretation of the physical event is adequate in a strict sense, or whether it gives only an inadequate image of physical reality. Sometimes such experiments, even if performed only in the imagination, are convincing even to mathematicians. In the nineteenth century many of the fundamental theorems of function theory were discovered by Riemann by thinking of simple experiments concerning the flow of electricity in thin metallic sheets.

In this section we wish to discuss, on the basis of experimental demonstrations, one of the deeper problems of the calculus of variations. This problem has been called Plateau's problem, because Plateau (1801–1883), a Belgian physicist, made interesting experiments on this subject. The problem itself is much older and goes back to the initial phases of the calculus of variations. In its simplest form it is the following: to find the surface of smallest area bounded by a given closed contour in space. We shall also discuss experiments connected with some related questions, and it will turn out that much light can thus be thrown on some of our previous results as well as on certain mathematical problems of a new type.

SOAP FILM EXPERIMENTS

Mathematically, Plateau's problem is connected with the solution of a "partial differential equation," or a system of such equations. Euler showed that all (non-plane) minimal surfaces must be saddle-shaped and that the mean curvature[1] at every point must be zero. The solution was shown to exist for many special cases during the last century, but the existence of the solution for the general case was proved only recently, by J. Douglas and by T. Radó.

Plateau's experiments immediately yield physical solutions for very general contours. If one dips any closed contour made of wire into a liquid of low surface tension and then withdraws it, a film in the form of a minimal surface of least area will span the contour. (We assume that we may neglect gravity and other forces which interfere with the tendency of the film to assume a position of stable equilibrium by attain-

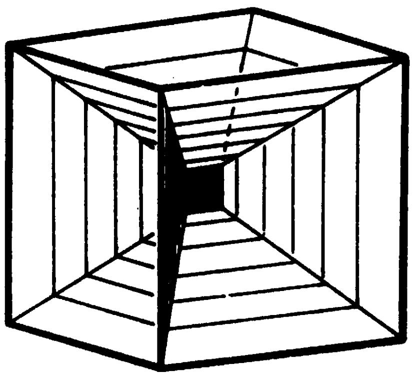

FIGURE 1—Cubic frame spanning a soap film system of 13 nearly plane surfaces.

ing the smallest possible area and thus the least possible value of the potential energy due to surface tension.) A good recipe for such a liquid is the following: Dissolve 10 grams of pure dry sodium oleate in 500 grams of distilled water, and mix 15 cubic units of the solution with 11 cubic units of glycerin. Films obtained with this solution and with frames

[1] The mean curvature of a surface at a point *P* is defined in the following way: Consider the perpendicular to the surface at *P*, and all planes containing it. These planes will intersect the surface in curves which in general have different curvatures at *P*. Now consider the curves of minimum and maximum curvature respectively. (In general, the planes containing these curves will be perpendicular to each other.) One-half the sum of these two curvatures is the mean curvature of the surface at *P*.

of brass wire are relatively stable. The frames should not exceed five or six inches in diameter.

With this method it is very easy to "solve" Plateau's problem simply by shaping the wire into the desired form. Beautiful models are obtained in polygonal wire frames formed by a sequence of edges of a regular polyhedron. In particular, it is interesting to dip the whole frame of a cube into such a solution. The result is first a system of different surfaces meeting each other at angles of 120° along lines of intersection. (If the cube is withdrawn carefully, there will be thirteen nearly plane surfaces.) Then we may pierce and destroy enough of these different surfaces so that only one surface bounded by a closed polygon remains. Several beautiful surfaces may be formed in this way. The same experiment can also be performed with a tetrahedron.

NEW EXPERIMENTS ON PLATEAU'S PROBLEM

The scope of soap film experiments with minimal surfaces is wider than these original demonstrations by Plateau. In recent years the problem of minimal surfaces has been studied when not only one but any number of contours is prescribed, and when, in addition, the topological structure of the surface is more complicated. For example, the surface might be one-sided or of genus different from zero. These more general problems produce an amazing variety of geometrical phenomena that can be exhib-

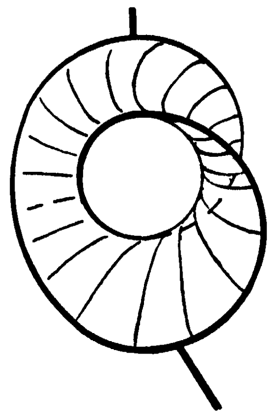

FIGURE 2—One-sided surface (Moebius strip).

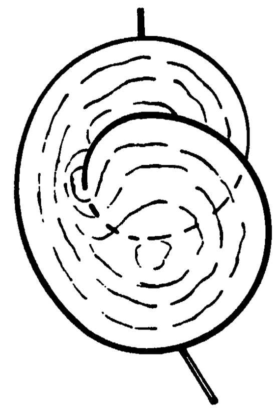

FIGURE 3—Two-sided surface.

ited by soap film experiments. In this connection it is very useful to make the wire frames flexible, and to study the effect of deformations of the prescribed boundaries on the solution.

We shall describe several examples:

1) If the contour is a circle we obtain a plane circular disk. If we continuously deform the boundary circle we might expect that the minimal surface would always retain the topological character of a disk. This is not the case. If the boundary is deformed into the shape indicated by Figure 2, we obtain a minimal surface that is no longer simply connected, like the disk, but is a one-sided Moebius strip. Conversely, we might start with this frame and with a soap film in the shape of a Moebius strip. We may deform the wire frame by pulling handles soldered to it (Figure 2). In this process we shall reach a moment when suddenly the topological character of the film changes, so that the surface is again of the type of a simply connected disk (Figure 3). Reversing the deformation we again obtain a Moebius strip. In this alternating deformation process the mutation of the simply connected surface into the Moebius strip takes place at a later stage. This shows that there must be a range of shapes of the contour for which both the Moebius strip and the simply connected surface are stable, i.e., furnish relative minima. But when the Moebius strip has a much smaller area than the other surface, the latter is too unstable to be formed.

2) We may span a minimal surface of revolution between two circles. After the withdrawal of the wire frames from the solution we find, not one simple surface, but a structure of three surfaces meeting at angles of 120°,

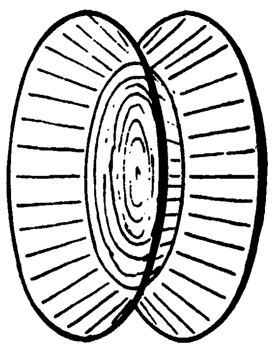

FIGURE 4—System of three surfaces.

one of which is a simple circular disk parallel to the prescribed boundary circles (Figure 4). By destroying this intermediate surface the classical catenoid is produced (the catenoid is the surface obtained by revolving a catenary about a line perpendicular to its axis of symmetry). If the two boundary circles are pulled apart, there is a moment when the doubly connected minimal surface (the catenoid) becomes unstable. At this moment the catenoid jumps discontinuously into two separated disks. This process is, of course, not reversible.

3) Another significant example is provided by the frame of Figures 5–7 in which can be spanned three different minimal surfaces. Each is bounded by the same simple closed curve; one (Figure 5) has the genus 1, while the other two are simply connected, and in a way symmetrical to each other. The latter have the same area if the contour is completely symmetrical. But if this is not the case then only one gives the absolute minimum of the area while the other will give a relative minimum, provided that the minimum is sought among simply connected surfaces. The possibility of the solution of genus 1 depends on the fact that by admitting surfaces of genus 1 one may obtain a smaller area than by requiring that the surface be simply connected. By deforming the frame we must, if the deformation is radical enough, come to a point where this is no longer true. At that moment the surface of genus 1 becomes more and more unstable and suddenly jumps discontinuously into the simply connected stable solution represented by Figure 6 or 7. If we start with one of these simply connected solutions, such as Figure 7, we may deform it in such a way that the other simply connected solution of Figure 6 becomes much more stable. The consequence is that at a certain moment a discontinuous transition from one to the other will take place. By slowly reversing the

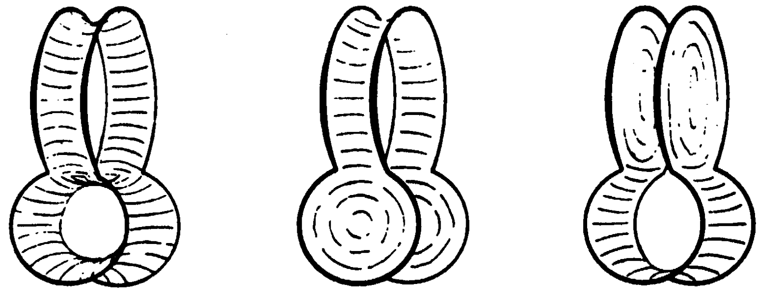

FIGURE 5 FIGURE 6 FIGURE 7

Frame spanning three different surfaces of genus 0 and 1.

deformation, we return to the initial position of the frame, but now with the other solution in it. We can repeat the process in the opposite direction, and in this way swing back and forth by discontinuous transitions between the two types. By careful handling, one may also transform discontinuously either one of the simply connected solutions into that of genus 1. For this purpose we have to bring the disk-like parts very close to each other, so that the surface of genus 1 becomes markedly more stable. Sometimes in this process intermediate pieces of film appear first and have to be destroyed before the surface of genus 1 is obtained.

This example shows not only the possibility of different solutions of the same topological type, but also of another and different type in one and

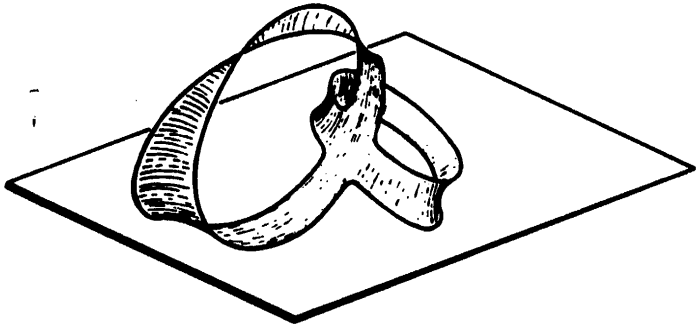

FIGURE 8—One-sided minimal surface of higher topological structure in a single contour.

the same frame; moreover, it again illustrates the possibility of discontinuous transitions from one solution to another while the conditions of the problem are changed continuously. It is easy to construct more complicated models of the same sort and to study their behavior experimentally.

An interesting phenomenon is the appearance of minimal surfaces bounded by two or more interlocked closed curves. For two circles we obtain the surface shown in Figure 9. If, in this example, the circles are

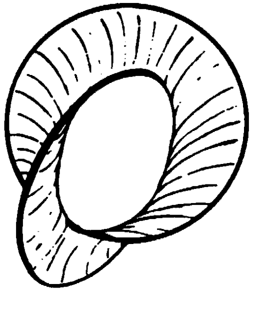

FIGURE 9—Interlocked curves.

perpendicular to each other and the line of intersection of their planes is a diameter of both circles, there will be two symmetrically opposite forms of this surface with equal area. If the circles are now moved slightly with respect to each other, the form will be altered continuously, although for each position only one form is an absolute minimum, and the other one a relative minimum. If the circles are moved so that the relative minimum is formed, it will jump over into the absolute minimum at some point. Here both of the possible minimal surfaces have the same topological

character, as do the surfaces of Figures 6 and 7, one of which can be made to jump into the other by a slight deformation of the frame.

EXPERIMENTAL SOLUTIONS OF OTHER MATHEMATICAL PROBLEMS

Owing to the action of surface tension, a film of liquid is in stable equilibrium only if its area is a minimum. This is an inexhaustible source of mathematically significant experiments. If parts of the boundary of a film are left free to move on given surfaces such as planes, then on these boundaries the film will be perpendicular to the prescribed surface.

We can use this fact for striking demonstrations of Steiner's problem and its generalizations. Two parallel glass or transparent plastic plates are joined by three or more perpendiculars bars. If we immerse this object in a soap solution and withdraw it, the film forms a system of vertical planes between the plates and joining the fixed bars. The projection appearing on the glass plates is the solution of the problem of finding the shortest straight line connection between a set of fixed points.

If the plates are not parallel, the bars not perpendicular to them, or the

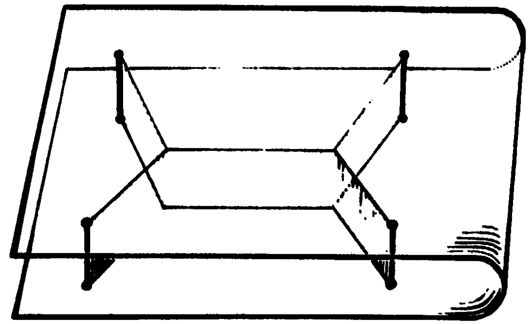

FIGURE 10—Demonstration of the shortest connection between 4 points.

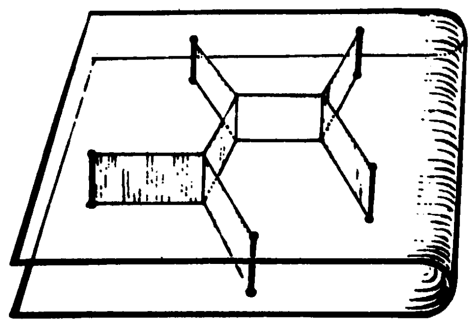

FIGURE 11—Shortest connection between 5 points.

plates curved, then the curves formed by the film on the plates will not be straight, but will illustrate new variational problems.

The appearance of lines where three sheets of a minimal surface meet at angles of 120° may be regarded as the generalization to more dimensions of the phenomena connected with Steiner's problem. This becomes clear e.g. if we join two points A, B in space by three curves, and study the corresponding stable system of soap films. As the simplest case we take for one curve the straight segment AB, and for the others two

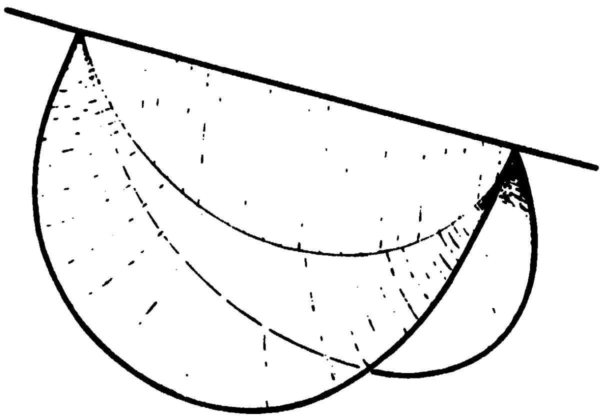

FIGURE 12—Three surfaces meeting at 120° spanned between three wires joining two points.

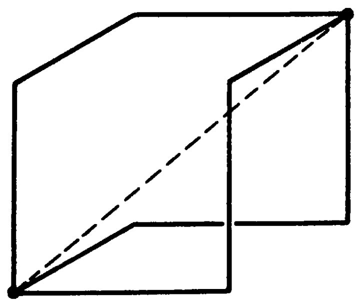

FIGURE 13—Three broken lines joining two points.

congruent circular arcs. The result is shown in Figure 12. If the planes of the arcs form an angle of less than 120°, we obtain three surfaces meeting at angles of 120°; if we turn the two arcs, increasing the included angle, the solution changes continuously into two plane circular segments.

Now let us join A and B by three more complicated curves. As an example we may take three broken lines each consisting of three edges

of the same cube that join two diagonally opposite vertices: we obtain three congruent surfaces meeting in the diagonal of the cube. (We obtain this system of surfaces from that depicted in Figure 1 by destroying the films adjacent to three properly selected edges.) If we make the three broken lines joining A and B movable, we can see the line of threefold intersection become curved. The angles of 120° will be preserved (Figure 13).

All the phenomena where three minimal surfaces meet in certain lines are fundamentally of a similar nature. They are generalizations of the plane problem of joining n points by the shortest system of lines.

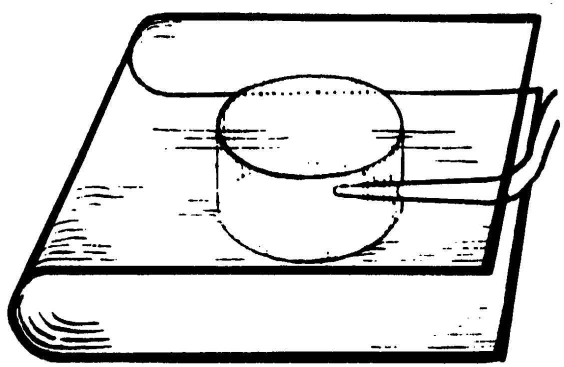

FIGURE 14—Demonstration that the circle has least perimeter for a given area.

Finally, a word about soap bubbles. The spherical soap bubble shows that among all closed surfaces including a given volume (defined by the amount of air inside), the sphere has the least area. If we consider soap bubbles of given volume which tend to contract to a minimum area but which are restricted by certain conditions, then the resulting surfaces will be not spheres, but surfaces of constant mean curvature, of which spheres and circular cylinders are special examples.

For example, we blow a soap bubble between two parallel glass plates which have previously been wetted by the soap solution. When the bubble touches one plate, it suddenly assumes the shape of a hemisphere; as soon as it also touches the other plate, it jumps into the shape of a circular cylinder, thus demonstrating the isoperimetric property of the circle in a most striking way. The fact that the soap film adjusts itself vertically to the bounding surface is the key to this experiment.