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Know Thy Math

A Collection of Some
Possibly Useful Tricks

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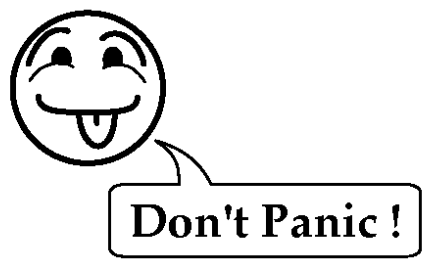

This is **not** a textbook of Mathematical Physics! There are quite enough of them listed in the 'References' list at the end of these notes. Rather, the purpose of these notes is to give a quick glance at some of the 'tricks of the trade' in the daily practice of mathematical physics, or physical mathematics—as you wish. The limited attention and span of these notes is focused on issues and topics that have arisen in office hours discussions with a wide variety of undergraduate and graduate students, with the motto:

The stupidest question is the one not asked,
for it will haunt you for the rest of your life.

—o—

1. Names, Noneclature, Notation ...

A number of (sometimes silly) names and expressions are in current use in physics, and we list a couple of them here, just in case the Reader is not aware [1]...

- ◇ *Prove* means 'derive by *logically consistent methods* and possibly using universally accepted facts'; however, it may behoove you to state the used 'universally accepted facts', just in case they are not universally, not accepted or not facts; often, stating these will clear your own thinking. While miracles may provide for a wonderful life, they are not a 'universally accepted' technique of proof. Also, 'prove statement A from statement B' suggests using "statement B", so by all means—do use it.
- ◇ A function is *even* (*odd*) if

$$f_{\text{even}}(-t) = +f_{\text{even}}(t)\ , \qquad \text{symmetric;} \tag{1.1$^+$}$$

$$f_{\text{odd}}(-t) = -f_{\text{odd}}(t)\ , \qquad \text{antisymmetric;} \tag{1.1$^-$}$$

[1] It is frustrating that experience proves the mention of these concepts necessary.

the latter are also called *skew-symmetric* or simply *skew*.

- *Invariant* means 'unchanging', and is used both as an adjective and as a noun. So, a *Lorentz-invariant* is an object which stays unchanged through any Lorentz transformation (pseudo-rotation in the 4-dimensional Minkowski spacetime). *Covariant* means 'changing accordingly' and is used only as an adjective; clearly, *covariant* must be a comparison and the reference object or transformation rule will vary from case to case. For example, a *Lorentz-covariant* object *does* change under Lorentz transformations, but precisely the same way as the gradient operator does. (Why the gradient operator has been accepted as *the* reference object here is a matter for historians; we only care that it *is*).
- *A general statement* cannot be proven by providing a particular example, or even a restricted collection of such. On the other hand, a general statement *can* be disproven by providing a single counter-example.

2. Integrals, Integrals

There are of course numerous tables of integrals (Ref. [10] is perhaps one of the most complete ones), readily available for the 'lookup & copy' methodology. Besides, algebraic manipulation computer programs such as *Mathematica*, *Maple*, *Macsyma*, *Theorist*, and to some extent also *MathCAD* and *MathLab*[2] can solve many thousands of rather general integrals. Yet, chances are that many of the integrals that one encounters will not be tabulated and the computer programs will choke on them[3]. Sometimes, a little massage will bring the integral into a form which can then be evaluated by one of these lazy-boy methods. Sometimes, the integral just won't yield. However, there still is a fairly broad area where tables are insufficient, computer programs are dumb, but human ingenuity + motivation just plain simply shines. As for this latter, there is a practical advice hidden in the observation:

If you believe that you lack ingenuity and motivation—you do.

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Quantum mechanics (but also many other branches of Physics and Engineering) abounds with integrals where the integrand is a product of and exponential function and polynomial. Very often, these integrals can be brought to the form

$$\Gamma(z) \stackrel{\text{def}}{=} \int_0^\infty \mathrm{d}t\; t^{z-1}\, e^{-t}\ . \tag{2.1}$$

In doing so, the following maneuvers may be useful:

- Reflecting the integration variable (replacing $t \to -t$ throughout in):

$$\int_{T_1}^{T_2} \mathrm{d}t\ f(t) \ =\ \int_{-T_1}^{-T_2} (-\mathrm{d}t)\ f(-t) \ =\ \int_{T_2}^{T_1} \mathrm{d}t\ f(-t)\ . \tag{2.2}$$

2) ... yes, there are numerous other programs, many of which written for those other, Mac-lookalike computers, but why bother with but the best?

3) Then again, *the* bible of all tables of integrals, Ref. [10] is being re-edited with corrections found by *Mathematica*!

◇ Dividing the symmetric integration range into two similar halves:

$$\begin{aligned}\int_{-T}^{+T} \mathrm{d}t\ f(t) &= \int_{-T}^{0} \mathrm{d}t\ f(t) + \int_{0}^{+T} \mathrm{d}t\ f(t) \overset{(2.2)}{=} \int_{0}^{+T} \mathrm{d}t\ f(-t) + \int_{0}^{+T} \mathrm{d}t\ f(t)\ , \\ &= \int_{0}^{+T} \mathrm{d}t\ \big[f(-t) + f(t)\big]\ .\end{aligned} \tag{2.3}$$

Since $f(-t)+f(t) = 2f(t)$ for even functions, while $f(-t)+f(t) = 0$ for odd functions,

$$\int_{-T}^{+T} \mathrm{d}t\ f_{\text{even}}(t) = 2\int_{0}^{+T} \mathrm{d}t\ f_{\text{even}}(t)\ , \qquad \int_{-T}^{+T} \mathrm{d}t\ f_{\text{odd}}(t) = 0\ . \tag{2.4}$$

◇ General change of the integration variable(s)—should be a "no-brainer":

$$\int_{x_0}^{x_1} \mathrm{d}x\, f(x) \quad = \quad \int_{t(x_0)}^{t(x_1)} \mathrm{d}t\ \Big(\frac{\mathrm{d}x}{\mathrm{d}t}\Big)\ f(x(t))\ . \tag{2.5}$$

◇ Integration by parts—should be another "no-brainer":

$$\int_{x_0}^{x_1} \mathrm{d}x\, f'(x)\, g(x) \quad = \quad \Big[f(x_1)\, g(x_1) - f(x_0)\, g(x_0)\Big] \ - \ \int_{x_0}^{x_1} \mathrm{d}x\, f(x)\, g'(x)\ . \tag{2.6}$$

It should be quite clear that this is a straightforward consequence of the 'product rule': $\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x)$. (Hint: move the integral on the right over to the left.)

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The benefit of changing an integral to the form (2.1) is seen upon noting that $\Gamma(z)$ satisfies a number of useful properties

$$\Gamma(1+z) \ = \ z\Gamma(z)\ , \tag{2.7a}$$

$$\Gamma(1-z) \ = \frac{\pi}{\Gamma(z)\sin(\pi z)}\ , \tag{2.7b}$$

$$\Gamma(k\,z) \ = (2\pi)^{\frac{1}{2}(1-k)}\, k^{kz-\frac{1}{2}} \prod_{r=0}^{k-1} \Gamma(z+\frac{r}{k})\ , \qquad k \text{ an integer.} \tag{2.7c}$$

The first of these implies that $\Gamma(z) = \Gamma(1+z)/z$, which we can substitute in the second one and obtain the frequently useful *reflection formula*

$$\Gamma(1+z)\,\Gamma(1-z) \ = \ \frac{\pi z}{\sin(\pi z)}\ . \tag{2.8}$$

Finally, for most physics applications it suffices to know that

$$\Gamma(1) = 1\ , \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}\ , \tag{2.9}$$

since most physics-related integrals (if they can be related to $\Gamma(z)$ at all) end up being expressed in terms of

$$\Gamma(n+1) = n!\ , \qquad \Gamma(n+\tfrac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}\ . \tag{2.10}$$

Here $n! \stackrel{\text{def}}{=} n{\cdot}(n{-}1)\cdots 2{\cdot}1$, while $n!! \stackrel{\text{def}}{=} n{\cdot}(n{-}2)\cdots 4{\cdot}2$ if n is even, or $n!! \stackrel{\text{def}}{=} n{\cdot}(n{-}2)\cdots 3{\cdot}1$ if n is odd. Also we have that

$$(2n)!! \;=\; 2^n\, n! \quad \text{and} \quad (2n{+}1)!! \;=\; \frac{(2n{+}1)!}{2^n\, n!} \tag{2.11}$$

The first result in (2.9) is elementary:

$$\Gamma(1) = \int_0^\infty \mathrm{d}t\; e^{-t} = \big(-e^{-t}\big)_{t=+\infty} - \big(-e^{-t}\big)_{t=0} \;=\; (0) - (-1) \;=\; 1\;. \tag{2.12}$$

The second one requires a small maneuver:

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty \mathrm{d}t\, t^{-\frac{1}{2}}\, e^{-t} = 2\int_0^\infty \mathrm{d}x\, e^{-x^2} = \int_{-\infty}^{+\infty} \mathrm{d}x\, e^{-x^2}\;, \tag{2.13}$$

where we changed the integration variable to $x = \sqrt{t}$, so $\frac{\mathrm{d}t}{\sqrt{t}} = 2\mathrm{d}x$ and used (2.3). Now comes a little trick (note the use of the distinct integration variables in the two factors of $\big[\Gamma(\frac{1}{2})\big]^2$):

$$\begin{aligned}\big[\Gamma(\tfrac{1}{2})\big]^2 &= \Big[\int_{-\infty}^\infty \mathrm{d}x\, e^{-x^2}\Big]\Big[\int_{-\infty}^\infty \mathrm{d}y\, e^{-y^2}\Big] \;=\; \int_{xy-\text{plane}} \mathrm{d}x\mathrm{d}y\, e^{-(x^2+y^2)}\;,\\ &= \int_0^\infty r\mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi\, e^{-r^2} \;=\; 2\pi\int_0^\infty r\mathrm{d}r\, e^{-r^2} \;=\; 2\pi\int_0^\infty (\tfrac{1}{2}\mathrm{d}u)\, e^{-u}\;,\\ &= \pi\Gamma(1) \;=\; \pi\;.\end{aligned} \tag{2.14}$$

whence $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, the second result in (2.9). (The second line began with the change of variables from Cartesian (x,y) to polar (r,ϕ), where $x = r\cos\phi$ and $y = r\sin\phi$; the last integral in the second line follows upon the change of variables $u = r^2$.)

Useful practice: Derive the master formula:

$$\int_0^\infty \mathrm{d}t\; e^{-(\alpha t)^\beta}\; t^\gamma \;=\; \frac{\Gamma(\frac{\gamma+1}{\beta})}{\beta\,\alpha^{\gamma+1}}\;. \tag{2.15}$$

The formula will apply even for complex α, β, γ, provided $\Re e(\alpha) > 0$. Integrals of the same type but the full range $-\infty < x < \infty$ are solved using (2.3) and this master formula. Integrals over the full range, but with a polynomial in the exponential instead of a simple power are solved by first completing the polynomial into a pure square, cube, etc., and then substituting so as to obtain a form of (2.15). So, for example,

$$\int_{-\infty}^\infty \mathrm{d}x\; e^{-a^2x^2+2abx}\, x^n = \int_{-\infty}^\infty \mathrm{d}x\; e^{-(ax+b)^2}\, e^{b^2}\, x^n = \frac{e^{b^2}}{a^{n+1}}\int_{-\infty}^\infty \mathrm{d}t\; e^{-t^2}\,(t-b)^n\;, \tag{2.16}$$

whereupon you expand $(t-b)^n$ and solve each integral separately using (2.15). Note that any finite limit would have been shifted in the change $x \mapsto t = ax{-}b$.

Related is Euler's beta function (for $\Re e(x), \Re e(y) > 0$)

$$\begin{aligned} B(x,y) \stackrel{\rm def}{=} \int_0^1 {\rm d}t\, t^{x-1}(1-t)^{y-1} \quad &= \quad \int_0^\infty {\rm d}u\, \frac{u^{x-1}}{(1+u)^{x+y}}\ , \\ = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \quad &= \quad \frac{x+y}{x\ y}\binom{x+y}{y}^{-1}\ . \end{aligned} \tag{2.17}$$

Here $\binom{x+y}{y}$ generalizes the (better be) well known binomial coefficient

$$\binom{n}{k} \stackrel{\rm def}{=} \frac{n!}{k!\,(n-k)!} \quad = \quad \frac{n}{1}\,\frac{(n-1)}{2}\cdots\frac{(n-k+1)}{k} \tag{2.18}$$

from integral to complex values arguments (with positive real part). Note that the latter formula applies even if n is not an integer, as long as k is an integer. The Euler beta function (2.17), however, holds for even complex arguments, so we can *define* the binomial coefficient to be

$$\binom{x+y}{y} \stackrel{\rm def}{=} \frac{(x+y)!}{x!\,y!} \quad = \quad \frac{x+y}{x\ y}\,\frac{\Gamma(x+y)}{\Gamma(x)\,\Gamma(y)} \quad = \quad \frac{x+y}{x\ y}\,\frac{1}{B(x+y)}\ , \tag{2.19}$$

which gives a well-defined result as long as $x{+}y$ is not a negative integer.

The (integral version of the) binomial coefficient appears in the *binomial expansion*:

$$(a+b)^n \quad = \quad \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k\ . \tag{2.20}$$

This is often used as

$$(a+b)^n \quad = \quad a^n\Big[1+\Big(\frac{b}{a}\Big)\Big]^n \quad = \quad a^n \sum_{k=0}^n \binom{n}{k}\Big(\frac{b}{a}\Big)^k\ , \tag{2.21}$$

and generalizes for cases when $n \to \nu$ is not an integer and/or $\nu < k$ into

$$(a+b)^\nu \quad = \quad a^\nu\Big[1+\Big(\frac{b}{a}\Big)\Big]^\nu \quad = \quad a^\nu \sum_{k=0}^\infty \binom{\nu}{k}\Big(\frac{b}{a}\Big)^k\ . \tag{2.22}$$

Since the k in these expressions are always integers, the last expression in (2.18) always applies and is also the quickest way to calculate. Note, however, that once the series becomes infinite, there is the issue of convergence! The series (2.22) converges (absolutely) precisely if $b < a$. So, clearly, to apply (2.22), one "pulls out" the larger of the two summands.

3. Seven Differential Equation Veils And Other Stories

Below are listed some standard types of differential equations and their solutions. (The Reader is hereby warned that the expressions below are quite possibly plagued by typos! So, one should use the formulae below as hints, but verify the correctness for each case separately.)

3.1. First Order Equations

1. *Bernuolli*'s equation: may involve one term non-linear in the dependent variable $f(x)$, but no source term:

$$\frac{\mathrm{d}f}{\mathrm{d}x} + p(x)f(x) + q(x)f^{\alpha}(x) \quad = \quad 0 \;, \qquad \alpha \text{ real} \;, \tag{3.1}$$

is solved by

$$f(x) = \begin{cases} e^{-\int \mathrm{d}x\, p(x)} \Big(C + (1-\alpha) \int \mathrm{d}x\, q(x)\, e^{(1-\alpha)\int \mathrm{d}x\, p(x)} \Big)^{\frac{1}{1-\alpha}} & \alpha \neq 1, \\ C\, e^{-\int \mathrm{d}x [p(x)+q(x)]} & \alpha = 1. \end{cases} \tag{3.2}$$

The gentle Reader is invited to verify that the second line indeed obtains as the limit $\alpha \to 1$ of the first, more general formula (up to a redefinition of the constant C).

2. *Ricatti*'s equation

$$\frac{\mathrm{d}f}{\mathrm{d}x} + p(x)f(x) + q(x)f^2(x) \quad = \quad s(x) \;, \tag{3.3}$$

is solved by

$$f(x) \quad = \quad -\frac{1}{2} \left[\frac{q(x)}{p(x)} - \frac{p'(x)}{p^2(x)} + \frac{g'(x)}{g(x)} \right] \tag{3.4}$$

where $g(x)$ satisfies a hopefully simpler equation

$$\frac{\mathrm{d}^2 g}{\mathrm{d}x^2} + r(x)g(x) \quad = \quad 0 \;, \tag{3.5}$$

$r(x)$ being an abbreviation for the monstrosity

$$r(x) \quad = \quad -\frac{1}{2} \left[q'(x) - \frac{p''(x)p(x) - p'^2}{p^2(x)} \right] - \frac{1}{4} \left[q(x) - \frac{p'(x)}{p(x)} \right]^2 - s(x) \;. \tag{3.6}$$

Besides this horrendous expression (to the complete accuracy of which no guarantee is given herein), another possibly useful fact is known about Ricatti's equation. If three independent solutions $f(x) = u, v, w$ should by any catch-as-catch-can methods be known, then the general 1-parameter family of solutions is obtained as

$$f_{\text{general}} \quad = \quad \frac{v(u-w) - C\, w(u-v)}{(u-w) - C\,(u-v)} \;, \qquad C = \mathit{const.} \tag{3.7}$$

(The general solution has only one parameter since the equation is of first order, so involves—in principle—a single integration.)

3. *Clairault*'s equation:

$$f \quad = \quad xp(f') \;+\; q(f') \;, \tag{3.8}$$

has what is called a *singular solution*, which is obtained in a parametric form:

$$\begin{aligned} x \;&=\; e^{-\int \mathrm{d}t \frac{p'(t)}{p(t)-t}} \left\{ C \;+\; \int \mathrm{d}t\, \frac{q'(t)}{p(t)-t}\, e^{\int \mathrm{d}t \frac{p'(t)}{p(t)-t}} \right\} \;, \\ f \;&=\; xp(t) + q(t) \;. \end{aligned} \tag{3.9}$$

In principle, y can now be obtained as $y = y(x)$ by eliminating t from this pair of equations. In practice, this is usually too impossible to achieve, but the parametric solution above is just as good. Amusingly, the latter of these two,

$$y \;=\; xp(t) + q(t) \;, \qquad t = const. \;, \tag{3.10}$$

is itself the *general solution*, where t is the integration constant. The curve described by Eqs. (3.9), with t eliminated, is the *envelope* of the 1-parameter family (parametrized by choices of t) of curves in Eq. (3.10). This relation between *general* and *singular* solutions is rather typical.

4. There are two general cases of

$$A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y \;=\; 0 \;, \tag{3.11}$$

depending on whether or not $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$. If yes, then the solution is given straightforwardly by

$$C \;=\; \int A\mathrm{d}x \;+\; \int B\mathrm{d}y \;-\; \int \mathrm{d}y\Big(\frac{\partial}{\partial y}\int \mathrm{d}xA\Big) \;. \tag{3.12}$$

In the very special case when $\frac{\partial A}{\partial y} = 0 = \frac{\partial B}{\partial x}$, the variables x, y are said to *separate*; the third term then drops from the solution.

The second one is the more general case where $\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x}$. However, this can be brought to the former more special case by means of an integrating factor $\lambda(x,y)$, such that

$$\frac{\partial(\lambda A)}{\partial x} = \frac{\partial(\lambda B)}{\partial y} \;, \tag{3.13}$$

so that the equivalent equation

$$\lambda A(x,y)\mathrm{d}x + \lambda B(x,y)\mathrm{d}y \;=\; 0 \;. \tag{3.14}$$

is solved as above. There is no general procedure for finding such a $\lambda(x,y)$. However, if λ is only a function of x, then it is

$$\lambda(x) \;=\; \exp\left\{\int \frac{\mathrm{d}x}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right)\right\} \;. \tag{3.15}$$

Similarly, if λ is only a function of y, it is calculated as

$$\lambda(y) \;=\; \exp\left\{\int \frac{\mathrm{d}y}{A}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)\right\} \;. \tag{3.16}$$

Clearly, if neither of these two definitions turns out to be a function of only x (or only y), you are stuck with the general case, $\lambda = \lambda(x,y)$, and the trial and error method—except in the rather special case when

$$A(\mu x, \mu y) = \mu^a A(x,y) \;, \quad \text{and} \quad B(\mu x, \mu y) = \mu^a B(x,y) \;. \tag{3.17}$$

In this case, the integrating factor is known to be

$$\lambda(x,y) \;=\; \frac{1}{xA + yB} \;. \tag{3.18}$$

This and some further special cases of Eq. (3.11) are discussed in Ref. [5].

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Further examples and classes of first order differential equations are discussed in the literature listed at the end. In general, there is just too many diverse categories of differential equations to succinctly classify them in a brief such as this.

3.2. Second Order Equations

5. The general (linear in f) equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}f}{\mathrm{d}x} + q(x)f(x) \quad = \quad 0 \tag{3.19}$$

is most of the time *not* solvable, but simplifies on writing

$$f(x) \quad \stackrel{\text{def}}{=} \quad \phi(x)e^{-\frac{1}{4}\int \mathrm{d}x p(x)} \;, \tag{3.20}$$

where $\phi(x)$ satisfies a hopefully simpler equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - \frac{1}{4}\Big(2p'(x) + p(x) - 4q(x)\Big)\phi(x) \quad = \quad 0 \;, \tag{3.21}$$

Another simplification may be obtained by a change of variable

$$t \quad \stackrel{\text{def}}{=} \quad \int \mathrm{d}x e^{-\int \mathrm{d}x\, p(x)} \;, \tag{3.22}$$

whereupon $f(x)$ becomes $f(t) = f\big(x(t)\big)$ and satisfies

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + \Big(e^{2\int \mathrm{d}x\, p(x)}q(x)\Big)f(x) \quad = \quad 0 \;, \tag{3.23}$$

6. *Euler*'s equation:

$$x^2\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + xp\frac{\mathrm{d}f}{\mathrm{d}x} + qf(x) \quad = \quad 0 \;, \tag{3.24}$$

where p, q are constant, is solved by

$$f(x) \quad = \quad x^{\frac{1-p}{2}}\Big(C_1 x^{\sqrt{\left(\frac{1-p}{2}\right)^2 - q}} + C_2 x^{-\sqrt{\left(\frac{1-p}{2}\right)^2 - q}}\Big) \;. \tag{3.25}$$

7. Finally, the homogeneous linear second order differential equation with *constant* coefficients p, q

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + p\frac{\mathrm{d}f}{\mathrm{d}x} + qf(x) \quad = \quad 0 \;, \tag{3.26}$$

is solved by

$$f(x) \quad = \quad \begin{cases} C_1 e^{k_1 x} + C_2 e^{k_2 x} & k_1 \neq k_2, \\[2ex] (C_1 + xC_2)e^{k_2 x} & k_1 = k_2, \end{cases} \tag{3.27}$$

where k_1, k_2 are the two solutions of the quadratic equation

$$k^2 + pk + q \quad = \quad 0 \;. \tag{3.28}$$

3.3. Arbitrary Order, Polynomial Coefficients

Differential equations of the form

$$\sum_{k=0}^{n} a_k(x)\frac{\mathrm{d}^k f}{\mathrm{d}x^k} \;=\; 0 \tag{3.29}$$

where $a_k(x)$ are polynomials in x can always be solved by the Frobenius's power series method. This proceeds by writing the solution in the form

$$f(x) \;\stackrel{\text{def}}{=}\; \sum_{j=0}^{\infty} c_j x^{j+s} \;; \tag{3.30}$$

calculating the required derivatives; inserting in the differential equation (3.29); rearranging the sums by possibly shifting the summation variable j (notice that the upper limit, $j=\infty$, does not change while the lower does) so as to obtain a single power series the coefficients of which combine contributions from each term. Since the powers of x are linearly independent, all coefficients of this new power series have to vanish, imposing conditions on s and recursion relations on the coefficients c_j. Finally, check convergence: if the series converges—it is the required solution. If the series does not converge, then a meaningful solution can be obtained by this method only if some parameters of the original differential equation can be chosen so as to terminate the infinite series for $f(x)$, that is, if the recursion relations for the c_j start producing zeros upon some high enough j. If this can be arranged, the series solution is said to terminate into a polynomial of finite order and no convergence issue arises.

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The natural extension of this type, where $a_k(x)$ are infinite power-series may be dealt with along the same lines. Two remarks are however in order. For one thing, even if the power-series $a_k(x)$ themselves and $\sum_{j=0}^{\infty} c_j x^{j+s}$ converge by themselves, the latter series does not qualify as a solution of the differential equation (3.29) unless the products $a_x(x)\phi^{(k)}$ also converge. On the other hand, we may safely assume that the coefficient power-series $a_k(x)$ are analytic and so admit a Taylor series (only non-negative powers of x); for, if they were not, we simply multiply the whole equation by a polynomial the zeros of which cancel each pole of the $a_k(x)$.

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Similarly, non-linear differential equations may sometimes be solved in a series form. The procedure is the same: starting with the Ansatz (3.30), inserting in the differential equation and obtaining a recursion relation for the constants c_j. Thereupon, it must be shown that the series converges to a well-defined function—but also that all non-linear terms in $f(x)$ are convergent non-linear expressions. As a somewhat forced example, consider the nonlinear 1st order differential equation

$$f'(x) - f^2(x) \;=\; 0 \;. \tag{3.31}$$

Rewriting this as

$$\frac{\mathrm{d}f}{f^2} \;=\; \mathrm{d}x \tag{3.32}$$

shows that the variables separate and the equation may be solved by straightforward integration, producing $f(x) = (C - x)^{-1}$. Frobenius' method would yield the well known

$$f(x) = \frac{1}{C} + \frac{x}{C^2} + \frac{x^2}{C^3} + \ldots \tag{3.33}$$

geometric series which converges (and the square of which also converges) to the above solution, and for $|x| < |C|$.

3.4. Series, Other Than Power

Frobenius's method employs a power series expansion $f(x) = \sum_{j=0}^{\infty} c_j x^{j+s}$. Also, other series may be attempted instead, and such substitutions are—in general—well adapted for solving *linear* differential equations. The reason for this ought to be obvious upon a moment's reflection on the caveats in the application of Frobenius's power series method, as mentioned above. That is, a purported solution must also be proven to provide a convergent expression for every non-linear term, and this is in general very difficult if at all possible.

However, *linear* differential equations (and even systems thereof) are amenable to series methods of solution. Consider the system of two first order linear differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a\,x(t) + b\,y(t) \ , \tag{3.34a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = c\,x(t) + d\,y(t) \ . \tag{3.34b}$$

The substitution $x(t) = \sum_{k=-\infty}^{+\infty} \xi_k e^{kt}$ and $y(t) = \sum_{k=-\infty}^{+\infty} \eta_k e^{kt}$ turns this into

$$\begin{aligned} \sum_{k=-\infty}^{+\infty} (k\,\xi_k - a\xi_k - b\eta_k)e^{ikt} &= 0 \ , \\ \sum_{k=-\infty}^{+\infty} (k\,\eta_k - c\xi_k - d\eta_k)e^{ikt} &= 0 \ . \end{aligned} \tag{3.35}$$

Now, again, we note that the functions (of t, parametrized by k) e^{kt} are linearly independent; in fact, e^{kt} is the kth power of $q \stackrel{\text{def}}{=} e^t$, so the above substitution may be regarded as a disguised application of Frobenius's method. At any rate, since e^{kt} and $e^{k't}$ are linearly independent if $k \neq k'$, the above sums vanish only if the coefficients do:

$$\left.\begin{aligned} k\,\xi_k - a\xi_k - b\eta_k &= 0 \ , \\ k\,\eta_k - c\xi_k - d\eta_k &= 0 \ , \end{aligned}\right. \qquad \text{fixed } k \ . \tag{3.36}$$

It should be obvious that this is the same as

$$\begin{bmatrix} k-a & -b \\ -c & k-d \end{bmatrix} \begin{bmatrix} \xi_k \\ \eta_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ , \tag{3.37}$$

which better be familiar! For there to be nonzero solutions ξ_k, η_k, it must be that the determinant of the matrix on the right hand side vanishes, which imposes a quadratic equation on k, with solutions:

$$k_{\pm} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}\ , \tag{3.38}$$

and the above (unspecified!) summations over k collapse to a summations over just two terms—(k_+, k_-). That is to say, ξ_k, η_k vanish for all other choices of k. To each of these two eigenvalues there corresponds an eigenvector, these being two linear combinations of $x(t)$ and $y(t)$ for which the above system diagonalizes (decouples). For giggles, the unimpressed Reader is invited to solve the above system by using plain vanilla power series in t, and then proving that the result converges to produce linear combinations of $e^{k_+ t}$ and $e^{k_- t}$.

The moral of the story may be summarized in the following points:

1. expand the sought-for functions into a series of linearly independent functions;
2. allow the series to range over a complete such set;
3. manipulate the summations so as to obtain a collection of *algebraic* relations for the coefficients;
4. make sure that the resulting series converges.

As we have seen, by a judicious choice of functions in which to expand, the fourth point may become trivial (finite series diverge only where at least one of their summands does). The whole Sturm-Liouville theory is a formal and practical development of this idea. Furthermore, there is no particular reason for restricting to discrete sums; that is, instead of series Ansatz, one may as easily consider an *integral transform*, $x(t) = \int \mathrm{d}k\, \xi(k)\phi_k(t)$, where $\phi_k(t)$ are the suitable functions of t, labeled by k. The Fourier and Laplace integral transforms (with $\phi_k(t) = e^{ikt}$, $-\infty < k < +\infty$, and $\phi_k(t) = e^{-kt}$, $0 \leq k < +\infty$, respectively) have gained most popularity, although in fact, any complete set of linearly independent functions may be used to define an integral (or series) transform.

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Note that a system of differential equations may be "simplified" by eliminating some of the functions, to obtain a single differential equation for each one function. For example, taking the derivative of Eq. (3.34a) and eliminating $\dot{y}(t)$ and $y(t)$ with the aid of the Eqs. (3.34)produces

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - (a+d)\frac{\mathrm{d}x}{\mathrm{d}t} + (ad-bc)x \ = \ 0\ , \tag{3.39}$$

for which the above recipes for solving second order differential equations yield the same solution which we obtained here by means of expansion into series. However, the simplicity of the example should not deceive the Reader: an $n \times n$ system of first order linear differential equations produces n (decoupled) nth order differential equation—one for each of the n functions. Reducing the system by eliminating functions raises the order of the resulting equations.

This then can also be used the other way around: for example, the second order differential equation

$$f''(x) + P(x)\, f'(x) + Q(x)\, f(x) \ = \ 0\ , \tag{3.40}$$

is equivalent to the system of first order equations

$$\begin{aligned} f'(x) - g(x) = & \ 0 \ , \\ g'(x) + P(x)\, g(x) + Q(x)\, f(x) = & \ 0 \ . \end{aligned} \tag{3.41}$$

This equivalence clearly exists regardless of linearity, or any other kind of homogeneity for that matter. Whether it is easier to solve the system of linear differential equations of the higher order (decoupled) equations depends very much on the particular case. However, being able toggle between these two extremal representations (and perhaps a number of intermediate ones) is often of some practical use.

Often, the system of first order differential equations tends to have a physical interpretation, and is sometimes better suited to address certain questions. An example are provided by the first order Maxwell's equations which couple the $\vec{E}$ and $\vec{B}$ fields, and where each coupling term has a physical significance. Equivalent to these are of course the two (decoupled) second order wave equations—one for $\vec{E}$ and one for $\vec{B}$, and with the source terms provided by the charge density and the charge current. For application of Gauss's theorem, the original Maxwell's equations are clearly more appropriate; wave phenomena of course favor the latter choice, second order decoupled equations.

Another example is provided by the whole Lagrangian and Hamiltonian formalisms: the former typically produces second order equations of motion, while the latter produces first order pairs.

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Now that we have scratched the tip of the iceberg—onward!

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