Montucla, J. F.: *Histoire des mathématiques* (1802), Albert Blanchard (reprint), 1960, Vol. 3, pp. 63–102.

Struik, D. J.: *A Source Book in Mathematics, 1200–1800*, Harvard University Press, 1969, pp. 168–78, 180–83, 263–69, 413–19.

Taton, René: *L'Œuvre scientifique de Monge*, Presses Universitaires de France, 1951, Chap. 4.

Whiteside, Derek T.: "Patterns of Mathematical Thought in the Later Seventeenth Century," *Archive for History of Exact Sciences*, 1, 1961, 179–388. See pp. 202–5 and 270–311.

———: *The Mathematical Works of Isaac Newton*, Johnson Reprint Corp., 1967, Vol. 2, 137–61. This contains Newton's *Enumeratio* in English.

24 The Calculus of Variations in the Eighteenth Century

> For since the fabric of the universe is most perfect and the work of a most wise Creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear.
>
> LEONHARD EULER

1. *The Initial Problems*

As in the areas of series and differential equations, the early work on the calculus of variations could hardly be distinguished from the calculus proper. But within a few years after Newton's death in 1727 it was clear that a totally new branch of mathematics, with its own characteristic problems and methodology, had come into being. This new subject, almost comparable in importance with differential equations for mathematics and science, supplied one of the grandest principles in all of mathematical physics.

To gain some preliminary notion of the nature of the calculus of variations, let us consider the problems that launched the mathematicians into the subject. Historically the first significant problem was posed and solved by Newton. In Book II of his *Principia*, he studied the motion of objects in water; then, in the Scholium to Proposition 34 of the third edition, he considered the shape that a surface of revolution moving at a constant velocity in the direction of its axis must have if it is to offer the least resistance to the motion. Newton assumed that the resistance of the fluid at any point on the surface of the body is proportional to the component of the velocity normal to the surface. In the *Principia* itself he gave only a geometrical characterization of the desired shape, but in a letter presumably written to David Gregory in 1694 he gave his solution.

In modern form, Newton's problem is to find the minimum value of the integral

$$J = \int_{x_1}^{x_2} \frac{y(x)[y'(x)]^3}{1 + [y'(x)]^2}\, dx$$

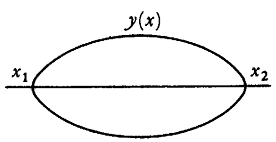

Figure 24.1

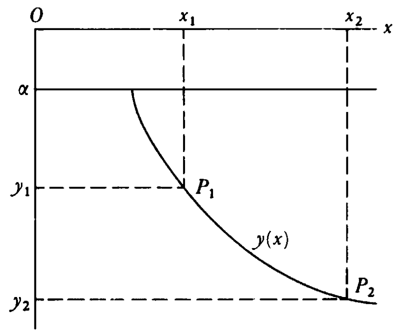

Figure 24.2

by choosing the proper function $y(x)$ for the shape of the curve that is to be rotated around the x-axis (Fig. 24.1). The peculiar feature of this problem (and of calculus of variations problems generally) is that it poses an integral whose value depends upon an unknown function $y(x)$ which appears in the integrand and which is to be determined so as to make the integral a minimum or a maximum.

Newton's solution, though it did use the idea of introducing a change in the shape of a part of the meridian arc $y(x)$, which is almost what the essential method of the calculus of variations involves, is not typical of the technique of the subject and so we shall not look into it. It may be of interest that the parametric equations of the proper $y(x)$ are

$$x = \frac{c}{p}(1 + p^2)^2, \qquad y = a + c\left(-\log p + p^2 + \frac{3}{4}p^4\right),$$

where p is the parameter. Of this work Newton says, "This proposition I conceive may be of use in the building of ships." Problems of this nature have become important in the design not only of ships but submarines and airplanes.

In the *Acta Eruditorum* of June 1696[1] John Bernoulli proposed as a challenge to other mathematicians the now famous brachistochrone problem. The problem is to determine the path down which a particle will slide from one given point to another not directly below in the shortest time. The initial velocity v_1 at P_1 (Fig. 24.2) is given; friction and air resistance are to be neglected. In modern form, this problem is to minimize the integral J which represents the time of descent where

$$J = \frac{1}{\sqrt{2g}}\int_{x_1}^{x_2}\sqrt{\frac{1 + [y'(x)]^2}{y(x) - \alpha}}\,dx.$$

1. Page 269 = *Opera*, 1, 161.

Here g is the gravitational acceleration and $\alpha = y_1 - v_1^2/2g$. Again the $y(x)$ in the integrand must be chosen so as to make J a minimum. The problem had been formulated and incorrectly solved by Galileo (1630 and 1638), who gave the arc of a circle as the answer. The correct answer is the arc of the unique cycloid joining P_1 and the second point P_2, which is concave upward; the line l on which the generating circle rolls must be at just the proper height, $y = \alpha$, above the given initial point of fall. Then there is one and only one cycloid through the two points.

Newton, Leibniz, L'Hospital, John Bernoulli, and his elder brother James found the correct solution. All these were published in the May issue of the *Acta Eruditorum* of 1697. The solutions of the two Bernoullis warrant further comment. John's method[2] was to see that the path of quickest descent is the same as the path of a ray of light in a medium with a suitably selected variable index of refraction, $n(x, y) = c/\sqrt{y - \alpha}$. The law of refraction at a sharp discontinuity (Snell's law) was known; so John broke up the medium into a finite number of layers with a sharp change in index from layer to layer and then let the number of layers go to infinity. James's method[3] was much more laborious and more geometrical. But it was also more general and was a bigger step in the direction of method for the calculus of variations.

The cycloid was well known through the work of Huygens and others on the pendulum problem (Chap. 23, sec. 5). When the Bernoulli brothers found that it was also the solution of the brachistochrone problem, they were amazed. John Bernoulli said,[4] "With justice we admire Huygens because he first discovered that a heavy particle traverses a cycloid in the same time, no matter what the starting point may be. But you will be struck with astonishment when I say that this very same cycloid, the tautochrone of Huygens, is the brachistochrone we are seeking."

Another important class of problems calls for geodesics, that is, paths of minimum length between two points on a surface. If the surface is a plane then the integral involved is

$$J = \int_{x_1}^{x_2}\sqrt{1 + [y'(x)]^2}\,dx,$$

and the answer is of course a line segment. In the eighteenth century the geodesic problem of most interest concerned the shortest paths on the surface of the earth, whose precise shape was not known, though the mathematicians believed it was some form of ellipsoid and most likely a figure of revolution. The early work on geodesics already noted (Chap. 23, sec. 7) did not use the method of the calculus of variations but it was clear that special devices would not be powerful enough to treat the general geodesic problem.

2. *Acta Erud.*, 1697, 206–11 = *Opera*, 1, 187–93.
3. *Acta Erud.*, 1697, 211–17 = *Opera*, 2, 768–78.
4. *Opera*, 1, 187–93.

Analytically the problems thus far formulated are of the form

$$J = \int_{x_1}^{x_2} f(x, y, y')\, dx$$

and call for finding the $y(x)$ that extends from (x_1, y_1) to (x_2, y_2) and that minimizes or maximizes J. Another class of problems, called isoperimetrical problems, also entered the history of the calculus of variations at the end of the seventeenth century. The progenitor of this class of problems, of all closed plane curves with a given perimeter to find the one that bounds maximum area, may date back to pre-Greek times. There is a story that Princess Dido of the ancient Phoenician city of Tyre ran away from her home to settle on the Mediterranean coast of North Africa. There she bargained for some land and agreed to pay a fixed sum for as much land as could be encompassed by a bull's hide. The shrewd Dido cut the hide into very thin strips, tied the strips end to end and proceeded to enclose an area having the total length of these strips as its perimeter. Moreover, she chose land along the sea so that no hide would be needed along the shore. According to the legend Dido decided that the length of hide should form a semicircle—the correct shape to enclose maximum area.

Apart from the work of Zenodorus (Chap. 5, sec. 7), there was practically no work on isoperimetrical problems until the end of the seventeenth century. In a move to challenge and embarrass his brother, James Bernoulli posed a rather complicated isoperimetrical problem involving several cases in the *Acta Eruditorum* of May 1697.[5] James even offered John a prize of fifty ducats for a satisfactory solution. John gave several solutions, one of which was obtained in 1701,[6] but all were incorrect. James gave a correct solution.[7] The brothers quarreled about the correctness of each other's solutions. Actually James's method, as in the case of the brachistochrone problem, was a major step toward the general technique soon to be fashioned. In 1718 John[8] considerably improved his brother's solution.

Analytically the basic isoperimetric problem is formulated thus. The possible curves are represented parametrically by

$$x = x(t), \qquad y = y(t), \qquad t_1 \le t \le t_2$$

and because they are closed curves, $x(t_1) = x(t_2)$ and $y(t_1) = y(t_2)$. Moreover no curve must intersect itself. The problem then calls for determining the $x(t)$ and $y(t)$ such that the length

$$L = \int_{t_1}^{t_2} \sqrt{(x')^2 + (y')^2}\, dt$$

5. Page 214.
6. *Mém. de l'Acad. des. Sci., Paris,* 1706, 235 = *Opera*, 1, 424.
7. *Acta Erud.*, 1701, 213 ff. = *Opera*, 2, 897–920.
8. *Mém. de l'Acad. des Sci., Paris,* 1718, 100 ff. = *Opera*, 2, 235–69.

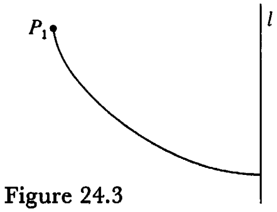

Figure 24.3

is a given constant and such that the area integral

$$J = \int_{t_1}^{t_2} (xy' - x'y)\, dt$$

is a maximum. There are two new features in this isoperimetrical problem. One, the use of the parametric representation, is incidental. The other is the presence of the auxiliary condition that L must be a constant.

Another problem that James posed in the May 1697 issue of the *Acta* was to determine the shape of the curve along which a particle slides from a given point P_1 with given initial velocity v_1 to any point of a line l (Fig. 24.3) so as to make the time of sliding from P_1 to l a minimum. This problem differs from the preceding ones in that the possible curves do not extend from one fixed point to another but from a fixed point to some line. The answer, given by James in the *Acta* of 1698 (though possessed by John in 1697 but not published by him), is an arc of a cycloid that cuts the line l at right angles. This problem was later generalized to the cases where l can be any given curve and where in place of P_1 another curve is given, so that the problem is to find the path requiring least time to slide from some point on one given curve to some point on another. This class of problems is described by the phrase "problems with variable endpoints."

2. *The Early Work of Euler*

In 1728 John Bernoulli proposed to Euler the problem of obtaining geodesics on surfaces by using the property that the osculating planes of geodesics cut the surface at right angles (Chap. 23, sec. 7). This problem started Euler off on the calculus of variations. He solved it in 1728.[9] In 1734 Euler generalized the brachistochrone problem to minimize quantities other than time and to take into account resisting media.[10]

Then Euler undertook to find a more general approach to problems in the subject. His method, which was a simplification of James Bernoulli's, was to replace the integral of a problem by a finite sum and to replace

9. *Comm. Acad. Sci. Petrop.*, 3, 1728, 110–24, pub. 1732 = *Opera*, (1), 25, 1–12.
10. *Comm. Acad. Sci. Petrop.*, 7, 1734/35, 135–49, pub. 1740 = *Opera*, (1), 25, 41–53.

derivatives in the integrand by difference quotients, thus making the integral a function of a finite number of ordinates of the arc $y(x)$. He then varied one or more arbitrarily selected ordinates and calculated the variation in the integral. By equating the variation of the integral to zero and by using a crude limiting process to transform the resulting difference equation, he obtained the differential equation which must be satisfied by the minimizing arc.

By the above described method applied to integrals of the form

$$J = \int_{x_1}^{x_2} f(x, y, y')\, dx, \tag{1}$$

Euler succeeded in showing that the function $y(x)$ that minimizes or maximizes the value of J must satisfy the ordinary differential equation

$$\boxed{f_y - \frac{d}{dx}(f_{y'}) = 0.} \tag{2}$$

This notation must be understood in the following sense. The integrand $f(x, y, y')$ is to be regarded as a function of the independent variables x, y, and y' insofar as f_y and $f_{y'}$ are concerned. However $df_{y'}/dx$ must be taken to be the derivative of $f_{y'}$ wherein $f_{y'}$ depends on x through x, y, and y'. That is, Euler's differential equation is equivalent to

$$f_y - f_{y'x} - f_{y'y}y' - f_{y'y'}y'' = 0. \tag{3}$$

Since f is known, this equation is a second order, generally nonlinear, ordinary differential equation in $y(x)$. This famous equation, which Euler published[11] in 1736, is still the basic differential equation of the calculus of variations. It is, as we shall see more clearly later, a necessary condition that the minimizing or maximizing function $y(x)$ must satisfy.

Euler then tackled more difficult problems that involved special side conditions, as in the isoperimetric problems, but his procedure was still to solve the differential equation (3) in order to get first the possible minimizing or maximizing arcs and then to determine from the number of constants in the general solution of (2) or (3) what side conditions he could apply. One of the problems he tackled was called to his attention by Daniel Bernoulli in a letter of 1742. Bernoulli proposed to find the shape of an elastic rod subject to pressure at both ends by assuming that the square of the curvature along the curve taken by the bent rod, that is, $\int_0^L ds/R^2$, where s is arc length and R is the radius of curvature, is a minimum. This condition amounts to assuming that the potential energy stored up in the shape taken by the rod is a minimum.

11. *Comm. Acad. Sci. Petrop.*, 8, 1736, 159–90, pub. 1741 = *Opera*, (1), 25, 54–80.

The differential equation (3) is not the proper one when the integrands of the integrals to be minimized or maximized are more complicated than in (1). In the years from 1736 to 1744 Euler improved his methods and obtained the differential equations analogous to (3) for a large number of problems. These results he published in 1744 in a book, *Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes* (The Art of Finding Curved Lines Which Enjoy Some Maximum or Minimum Property).[12] Euler's work in his *Methodus* was cumbersome because he used geometric considerations, successive differences, and series and he changed derivatives to difference quotients and integrals to finite sums. He failed, in other words, to make most effective use of the calculus. But he ended with simple and elegant formulas applicable to a large variety of problems; and he treated a large number of examples to show the convenience and generality of his method. One example deals with minimal surfaces of revolution. Here the problem is to determine the plane curve $y = f(x)$ lying between (x_0, y_0) and (x_1, y_1) such that when revolved around the x-axis it generates the surface of least area. The integral to be minimized is

$$A = \int_{x_0}^{x_1} 2\pi y\sqrt{1 + y'^2}\, dx. \tag{4}$$

Euler proved that the function $f(x)$ must be an arc of a catenary; the surface so generated is called a catenoid. In an appendix to his 1744 book Euler also gave a definitive solution of the elastic-rod problem referred to above. He not only deduced that the shape of the rod took the form of an elliptic integral, but also gave solutions for different kinds of end-conditions. This book brought him immediate fame and recognition as the greatest living mathematician.

With this work the calculus of variations came into existence as a new branch of mathematics. However, geometrical arguments were used extensively, and the combined analytical and geometrical arguments were not only complicated but hardly provided a systematic general method. Euler was fully aware of these limitations.

3. *The Principle of Least Action*

While progress in the solution of problems of the calculus of variations was being made, a new motivation for work in the subject came directly from physics. The contemporary development was the Principle of Least Action. To explain the basis for this principle we must go back a bit. Euclid had proved in his *Catoptrica* (Chap. 7, sec. 7) that light traveling from P to a mirror (Fig. 24.4) and then to Q takes the path for which $\sphericalangle 1 = \sphericalangle 2$. Then the Alexandrian Heron proved that the path PRQ, which light actually

12. *Opera*, (1), 24.

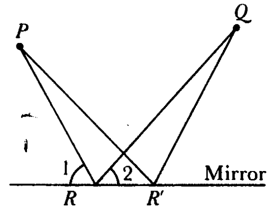

Figure 24.4

takes, is shorter than any other path, such as $PR'Q$, which it could conceivably take. Since the light takes the shortest path, if the medium on the upper side of the line RR' is homogeneous, then the light travels with constant velocity and so takes the path requiring least time. Heron applied this principle of shortest path and least time to problems of reflection from concave and convex spherical mirrors.

Basing their case on this phenomenon of reflection and on philosophic, theological, and aesthetic principles, philosophers and scientists after Greek times propounded the doctrine that nature acts in the shortest possible way or, as Olympiodorus (6th cent. A.D.) said in his *Catoptrica*, "Nature does nothing superfluous or any unnecessary work." Leonardo da Vinci said nature is economical and her economy is quantitative, and Robert Grosseteste believed that nature always acts in the mathematically shortest and best possible way. In medieval times it was commonly accepted that nature behaved in this manner.

The seventeenth-century scientists were at least receptive to this idea but, as scientists, tried to tie it to phenomena that supported it. Fermat knew that under reflection light takes the path requiring least time and, convinced that nature does indeed act simply and economically, affirmed in letters of 1657 and 1662[13] his Principle of Least Time, which states that light always takes the path requiring least time. He had doubted the correctness of the law of refraction of light (Chap. 15, sec. 4) but when he found in 1661[14] that he could deduce it from his Principle, he not only resolved his doubts about the law but felt all the more certain that his Principle was correct.

Fermat's Principle is stated mathematically in several equivalent forms. According to the law of refraction

$$\frac{\sin i}{\sin r} = \frac{v_1}{v_2},$$

where v_1 is the velocity of light in the first medium and v_2 in the second. The ratio of v_1 to v_2 is denoted by n and is called the index of refraction of the

13. *Œuvres*, 2, 354–59, 457–63.
14. *Œuvres*, 2, 457–63.

second medium relative to the first, or, if the first is a vacuum, n is called the absolute index of refraction of the nonvacuous medium. If c denotes the velocity of light in a vacuum, then the absolute index $n = c/v$ where v is the velocity of light in the medium. If the medium is variable in character from point to point, then n and v are functions of x, y, and z. Hence the time required for light to travel from a point P_1 to a point P_2 along a curve $x(\sigma)$, $y(\sigma)$, $z(\sigma)$ is given by

$$(5) \qquad J = \int_{\sigma_1}^{\sigma_2} \frac{ds}{v} = \int_{\sigma_1}^{\sigma_2} \frac{n}{c}\, ds = \frac{1}{c}\int_{\sigma_1}^{\sigma_2} n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, d\sigma,$$

where σ_1 is the value of σ at P_1 and σ_2 the value at P_2. Thus the Principle states that the path light actually takes in traveling from P_1 to P_2 is given by the curve which makes J a minimum.[15]

By the early eighteenth century the mathematicians had several impressive examples of the fact that nature does attempt to maximize or minimize some important quantities. Huygens, who had at first objected to Fermat's Principle, showed that it does hold for the propagation of light in media with variable indices of refraction. Even Newton's first law of motion, which states that the straight line or shortest distance is the natural motion of a body, showed nature's desire to economize. These examples suggested that there might be some more general principle. The search for such a principle was undertaken by Maupertuis.

Pierre-Louis Moreau de Maupertuis (1698–1759), while working with the theory of light in 1744, propounded his famous Principle of Least Action in a paper entitled "Accord des différentes lois de la nature qui avaient jusqu'ici paru incompatibles."[16] He started from Fermat's Principle, but in view of disagreements at that time as to whether the velocity of light was proportional to the index of refraction as Descartes and Newton believed, or inversely proportional as Fermat believed, Maupertuis abandoned least time. In fact he did not believe that it was always correct.

Action, Maupertuis said, is the integral of the product of mass, velocity, and distance traversed, and any changes in nature are such as to make the action least. Maupertuis was somewhat vague because he failed to specify the time interval over which the product of m, v, and s was to be taken and because he assigned a different meaning to action in each of the applications he made to optics and some problems of mechanics.

Though he had some physical examples to support his Principle, Maupertuis advocated it also for theological reasons. The laws of behavior of matter had to possess the perfection worthy of God's creation; and the

15. There are instances, as for example in the reflection of light from a concave mirror, where light takes the path requiring maximum time. This fact was known to Fermat and was explicitly stated by William R. Hamilton.
16. *Mém. de l'Acad. des Sci., Paris*, 1744.

$S = \int p\,dq = \int mv\,ds$

least action principle seemed to satisfy this criterion because it showed that nature was economical. Maupertuis proclaimed his principle to be a universal law of nature and the first scientific proof of the existence of God. Euler, who had corresponded with Maupertuis on this subject between 1740 and 1744, agreed with Maupertuis that God must have constructed the universe in accordance with some such basic principle and that the existence of such a principle evidenced the hand of God.

In the second appendix to his 1744 book Euler formulated the Principle of Least Action as an exact dynamical theorem. He limited himself to the motion of a single particle moving along plane curves. Moreover, he supposed that the speed is dependent upon position or, in modern terms, that the force is derivable from a potential. Whereas Maupertuis wrote

$$mvs = \text{min.},$$

Euler wrote

$$\partial \int v\, ds = 0,$$

by which he meant that the rate of change of the integral for a change in the path must be zero. He also wrote that, since $ds = v\, dt$,

$$\partial \int v^2\, dt = 0.$$

Just what Euler meant by the rate of change of the integral was still vague here even though he applied the principle correctly in specific problems by using his technique of the calculus of variations. At least he showed that Maupertuis's action was least for motions along plane curves.

Euler went further than Maupertuis in believing that all natural phenomena behave so as to maximize or minimize some function, so that the basic physical principles should be expressed to the effect that some function is maximized or minimized. In particular this should be true in dynamics, which studies the motions of bodies propelled by forces. Euler was not too far from the truth.

4. *The Methodology of Lagrange*

Euler's work attracted the attention of Lagrange, who began to concern himself with problems of the calculus of variations in 1750 when he was nineteen. He discarded the geometric-analytic arguments of the Bernoullis and Euler and introduced purely analytical methods. In 1755 he obtained a general procedure, systematic and uniform for a wide variety of problems, and worked on it for several years. His famous publication on the subject was the "Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies."[17] In a letter to Euler of August

17. *Misc. Taur.*, 2, 1760/61, 173–95, pub. 1762 = *Œuvres*, 1, 333–62.

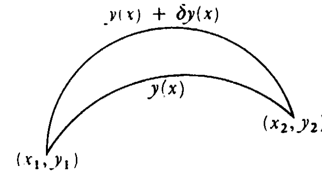

Figure 24.5

1755 he described the method, which he termed the method of variations, but which Euler in a paper presented to the Berlin Academy in 1756[18] named the calculus of variations.

Let us note Lagrange's method for the basic problem of the calculus of variations, namely, to minimize or maximize the integral

$$J = \int_{x_1}^{x_2} f(x, y, y')\, dx, \tag{6}$$

where $y(x)$ is to be determined. One of Lagrange's innovations was not to vary individual ordinates of the minimizing or maximizing curve $y(x)$ but to introduce new curves running between the endpoints (x_1, y_1) and (x_2, y_2). These new curves (Fig. 24.5) were represented by Lagrange in the form $y(x) + \delta y(x)$, the δ being a special symbol introduced by Lagrange to indicate a variation of the *entire curve* $y(x)$. The introduction of a new curve in the integrand of (6) of course changes the value of J. The increment in J, which we shall denote by ΔJ, is then

$$\Delta J = \int_{x_1}^{x_2} \{f(x, y + \delta y, y' + \delta y') - f(x, y, y')\}\, dx.$$

Now Lagrange regards f as a function of three independent variables, but since x is not changed, the integrand can be expanded by means of Taylor's theorem applied to a function of two variables. The expansion gives first degree terms in δy and $\delta y'$, second degree terms in these increments, and so forth. Lagrange then writes

$$\Delta J = \delta J + \frac{1}{2}\,\delta^2 J + \frac{1}{3!}\,\delta^3 J + \cdots \tag{7}$$

where δJ indicates the integral of first degree terms in δy and $\delta y'$, $\delta^2 J$ indicates the integral of the second degree terms, and so forth. Thus

$$\delta J = \int_{x_1}^{x_2} (f_y\, \delta y + f_{y'}\, \delta y')\, dx$$

$$\delta^2 J = \int_{x_1}^{x_2} \{f_{yy}(\delta y)^2 + 2f_{yy'}(\delta y)(\delta y') + f_{y'y'}(\delta y')^2\}\, dx.$$

18. "Elementa Calculi Variationum," *Novi Comm. Acad. Sci. Petrop.*, 10, 1764, 51–93, pub. 1766 = *Opera*, (1), 25, 141–76.

δJ is called the first variation of J; $\delta^2 J$, the second variation; and so forth.

Lagrange now argues that the value of δJ, since it contains the first order terms in the small variations δy and $\delta y'$, dominates the right side of (7), so that when δJ is positive or negative ΔJ will be positive or negative. But at a maximum or minimum of J, ΔJ must have the same sign, as in the case of ordinary maxima and minima of a function $f(x)$ of one variable, so that for $y(x)$ to be a maximizing function, δJ must be 0. Moreover, Lagrange says,

$$\delta y' = \frac{d(\delta y)}{dx}; \tag{8}$$

that is, the order of the operations d and δ can be interchanged. This is correct, though the reason was not clear to Lagrange's contemporaries and Euler clarified it later. [It is easily seen to be correct if we write $y + \delta y$ as $y + n(x)$, where $n(x)$ is the variation of $y(x)$. Then $\delta y = y + n(x) - y = n(x)$ and $\delta y' = y' + n'(x) - y' = n'(x)$. But $n'(x) = dn(x)/dx = d(\delta y)/dx$.] Using (8) Lagrange writes the first variation as

$$\delta J = \int_{x_1}^{x_2} \left[f_y \,\delta y + f_{y'} \frac{d}{dx}(\delta y)\right] dx.$$

Integrating the second term by parts and using the fact that δy must vanish at x_1 and at x_2 yields

$$\delta J = \int_{x_1}^{x_2} \left(f_y \,\delta y - \left(\frac{d}{dx} f_{y'}\right) \delta y\right) dx. \tag{9}$$

Now δJ must be 0 for every variation δy. Hence Lagrange concludes that the coefficient of δy must be 0,[19] or that

$$f_y - \frac{d}{dx}(f_{y'}) = 0. \tag{10}$$

Thus Lagrange arrived at the same ordinary differential equation for $y(x)$ that Euler had obtained. Lagrange's method of deriving (10) (except for his use of differentials) and even his notation are used today. Of course, (10) is a necessary condition on $y(x)$ but not sufficient.

In this paper of 1760/61 Lagrange also deduced for the first time end-conditions that must be satisfied by a minimizing curve for problems with variable endpoints. He found the transversality conditions that must hold at the intersections of the minimizing curve with the fixed curves or

19. The fact that the coefficient of δy must be 0 was intuitively accepted or incorrectly proven by every writer on the subject for one hundred years after Lagrange's work. Even Cauchy's proof was inadequate. The first correct proof was given by Pierre Frédéric Sarrus (1798–1861) (*Mém. divers Savans*, (2), 10, 1848, 1–128). The result is now known as the fundamental lemma of the calculus of variations.

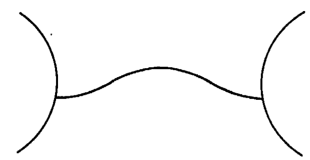

Figure 24.6

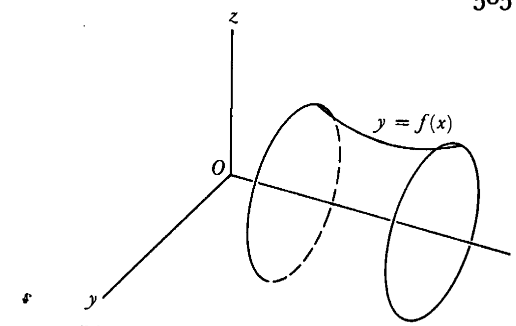

Figure 24.7

surfaces on which the endpoints of the comparison curves are allowed to vary (Fig. 24.6).

Though much more remains to be said about maximizing and minimizing integrals of the form (6), historically the next step, made by Lagrange in his 1760/61 paper and in a following one,[20] was to consider problems leading to multiple integrals. The integral to be maximized or minimized is of the form

$$J = \iint f(x, y, z, p, q)\, dx\, dy \tag{11}$$

wherein z is a function of x and y, $p = \partial z/\partial x$, and $q = \partial z/\partial y$. The integration is over some area in the xy-plane. The problem, then, is to find the function $z(x, y)$ that maximizes or minimizes the value of J. One of the most important problems that comes under this class of double integrals is to find the surface of least area among all surfaces whose boundary is fixed in some way. Thus one might be given two closed non-intersecting curves in space and seek the surface of minimum area bounded by these two curves. As a special case of the minimal surface problem, the two curves can be circles parallel to the yz-plane (Fig. 24.7) and with centers on the x-axis. Then the possible minimal surfaces are necessarily surfaces of revolution bounded by the two curves and the problem is to find the surface of revolution of minimum area. This last problem, as we noted above, had already been solved by Euler in 1744. However, this special case of a surface of revolution can be treated by the theory applicable to the integral (11).

By a method similar to the one he had used for the simpler integral (6), Lagrange obtained the differential equation that the function $z(x, y)$ minimizing (11) must satisfy. If we use the common notation

$$\frac{\partial z}{\partial x} = p, \qquad \frac{\partial z}{\partial y} = q, \qquad \frac{\partial^2 z}{\partial x^2} = r, \qquad \frac{\partial^2 z}{\partial x \partial y} = s, \qquad \frac{\partial^2 z}{\partial y^2} = t,$$

20. *Misc. Taur.*, 4, 1766/69 = *Œuvres*, 2, 37–63.

then the equation is

$$Rr + Ss + Tt = U \tag{12}$$

wherein R, S, T, and U are functions of x, y, z, p, and q. This nonlinear second order partial differential equation, called the equation of Monge, is not easy to solve; equations of this form had been the subject of research from the days of Euler onward (Chap. 22, sec. 7).

In the case of the minimal surface problem, the integral (11) becomes

$$\iint (1 + p^2 + q^2)^{1/2}\, dx\, dy, \tag{13}$$

and for this special class of problems the partial differential equation (12) becomes

$$(1 + q^2)r - 2pqs + (1 + p^2)t = 0. \tag{14}$$

This equation was given by Lagrange in his 1760/61 paper (though not quite in this form) and is a major analytical result in the theory of minimal surfaces. Geometrically, as Meusnier pointed out in a paper of 1785,[21] this partial differential equation expresses the fact that at any point on the minimizing surface the principal radii of curvature are equal and opposite or that the mean curvature, that is, the average of the principal curvatures, is zero.

In a later paper (1770)[22] Lagrange also considered single and multiple integrals in which higher derivatives than the first ones appear in the integrand. This topic has been well developed since Lagrange's time and is now standard material in the calculus of variations. However, since the principles are not basically different from the cases already considered, we shall not go into this extension of the subject. The content of Lagrange's papers on the calculus of variations is incorporated in his *Mécanique analytique*.

The calculus of variations was not well understood by the contemporaries of Lagrange and Euler. Euler explained Lagrange's method in numerous writings and used it to re-prove a number of old results. Though he realized that the calculus of variations was a new branch or technique, which he says is symbolized by the new operational symbol δ, he, like Lagrange, tried to base the logic of the calculus of variations on the ordinary calculus. Euler's idea[23] was to introduce a parameter t such that the curves of the family considered in a variations problem would vary with t, that is, for each t of some range there would be a curve $y_t(x)$. Then, says Euler, whereas $dy = (dy/dx)\, dx$, $\delta y = (dy/dt)\, dt$. Hence the variation δy is expressed by a partial differentiation with respect to t. He then formulated the technique

21. *Mém. divers Savans*, 10, 1785, 477–85.
22. *Nouv. Mém. de l'Acad. de Berlin*, 1770 = *Œuvres*, 3, 157–86.
23. *Novi Comm. Acad. Sci. Petrop.*, 16, 1771, 35–70, pub. 1772 = *Opera*, (1), 25, 208–35.

of the calculus of variations in terms of this new concept of differentiation with respect to t. His final results were, of course, the same as those already obtained.

Euler proceeded (1779)[24] to consider space curves with maximum or minimum properties and (1780) extensions of the brachistochrone problem when the applied force (which is gravity in the usual problem) operates in three dimensions or when a resisting medium is present.[25]

5. *Lagrange and Least Action*

Lagrange applied the calculus of variations to dynamics. He took over from Euler the Principle of Least Action and became the first to express the principle in concrete form, namely, that for a single particle the integral of the product of mass, velocity, and distance taken between two fixed points is a maximum or a minimum; that is, $\int mv\, ds$ must be a maximum or a minimum for the actual path taken by the particle. Alternatively, since $ds = v\, dt$, then $\int mv^2\, dt$ must be a maximum or a minimum. The quantity mv^2 [today $(1/2)mv^2$] is called the kinetic energy; in Lagrange's day it was called living force. Lagrange also asserted that the principle is true for a collection of particles and even for extended masses, though he was not clear on the last case.

Using the Principle of Least Action and the method of the calculus of variations, Lagrange obtained his famous equations of motion. Let us consider the case where the kinetic energy is a function of x, y, and z. Then for a single particle the kinetic energy T is given by

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{15}$$

Lagrange also supposed that the forces acting to cause the motion are derivable from a potential function V, which depends on x, y, and z. An additional condition, then, is that $T + V = \text{const.}$, that is, the total energy is constant. Lagrange's action is

$$\int_{t_0}^{t_1} T\, dt \tag{16}$$

and his Principle of Least Action states that this action must be a minimum or a maximum, that is

$$\delta \int_{t_0}^{t_1} T\, dt = 0. \tag{17}$$

24. *Mém. de l'Acad. des Sci. de St. Peters.*, 4, 1811, 18–42, pub. 1813 = *Opera*, (1), 25, 293–313.
25. *Mém. de l'Acad. des Sci. de St. Peters.*, 8, 1817/18, 17–45, pub. 1822 = *Opera*, (1), 25, 314–42.

In a minimizing or maximizing action, even though the motion takes place between two fixed points in space and the two fixed time values t_0 and t_1, the space and time variables must be varied.

By applying the method of the calculus of variations to the action integral, Lagrange derived equations analogous to Euler's equation (2), namely,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) + \frac{\partial V}{\partial x} = 0 \tag{18}$$

and the two corresponding equations with y and z. These equations are the equivalent of Newton's second law of motion.

Lagrange made the further step of introducing what are now called generalized coordinates. That is, in place of rectangular coordinates one may use polar coordinates or, in fact, any set of coordinates q_1, q_2, q_3, which are needed to fix the position of the particle (or extended mass). Then

$$\begin{aligned} x &= x(q_1, q_2, q_3) \\ y &= y(q_1, q_2, q_3) \\ z &= z(q_1, q_2, q_3), \end{aligned}$$

where the q_i are now functions of t. In terms of the new coordinates, T becomes a function of the q_i and $\dot{q}_i$ while V becomes a function of the q_i. Then the equations (18) become

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0, \qquad i = 1, 2, 3. \tag{19}$$

This is a set of 3 simultaneous second order ordinary differential equations in the q_i. They are the Euler (characteristic) equations for the action integral. If n coordinates are needed to fix the position of the moving object, for example, 2 particles require 6 coordinates, then equations (19) are replaced by n equations.[26]

These generalized coordinates need not have either geometrical or physical significance. One speaks of them today as coordinates in a configuration space and then the $q_i(t)$ are the equations of a path in configuration space. Thus Lagrange recognized that the variational principle, namely, that the action must be a minimum or maximum, can be used with any set

26. Lagrange is explicit that the number of variables in T and in V are the number required to determine the position of the mechanical system. Thus if there are N independent particles and each requires three coordinates (x_i, y_i, z_i) to describe its path in space, then $3N$ coordinates are required. In this case there will be $3N$ coordinates q_i, $3N$ equations relating the Cartesian coordinates to the q_i and $3N$ equations (19). The number of independent coordinates or the number of degrees of freedom, as physicists put it, depends on the system being treated and on the constraints in the motion.

of coordinates and the Lagrangian equations of motion (19) are invariant in form with respect to any coordinate transformation.

Lagrange's Principle, though it amounts to Newton's second law of motion, has several advantages over Newton's formulation. First of all, any convenient coordinate system is already, so to speak, built into the formulation. Secondly, it is easier to handle problems with constraints on the motion. Thirdly, instead of a series of separate differential equations, which may be numerous if many particles are involved, there is—to start with, at least—one principle from which the differential equations follow. Finally, though his principle supposes a knowledge of the kinetic and potential energies of a problem, it does not require a knowledge of the forces acting. With his principle Lagrange deduced major laws of mechanics and solved new problems, though it was not broad enough to include all the problems mechanics deals with. His work on the action principle is fully treated in his *Mécanique analytique*. He also started the movement to deduce the laws of other branches of physics from variational principles that would be the analogues of least action. He himself gave a variational principle for a broad class of hydrodynamical problems. This subject will be resumed in our study of the nineteenth-century work.

From the mathematical standpoint Lagrange's work on least action gave major importance to the calculus of variations. In particular Lagrange had derived the Euler equations for an integral whose integrand contains one independent variable but several dependent variables and their derivatives. This is an extension of the original calculus of variations problem, which contains only one dependent variable and its derivative. In this extended case the Euler equations are a system of second order ordinary differential equations in the q_i.

6. *The Second Variation*

The Euler differential equation, as Euler and Lagrange realized, is only a necessary condition for the solution to furnish a maximum or a minimum. They used the differential equation to find the solution and then decided on intuitive or physical grounds whether it furnished a maximum or a minimum. The role of the Euler equation is entirely analogous to the condition $f'(x) = 0$ in the ordinary calculus. A value of x that maximizes or minimizes $y = f(x)$ must satisfy $f'(x) = 0$, but the converse is not necessarily true.

The question of what additional conditions a solution of the Euler equation must satisfy to actually furnish a maximum or a minimum value of an integral depending on $y(x)$ was tackled unsuccessfully by Laplace in 1782. It was then taken up by Legendre in 1786.[27] Guided by the fact that

27. *Hist. de l'Acad. des Sci., Paris*, 1786, 7–37, pub. 1788.

in the ordinary calculus the sign of $f''(x)$ at a value of x for which $f'(x) = 0$ determines whether $f(x)$ has a maximum or a minimum, Legendre considered the second variation $\delta^2 J$, recast its form, and concluded that J is a maximum for the curve $y(x)$ which satisfies Euler's equation and passes through (x_0, y_0) and (x_1, y_1) provided that $f_{y'y'} \leq 0$ at each x along $y(x)$. Likewise, J is a minimum for a $y(x)$ satisfying the first two conditions provided that $f_{y'y'} \geq 0$ at each x along $y(x)$. Legendre then extended this result to more general integrals than (6). However, Legendre realized in 1787 that the condition on $f_{y'y'}$ was just a necessary condition on $y(x)$ in order that it be a maximizing or minimizing curve. The problem of finding sufficient conditions that a curve $y(x)$ maximize or minimize an integral such as (6) was not solved in the eighteenth century.

Bibliography

Bernoulli, James: *Opera*, 2 vols., 1744, reprint by Birkhaüser, 1968.
Bernoulli, John: *Opera Omnia*, 4 vols., 1742, Georg Olms (reprint), 1968.
Bliss, Gilbert A.: *The Calculus of Variations*, Open Court, 1925.
———: "The Evolution of Problems in the Calculus of Variations," *Amer. Math. Monthly*, 43, 1936, 598–609.
Cantor, Moritz: *Vorlesungen über Geschichte der Mathematik*, B. G. Teubner, 1898 and 1924, Vol. 3, Chap. 117, and Vol. 4, 1066–74.
Caratheodory, C.: Introduction to Series (1), Vol. 24 of Euler's *Opera Omnia*, viii–lxii, Orell Füssli, 1952. Also in C. Caratheodory: *Gesammelte mathematische Schriften*, C. H. Beck, 1957, Vol. 5, pp. 107–74.
Darboux, Gaston: *Leçons sur la théorie générale des surfaces*, 2nd ed., Gauthier-Villars, 1914, Vol. 1, Book III, Chaps. 1–2.
Euler, Leonhard: *Opera Omnia*, (1), Vols. 24–25, Orell Füssli, 1952.
Hofmann, Joseph E.: "Über Jakob Bernoullis Beiträge zur Infinitesimalmathematik," *L'Enseignement Mathématique*, (2), 2, 1956, 61–171; published separately by Institut de mathématiques, Geneva, 1957.
Huke, Aline: *An Historical and Critical Study of the Fundamental Lemma in the Calculus of Variations*, University of Chicago Contributions to the Calculus of Variations, University of Chicago Press, Vol, 1, 1930, pp. 45–160.
Lagrange, Joseph-Louis: *Œuvres de Lagrange*, Gauthier-Villars, 1867–69, relevant papers in Vols. 1–3.
———: *Mécanique analytique*, 2 vols., 4th ed., Gauthier-Villars, 1889.
Lecat, Maurice: *Bibliographie du calcul des variations depuis les origines jusqu'à 1850*, Gand, 1916.
Montucla, J. F.: *Histoire des mathématiques*, 1802, Albert Blanchard (reprint), 1960, Vol. 3, 643–58.
Porter, Thomas Isaac: "A History of the Classical Isoperimetric Problem," *University of Chicago Contributions to the Calculus of Variations*, University of Chicago Press, 1933, Vol. 2, pp. 475–517.
Smith, David E.: *A Source Book in Mathematics*, Dover (reprint), 1959, pp. 644–55.

Struik, D. J.: *A Source Book in Mathematics, 1200–1800*, Harvard University Press, 1969, pp. 391–413.
Todhunter, Isaac: *A History of the Calculus of Variations during the Nineteenth Century*, 1861, Chelsea (reprint), 1962.
Woodhouse, Robert: *A History of the Calculus of Variations in the Eighteenth Century*, 1810, Chelsea (reprint), 1964.