1 Second-order differential equations in the phase plane

IT IS NOT in general possible to obtain analytic solutions to an arbitrary differential equation. This is not simply because ingenuity fails, but because the repertory of standard functions (polynomials, exp, sin, and so on) in terms of which solutions may be expressed is too limited to accommodate the variety of differential equations encountered in practice. Even if an analytic solution can be found, the 'formula' is often too complicated to display clearly the principal features of the solution; this is particularly true of implicit solutions and of solutions which are in the form of integrals or infinite series.

The qualitative study of differential equations is concerned with how to deduce important characteristics of the solutions of differential equations without actually solving them. Since these considerations apply equally to linear equations and to the much greater variety of nonlinear equations, this book is mainly about nonlinear equations. In this chapter we introduce a geometrical device, the phase plane, which is used extensively for obtaining directly from the differential equation such properties as equilibrium, periodicity, unlimited growth, stability, and so on. The classical pendulum problem shows how the phase plane may be used to reveal all the main features of the solutions of a particular differential equation.

1.1. Phase diagram for the pendulum equation

The simple pendulum consists of a bob of mass m suspended from a fixed point O by a light string or rod of length a, which is allowed to swing in a vertical plane. If no friction is present then the equation of motion is

$$ma^2\ddot{\theta} + mga\sin\theta = 0, \tag{1.1}$$

where θ is the inclination of the string to the downward vertical (Fig. 1.1). Since $\ddot{\theta} = \dot{\theta}(\mathrm{d}\dot{\theta}/\mathrm{d}\theta)$, eqn (1.1) becomes

$$ma^2\dot{\theta}\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta} + mga\sin\theta = 0.$$

This equation relates $\dot{\theta}$ and θ instead of θ and t. By integrating with

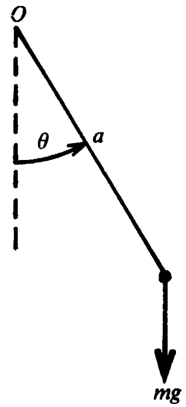


FIG. 1.1. The simple pendulum

respect to θ we obtain

$$ma^2 \int \dot{\theta}\,\mathrm{d}\dot{\theta} + mga \int \sin\theta\,\mathrm{d}\theta = C$$

where C is a constant. Therefore

$$\tfrac{1}{2}ma^2\dot{\theta}^2 - mga\cos\theta = C. \tag{1.2}$$

Equation (1.2) expresses conservation of energy, the terms on the left being, in order, the kinetic energy and potential energy of the pendulum. The value of C for a particular motion can be settled by using the initial conditions: the values of θ and $\dot{\theta}$ at $t = 0$. Equation (1.2) then gives the relation between $\dot{\theta}$ and θ for the motion corresponding to these initial conditions. By choosing different values for C we can obtain this relation for any possible motion.

Equation (1.2) is also a differential equation for θ in terms of t, but it cannot be solved in terms of elementary functions (see McLachlan 1956). It is therefore not easy to obtain a useful representation of θ as a function of time. We shall show how it is possible, by working directly with eqn (1.2), to reveal the main characteristics of the solutions.

The relation (1.2) can be represented in a diagram. Set up a cartesian *phase plane* having θ and $\dot{\theta}$ as its axes (Fig. 1.2) and plot the one-parameter family of curves generated by (1.2) for different values of C.

A given pair of values $(\theta, \dot{\theta})$ is called a *state* of the 'system' (in this case a pendulum), and the diagram shows how any state evolves as time progresses. We know that a given state determines all subsequent states,

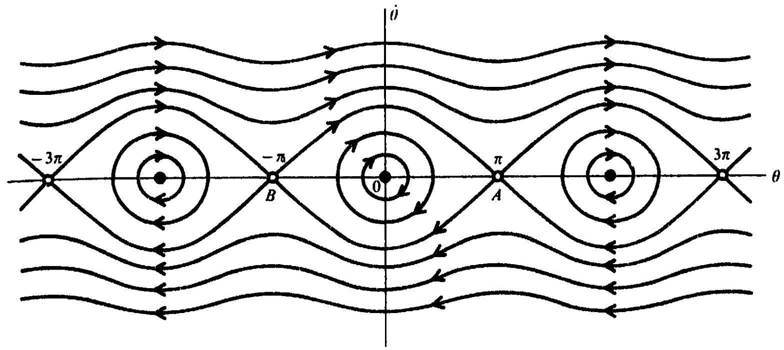


FIG. 1.2. Phase diagram for the simple pendulum

since it serves as initial conditions for the subsequent motion.

The curves depicted in Fig. 1.2 are known as the *phase paths*, *trajectories*, or *integral curves* corresponding to eqn (1.1), and the complete figure is called the *phase diagram* or *phase portrait* for the system. Each phase path corresponds to a particular possible motion of the system. Associated with each path is a *direction*, indicated by an arrow in Fig. 1.2, showing how the state of the system changes as time *increases*; the direction of the arrows is settled by observing that when $\dot{\theta}$ is positive, θ must be increasing with time and when $\dot{\theta}$ is negative, θ must be decreasing with time.

We shall see that despite the non-appearance of the time variable in the phase plane display, we can deduce several physical features of the pendulum's possible motions from Fig. 1.2. Consider first the possible states of physical equilibrium of the pendulum. The obvious one is when the pendulum hangs without swinging; then $\theta = 0$, $\dot{\theta} = 0$, which corresponds to the origin in Fig. 1.2. The corresponding function $\theta(t) = 0$ is a perfectly legitimate *constant solution* of (1.1): the phase path degenerates to a *single point*. If the suspension consists of a light rod there is a second position of equilibrium, where it is balanced vertically on end. This is the state $\theta = \pi$, $\dot{\theta} = 0$, another constant solution, represented by the point A on the phase diagram. The same physical condition is described by $\theta = -\pi$, $\dot{\theta} = 0$, represented by the point B, and indeed the state $\theta = n\pi$, $\dot{\theta} = 0$, where n is any integer, corresponds to one of these two equilibrium conditions. In fact we have displayed in Fig. 1.2 only part of the phase diagram, whose pattern repeats periodically: there is not in this case a one-to-one relationship between the physical condition of the pendulum and points on its phase diagram.

Since the points O, A, B represent states of physical equilibrium, they are called *equilibrium points* or *critical points* on the phase diagram.

Now consider the family of closed curves immediately surrounding the origin in Fig. 1.2. These indicate periodic motions, in which the pendulum swings to and fro about the vertical. The amplitude of the swing is the maximum value of θ encountered on the curve. For small enough amplitudes, the curves represent the usual 'small amplitude' solutions of the pendulum equation in which eqn (1.1) is simplified by writing $\sin\theta \approx \theta$. The phase paths are nearly ellipses in the small amplitude region.

The wavy lines at the top and bottom of Fig. 1.2, on which $\dot\theta$ is of constant sign and θ continuously increases or decreases, correspond to whirling motions of the pendulum. The fluctuations in $\dot\theta$ are due to the gravitational influence, and for such phase paths on which $\dot\theta$ is very large these fluctuations become imperceptible: the phase paths become nearly straight lines parallel to the θ axis.

We can discuss also the *stability* of the two typical equilibrium points O and A. If the initial state is displaced slightly from O, it goes on to one of the nearby closed curves and the pendulum oscillates with small amplitude about O. We describe the equilibrium point at O as being *stable.* If the initial state is slightly displaced from A (the vertically upward equilibrium position) however, it will normally fall on a phase path which carries the state far from the equilibrium state A into a large oscillation or a whirling condition (see Fig. 1.3). This equilibrium point is therefore described as *unstable.*

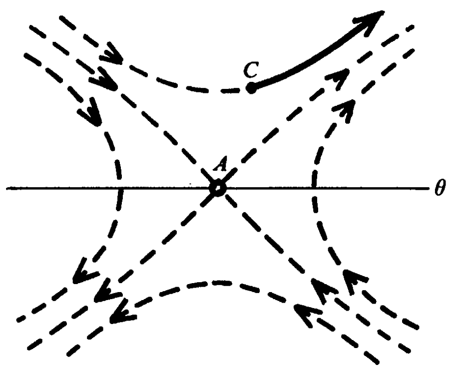


FIG. 1.3. Unstable equilibrium point for the pendulum; displaced state C

1.2. Autonomous equations in the phase plane

The second-order differential equation of general type

$$\ddot{x} = f(x, \dot{x}, t) \tag{1.3}$$

can be interpreted as an equation of motion for a mechanical system, in which x represents displacement of a particle of unit mass, $\dot x$ its velocity, $\ddot x$ its acceleration, and f the applied force, so that (1.3) expresses Newton's law of motion for the particle:

$$\text{acceleration} = \text{force per unit mass.}$$

A mechanical system is in equilibrium if its state does not change with time. This implies that *an equilibrium state corresponds to a constant solution* of (1.3), and conversely. A constant solution implies in particular that $\dot x$ and $\ddot x$ must be simultaneously zero. Note that $\dot x = 0$ is not alone sufficient for equilibrium: a swinging pendulum is instantaneously at rest at its maximum amplitude, but this is obviously not a state of equilibrium. Such constant solutions are therefore the constant solutions (if any) of

$$f(x, 0, t) = 0. \tag{1.4}$$

We distinguish between two types of equation:

(i) the *autonomous* type in which f does not depend explicitly on t;

(ii) the *non-autonomous* or *forced* equation where t appears explicitly in the function f.

A typical non-autonomous equation is the linear oscillator with a harmonic forcing term

$$\ddot x + k\dot x + \omega_0^2 x = F\cos\omega t,$$

in which $f(x, \dot x, t) = -k\dot x - \omega_0^2 x + F\cos\omega t$. There are no equilibrium states. Equilibrium states are not usually associated with non-autonomous equations although they can occur as, for example, in the equation (Mathieu's equation, Chapter 9)

$$\ddot x + (\alpha + \beta\cos t)x = 0,$$

which has an equilibrium state at $x = 0$.

In the present chapter we shall consider only autonomous equations; in this case t is absent and we shall write, instead of (1.3),

$$\ddot x = f(x, \dot x). \tag{1.5}$$

The constant solutions (which imply equilibrium states) are the solutions (if any) of

$$f(x,0) = 0. \tag{1.6}$$

For example, the equation $\ddot{x} = (1-x^2)+x\dot{x}$ has the constant solutions $x = 1$ and $x = -1$.

Now consider the representation of eqn (1.5) on a phase plane. The state of the system at a time $t = t_0$ consists of the pair of numbers $(x(t_0), \dot{x}(t_0))$. This state can be regarded as *a pair of initial conditions*, and therefore determines subsequent (and earlier) states. The succession of states can be depicted on a phase plane, as in Section 1.1 for the pendulum, in which the axes are for x and $\dot{x}$. We will relabel the $\dot{x}$ axis as y, thus defining y by

$$\dot{x} = y; \tag{1.7}$$

then $\ddot{x} = \dot{y}$ and (1.5) becomes

$$\dot{y} = f(x,y). \tag{1.8}$$

Equations (1.7) and (1.8) can be looked on as two simultaneous first-order equations for the functions $x(t)$, $y(t)$ (or $\dot{x}(t)$). To obtain the relation between y and x which will give the phase paths, eliminate t by dividing (1.8) by (1.7): $\dot{y}/\dot{x} = f(x,y)/y$, or

$$\frac{dy}{dx} = \frac{f(x,y)}{y}. \tag{1.9}$$

The solutions of this equation are the phase paths.

Example 1.1 *Find the phase paths for the equation $\ddot{x}+\alpha\sin x = 0$.*

This is essentially eqn (1.1). We define

$$\dot{x} = y$$

and the equation gives

$$\dot{y} = -\alpha\sin x.$$

The equation for the phase paths is

$$\frac{dy}{dx} = -\frac{\alpha\sin x}{y},$$

which separates to give

$$\int y\,dy = -\alpha\int\sin x\,dx + C,$$

or

$$\tfrac{1}{2}y^2 - \alpha\cos x = C,$$

where C is the parameter of the phase paths. The diagram in the plane is as Fig. 1.2.

The process of obtaining constant solutions of eqn (1.5) amounts to putting $x = 0$, $\ddot{x} = 0$, or to putting $\dot{x} = 0$, $\dot{y} = 0$ in (1.7) and (1.8). An equilibrium point (x_0, y_0) on the phase diagram is therefore any solution of the pair

$$y = 0, f(x,y) = 0. \tag{1.10}$$

Note that

(a) *equilibrium points are always situated on the x axis*;

(b) *except at equilibrium points the phase paths cut the x axis at right angles* (from (1.7), (1.8) and (1.10));

(c) *closed paths represent periodic solutions, since on completing a circuit the original state is returned to, and the motion simply repeats itself indefinitely.*

Now consider the time taken between two points on a phase path. Figure 1.4 shows a segment $\mathscr{C}$ of the phase path joining two points A and B. P represents any intermediate state: we call it a *representative point* of the path. P moves along $\mathscr{C}$ with velocity $(\dot{x}, \dot{y})$, or $(\dot{x}, \ddot{x})$ by (1.7). The time taken, T_{AB}, is therefore given by

$$T_{AB} = \int_{\mathscr{C}} dt \equiv \int_{\mathscr{C}} \left(\frac{dx}{dt}\right)^{-1}\left(\frac{dx}{dt}\right)dt = \int_{\mathscr{C}} \frac{dx}{y}, \tag{1.11}$$

which is calculable when $\mathscr{C}$ is given. The phase diagram therefore merely suppresses and does not completely lose track of the time variable since (1.11) depends only on the geometry of the phase path.

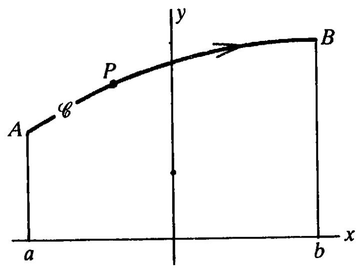


FIG. 1.4.

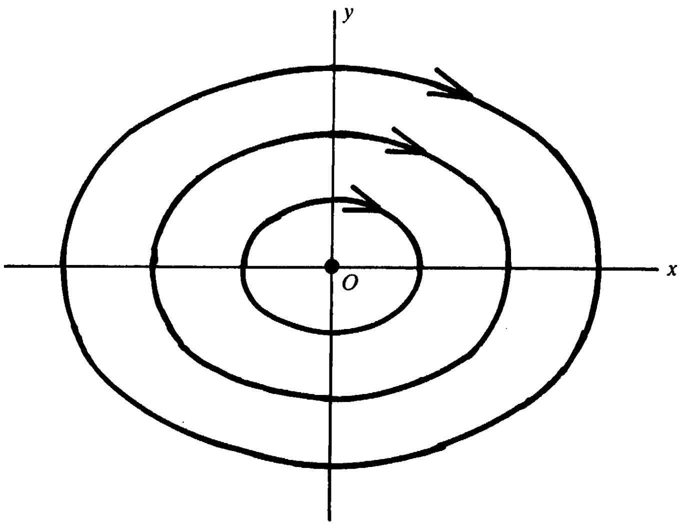


FIG. 1.5. Centre for the simple harmonic oscillator

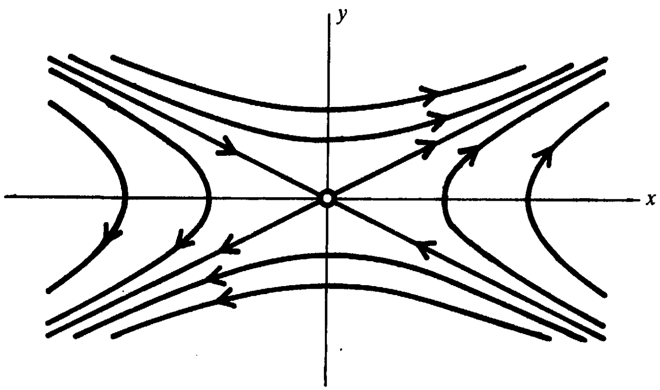


FIG. 1.6. Saddle point

Example 1.2 *Construct the phase diagram for the simple harmonic oscillator equation* $\ddot{x}+\omega^2 x = 0$.

This approximates to the pendulum equation for small-amplitude swings. Corresponding to eqns (1.7) and (1.8) we have

$$\dot{x} = y, \quad \dot{y} = -\omega^2 x.$$

There is one equilibrium point at

$$x = 0, \quad y = 0.$$

The phase paths are the solutions of

$$\frac{dy}{dx} = -\omega^2 \frac{x}{y},$$

which is separable, leading to

$$y^2 + \omega^2 x^2 = C.$$

The phase portrait therefore consists of a family of ellipses concentric with the origin (Fig. 1.5). All solutions are therefore periodic. The equilibrium point is stable.

An equilibrium point surrounded in its immediate neighbourhood (not necessarily over the whole plane) by closed paths is called a *centre*. A centre is a *stable* equilibrium point.

Example 1.3 *Construct the phase diagram for the equation* $\ddot{x} - \omega^2 x = 0$.

The equivalent first-order pair ((1.7) and (1.8)) are

$$\dot{x} = y, \quad \dot{y} = \omega^2 x.$$

There is a single equilibrium point at

$$x = 0, \quad y = 0.$$

The phase paths are the solutions of

$$\frac{dy}{dx} = \omega^2 \frac{x}{y};$$

therefore their equations are

$$y^2 - \omega^2 x^2 = C.$$

These paths are hyperbolas with asymptotes $y = \pm \omega x$ as shown in Fig. 1.6.

An equilibrium point with paths of this type in its neighbourhood is called a *saddle point*. Such a point is *unstable* since a small displacement from the equilibrium state will generally involve a solution which goes far from this state. (The question of stability is discussed precisely in Chapter 8.) In the figures, stable equilibrium points are usually indicated by a full dot •, and unstable ones by an 'open' dot o.

1.3. Conservative systems

In mechanical terms, consider a system with one degree of freedom and let x be a generalized coordinate (position, angle, etc.). Let $\mathscr{T}$ and $\mathscr{V}$ be the kinetic and potential energy functions, and assume that they take the form

$$\mathscr{T} = \tfrac{1}{2}m(x)\dot{x}^2, \quad \mathscr{V} = \mathscr{V}(x) \tag{1.12}$$

where m is another function, $m(x) > 0$. If the system is *conservative*, then the total energy, $\mathscr{E}$, is constant during motion:

$$\tfrac{1}{2}m(x)\dot{x}^2 + \mathscr{V}(x) = \mathscr{E}, \quad \text{constant,} \tag{1.13}$$

which gives the phase paths. The type of equation of motion which leads to (1.13) can be obtained by taking the time derivative of (1.13):

$$m(x)\ddot{x} + \tfrac{1}{2}m'(x)\dot{x}^2 + \mathscr{V}'(x) = 0. \tag{1.14}$$

Equation (1.14) can be simplified by introducing a new variable, u, in place of x (in effect another generalized coordinate) by

$$u = \int \sqrt{\{m(x)\}}\,dx.$$

Equation (1.14) becomes

$$\ddot{u} + \mathscr{Q}'(u) = 0, \tag{1.15}$$

where $\mathscr{Q}'(u) = \mathscr{V}'(x)m^{-1/2}(x)$, and the corresponding energy-type equation for the phase paths is

$$\tfrac{1}{2}\dot{u}^2 + \mathscr{Q}(u) = C. \tag{1.16}$$

We shall need only the form (1.15), but it should be remembered that equations of the form (1.14) also represent conservative systems.

Translating (1.15) and (1.16) back into the usual notation, we take the standard equation for a conservative system to be

$$\ddot{x} = f(x), \tag{1.17}$$

where the 'force per unit mass' f, is independent of $\dot{x}$. Suppose that f is continuous. Define for the new problem a function $\mathscr{V}$ by

$$\mathscr{V}(x) = -\int f(x)\,dx, \quad \text{or} \quad \mathscr{V}'(x) = -f(x). \tag{1.18}$$

Then $\mathscr{V}$ may be spoken of as the potential energy function (or just the 'potential') for (1.17). The equilibrium points of (1.17) are then given by

$$-f(x) = \mathscr{V}'(x) = 0 \tag{1.19}$$

(that is, the stationary points of the potential energy, as expected). By writing $\dot{x} = y$ as in (1.7), the phase paths are given by the 'energy equation'

$$\tfrac{1}{2}y^2 + \mathscr{V}(x) = C \tag{1.20}$$

for various values of C, or by

$$y = \pm\sqrt{(2C - 2\mathscr{V}(x))}. \tag{1.21}$$

Figure 1.7 shows the types of equilibrium point arising from the three types of turning point of $\mathscr{V}$: a local minimum always leads to a centre, a maximum to a saddle point, and a point of inflection to a cusp.

The construction can be thought of in the following way. For a fixed C, $2C - 2\mathscr{V}(x)$ can be read off from the top frame for the range of x for which this is non-negative, y calculated from (1.21) for each x, and the symmetrically-placed pair of points inserted in the lower frame.

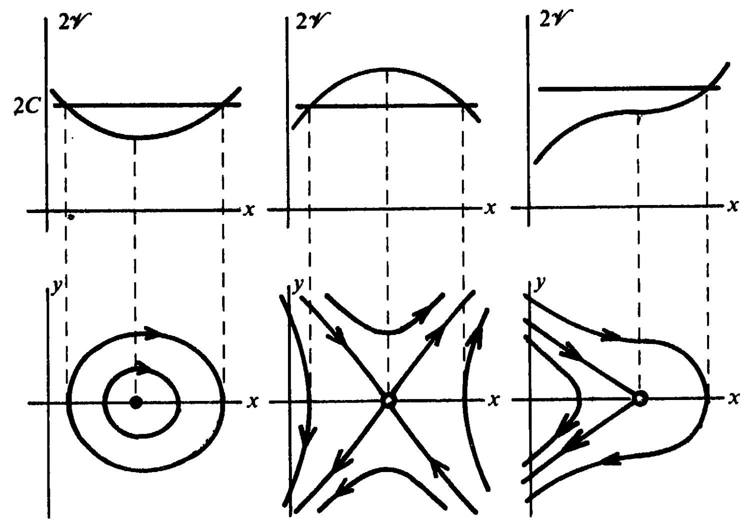


FIG. 1.7. • Stable equilibrium point, o unstable equilibrium point

Alternatively we can view the matter in terms of $f(x)$, regarded as the 'restoring force' for a particle on a nonlinear spring. If $f(x)$ changes sign from positive to negative on passing through the equilibrium point ($\mathscr{V}$ having a minimum), the system behaves like an ordinary linear spring and oscillates, implying a centre. If it changes from negative to positive, the particle is repelled from the equilibrium point and there is a saddle.

Example 1.4 *Sketch the phase diagram for the equation* $\ddot{x} = x^3 - x$.

This represents a conservative system (in fact the pendulum equation (1.1) for moderate amplitudes, writing $\sin\theta \approx \theta - \frac{1}{6}\theta^3$, leads to an equation reducible to this one). We have $\mathscr{V}'(x) = x - x^3$ and $\mathscr{V}(x) = -\frac{1}{4}x^4 + \frac{1}{2}x^2$ (to an additive constant). Figure 1.8 shows the construction of the phase diagram.

There are three equilibrium points: at (0, 0), a centre; at (1, 0), a saddle; and at (−1, 0), a saddle. The reconciliation between the types of phase path originating round these points is achieved by special paths called the *separatrices* (shown as broken lines). These correspond to values of C of 0 and $\frac{1}{4}$, the ordinates of the maxima and minimum of $\mathscr{V}$. They start or end at equilibrium points—they must not be mistaken for closed paths.

1.4. The damped linear oscillator

Generally speaking (apart from (1.14)) equations of the form

$$\ddot{x} = f(x, \dot{x}) \tag{1.22}$$

do not arise from conservative systems, and can be expected to show new phenomena. The simplest such system is a linear oscillator with linear damping, having the equation

$$\ddot{x} + k\dot{x} + cx = 0 \tag{1.23}$$

where $c > 0$, $k > 0$. An equation of this form describes, for example, a spring-mass system with a dashpot or a circuit containing inductance, capacitance, and resistance, and serves as a model for many other oscillating systems. We shall show how the familiar features of damped oscillations show up on the phase plane.

Equation (1.23) is a standard type of linear equation. The nature of its solutions depend on whether the roots of the auxiliary equation

$$m^2 + km + c = 0$$

are real and different, complex, or coincident and real. The roots are given by

$$\begin{matrix} m_1 \\ m_2 \end{matrix} = \tfrac{1}{2}\{-k \pm \sqrt{(k^2 - 4c)}\},$$

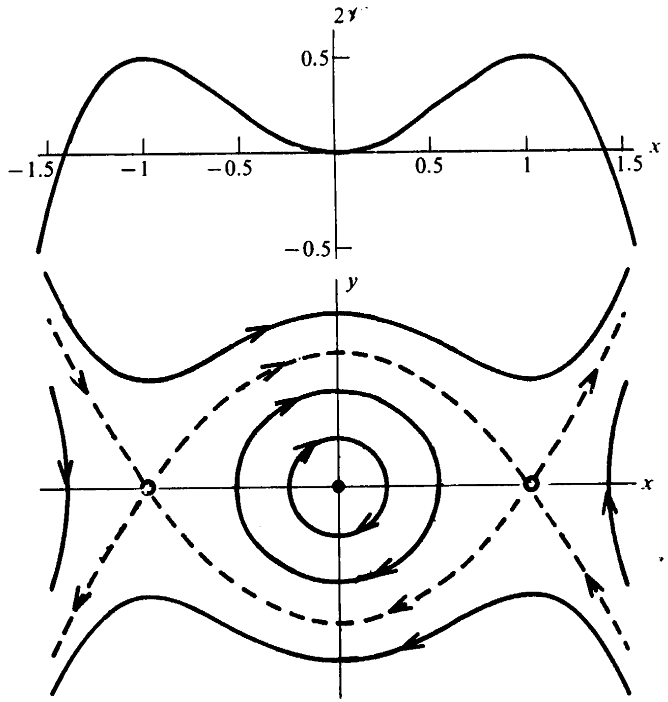


FIG. 1.8. Centre and saddle points. – – – – – separatrices

and the discriminant, Δ,

$$\Delta = k^2 - 4c, \tag{1.24}$$

is therefore the parameter which determines the general type of motion.

Strong damping (Δ > 0)

The solutions are given by

$$x(t) = A\,e^{m_1 t} + B\,e^{m_2 t}, \tag{1.25}$$

where m_1 and m_2 are real and negative, A and B being any constants. Figure 1.9 shows two typical solutions. There is no oscillation and the t axis is cut at most once.

To construct the phase paths we could write as usual

$$\dot{x} = y, \quad \dot{y} = -cx - ky. \tag{1.26}$$

There is a single equilibrium point at $x = 0, y = 0$. The equation for the phase paths is

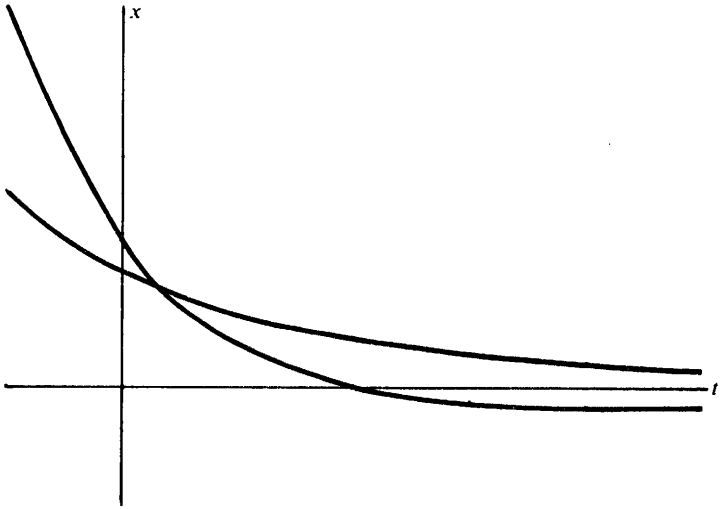


FIG. 1.9. Solution curves for a heavily-damped simple harmonic oscillator

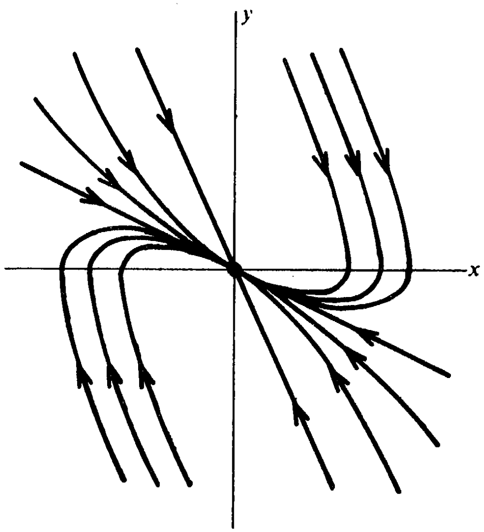


FIG. 1.10. Stable node

$$\frac{dy}{dx} = -c\frac{x}{y} - k. \tag{1.27}$$

A general approach to linear systems such as (1.26) will be described in Chapter 2. For the present we remark that the solutions of (1.27) are too complicated for simple interpretation. We therefore proceed in the following way. From (1.25)

$$x = A\,e^{m_1 t} + B\,e^{m_2 t}, \quad \text{so} \quad y = \dot{x} = Am_1\,e^{m_1 t} + Bm_2\,e^{m_2 t}, \tag{1.28}$$

and for fixed A and B, (1.28) constitutes a parametric representation of a phase path. The phase paths in Fig. 1.10 are plotted in this way for certain values of k and c.

This shows a new type of equilibrium point, called a *node*. For all slight displacements from $x = 0$, $\dot{x} = 0$, the state returns to the equilibrium point. It is therefore a *stable node*. That all the phase paths terminate at the origin can be seen by letting $t \to \infty$ in (1.28).

Weak damping ($\Delta < 0$)

The exponents are complex with negative real part, and the solutions are

$$x(t) = A\exp(-\tfrac{1}{2}kt)\cos\{\tfrac{1}{2}\surd(-\Delta)t + \alpha\} \tag{1.29}$$

where A, α are arbitrary constants. A typical solution is shown in Fig. 1.11(a): it represents an oscillation with exponentially decreasing amplitude, decaying more rapidly for larger k. Its image on the phase plane, plotted parametrically as before, is shown in Fig. 1.11(b).

The equilibrium point at the origin is called a *stable spiral* or a *stable focus.*

Critical damping ($\Delta = 0$)

In this case $m_1 = m_2 = -\tfrac{1}{2}k$ and the solutions are

$$x(t) = (A + Bt)\exp(-\tfrac{1}{2}kt).$$

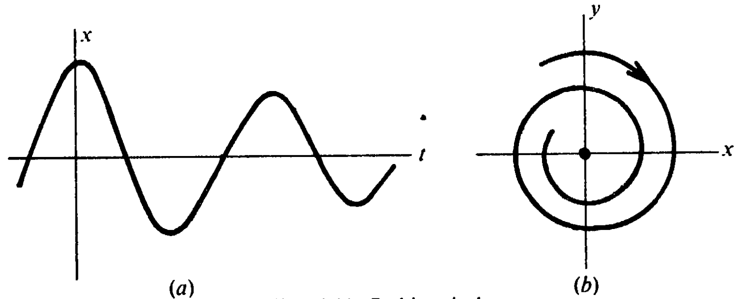


FIG. 1.11. Stable spiral

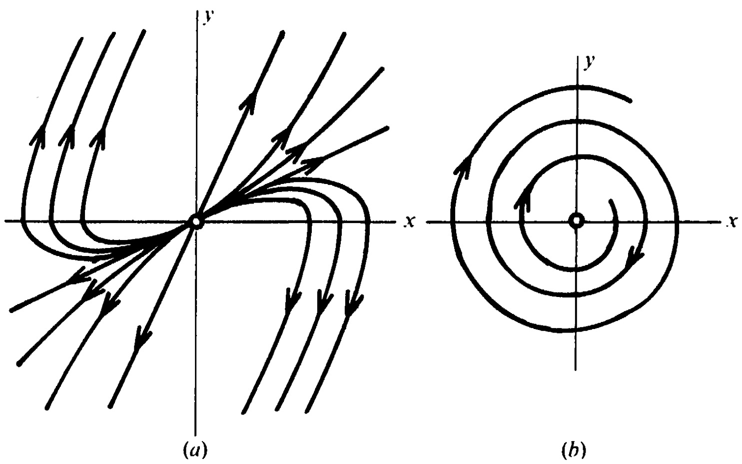


FIG. 1.12. (a) Unstable node; (b) Unstable spiral

The solutions resemble those for strong damping and the phase diagram shows a stable node.

Permutations of signs of the parameters k and c are possible.

(i) $c < 0, k \neq 0$. m_1 and m_2 are real but have different signs. The phase diagram shows a saddle point.

(ii) $c > 0, k < 0$. This case is called *negative damping*. Instead of energy being lost by the equivalent of a resistance or friction, energy is generated constantly in the system. The node or spiral is now *unstable* as shown in Fig. 1.12, since a slight disturbance from equilibrium leads to the system being carried far from the equilibrium state.

1.5. Nonlinear damping

Returning to the system

$$\ddot{x} = f(x, \dot{x}),$$

assume that f takes the form

$$f(x, \dot{x}) = -h(x, \dot{x}) - g(x), \tag{1.30}$$

where h does not contain an additive function of x. Then

$$\ddot{x} + h(x, \dot{x}) + g(x) = 0. \tag{1.31}$$

In mechanical terms the equation describes the displacement x of a particle of unit mass under a force system containing a conservative element, $g(x)$, and a dissipative or energy-generating component, $h(x, \dot{x})$.

In the case of a spring, $-g(x)$ is the restoring force. If g is of the appropriate type we should expect a tendency to oscillate, modified by the presence of the term $h(x, \dot{x})$.

The kinetic energy $\mathcal{T}$ is equal to $\frac{1}{2}\dot{x}^2$. We define a potential energy function (or, simply, a *potential*) for (1.31) by

$$\mathcal{V}(x) = \int g(x)\,\mathrm{d}x, \quad \text{so that} \quad g(x) = \mathcal{V}'(x). \tag{1.32}$$

The total energy $\mathcal{E}$ defined by

$$\mathcal{E} = \mathcal{T} + \mathcal{V} = \tfrac{1}{2}\dot{x}^2 + \int g(x)\,\mathrm{d}x \tag{1.33}$$

is not in general constant. Consider how $\mathcal{E}$ changes as a particular motion progresses: that is, along a phase path.

$$\begin{aligned}\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} &= \frac{\mathrm{d}\mathcal{T}}{\mathrm{d}t} + \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}t} = \dot{x}\ddot{x} + \mathcal{V}'(x)\dot{x} \\ &= \dot{x}\{f(x,\dot{x}) + g(x)\} \quad \text{(on the path)} \\ &= -\dot{x}h(x,\dot{x})\end{aligned} \tag{1.34}$$

by (1.31). Integrate (1.34) with respect to t from $t = \tau_0$ to $t = \tau$; then

$$\mathcal{E}(\tau) - \mathcal{E}(\tau_0) = -\int_{\tau_0}^{\tau} \dot{x}h(x,\dot{x})\,\mathrm{d}t \tag{1.35}$$

(where $x(t)$ is the solution of (1.31) on the chosen phase path).

It may be possible to say, for a phase path lying in some region of the phase plane, either that

(i) $\dot{x}h(x,\dot{x}) = yh(x, y) > 0$, in which case $\mathcal{E}(\tau) < \mathcal{E}(\tau_0)$: that is, the energy decreases and h has a damping effect, contributing to a general decrease in amplitude; or that

(ii) $\dot{x}h(x,\dot{x}) = yh(x, y) < 0$, in which case $\mathcal{E}(\tau) > \mathcal{E}(\tau_0)$: the effect is of an internal source injecting energy into the system.

A system may contain both characteristics. For example, a pendulum clock has energy stored in a weight which is supplied to the pendulum, and a balance is achieved between the rate of energy supplied and the loss through friction to maintain a steady oscillation.

Example 1.5 *Examine the equation* $\ddot{x} + |\dot{x}|\dot{x} + x = 0$ *for damping effects.*

$$h(x, \dot{x}) = |\dot{x}|\dot{x},$$

and

$$yh(x, y) = |y|y^2 > 0, \quad y \neq 0.$$

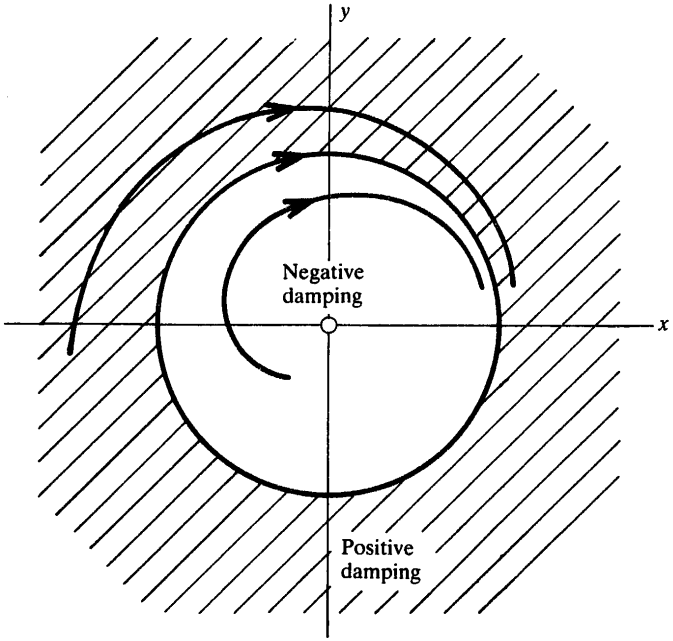


FIG. 1.13. Approach of two phase paths to a stable limit cycle

Except for the equilibrium point (0, 0) there is a loss of energy along every phase path no matter where it goes in the phase plane. We should therefore not be surprised if, from any initial state, the corresponding phase path eventually entered the origin, and motion ceased.

Example 1.6 *Examine the equation* $\ddot{x}+(x^2+\dot{x}^2-1)\dot{x}+x = 0$ *for damping or energy input effects.*

Putting $\dot{x} = y$,

$$yh(x, y) = (x^2+y^2-1)y^2.$$

When $x^2+y^2 < 1$, $yh(x, y) < 0$ and when $x^2+y^2 > 1$, $yh(x, y) > 0$.

The regions of energy loss and energy input on the phase plane are shown in Fig. 1.13. It can be verified that $x = \cos t$ is a solution of the differential equation, and this is represented by the circle $x^2+y^2 = 1$ on the phase plane. The phase diagram consists of this circle, together with paths approaching it from the outside, paths approaching it from the inside, and the equilibrium point at the origin. The *isolated* closed path $x^2+y^2 = 1$; isolated in the sense that there is no other closed path in its neighbourhood; is called a *limit cycle.* All paths approach the circle as $t \to \infty$. A limit cycle is therefore one of the most important features of a physical system.

A useful way to approach this and similar equations is to use polar coordinates (r, θ) where $x = r\cos\theta$, $y = r\sin\theta$. It follows that, since $r^2 = x^2+y^2$ and $\tan\theta = y/x$,

$$\dot{r} = (x\dot{x}+y\dot{y})/r, \quad \dot{\theta} = (x\dot{y}-\dot{x}y)/r^2.$$

Thus

$$\dot{r} = -r(r^2-1)\sin^2\theta, \tag{1.36}$$

$$\dot{\theta} = -1-(r^2-1)\sin\theta\cos\theta. \tag{1.37}$$

A particular solution of (1.36) is given by $r = 1$, the unit circle previously obtained. Also from (1.36) $\dot{r} > 0$ for $r < 1$ ($\sin\theta \neq 0$), and $\dot{r} < 0$ for $r > 1$ ($\sin\theta \neq 0$). We infer that outside the limit cycle r decreases and inside r increases with time, implying that the limit cycle is stable, the paths approaching it from both inside and outside.

A more decisive way of seeing what is happening in such cases, not involving reference to energy, is shown by the following example.

Example 1.7 *Show that all solutions of the equation* $\ddot{x}+|\dot{x}|\dot{x}+x^3 = 0$ *tend to zero as* $t \to \infty$.

It is easy to confirm the following important identity

$$\ddot{x} = \dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = \frac{1}{2}\frac{\mathrm{d}(\dot{x}^2)}{\mathrm{d}x}.$$

The equation may then be written

$$\frac{1}{2}\frac{\mathrm{d}(\dot{x}^2)}{\mathrm{d}x}+x^3 = -|\dot{x}|\dot{x},$$

or, in terms of the phase plane variables,

$$\frac{1}{2}\frac{\mathrm{d}(y^2)}{\mathrm{d}x}+x^3 = -|y|y. \tag{1.38}$$

Consider a segment of a path starting at the point A, where $x = a$, and ending at B, where $x = b$. By integrating (1.38) with respect to x from a to b we find that

$$[\tfrac{1}{2}y^2+\tfrac{1}{4}x^4]_{\mathrm{A}}^{\mathrm{B}} = -\int_a^b |y|y\,\mathrm{d}x = -\int_{t_{\mathrm{A}}}^{t_{\mathrm{B}}} |y|y^2\,\mathrm{d}t \tag{1.39}$$

(since $\mathrm{d}x = \dot{x}\,\mathrm{d}t = y\,\mathrm{d}t$). The right-hand side is negative everywhere, so (1.39) implies that, as the representative point moves along any path, the value of $\frac{1}{2}y^2 + \frac{1}{4}x^4$ (corresponding to the potential energy for this example) steadily decreases. But the family of curves defined by

$$\tfrac{1}{2}y^2+\tfrac{1}{4}x^4 = \textit{constant}$$

is a family of ovals closing in on the origin as the constant gets smaller. The above argument shows that a representative point crosses these successively in an inward direction, and so the path approaches the origin.

This method may be used in Examples 1.5 and 1.6, in which the analogous family consists of the circles $\frac{1}{2}y^2+\frac{1}{2}x^2 = constant$.

In the system of Example 1.6 the regions of positive and negative damping are separated by the circle $x^2+y^2 = 1$. In the important equation (van der Pol's equation)

$$\ddot{x}+e(x^2-1)\dot{x}+x = 0, \quad e > 0,$$

negative damping occurs in the strip $|x| < 1$ and positive damping in the half-planes $|x| > 1$. We shall see later that this type of damping also leads to a limit cycle. Van der Pol's equation originally arose as an idealization of a spontaneously oscillating, or self-excited, valve circuit (Minorsky 1962). It will provide for us the simplest form of equation having this pattern of positive and negative damping which gives rise to a limit cycle.

A closely-connected equation is Rayleigh's equation

$$\ddot{x}+e(\tfrac{1}{3}\dot{x}^2-1)\dot{x}+x = 0, \tag{1.40}$$

which originated in connection with a theory of the oscillation of a violin string. It has a similar distribution of positive and negative damping regions in the phase plane, and in fact reduces to van der Pol's equation by differentiating and putting $\dot{x} = z$.

1.6. Some applications

(i) *Dry friction*

Dry (or Coulomb) friction occurs when the surfaces of two solids are in contact and in relative motion without lubrication. The system shown in Fig. 1.14 illustrates dry friction. A continuous belt is driven by rollers at a constant speed v_0. A block of mass m connected to a fixed support by a spring of stiffness c rests on the belt. If F is the frictional force between

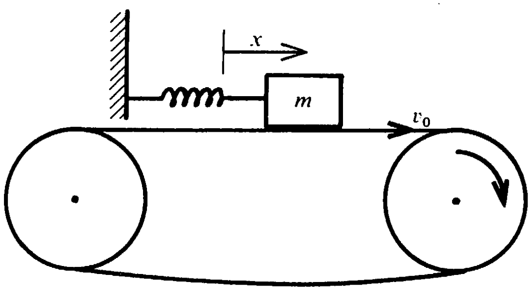


FIG. 1.14.

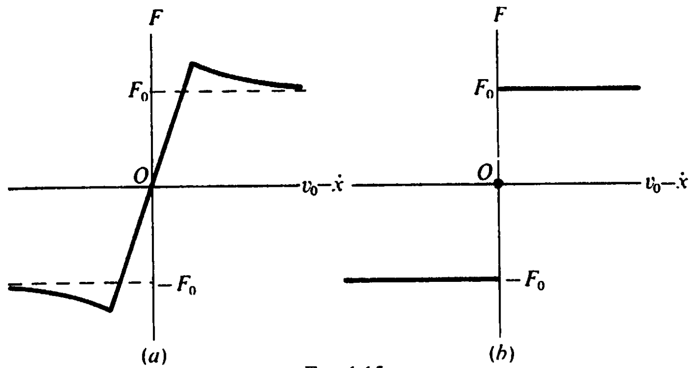


FIG. 1.15.

the block and the belt and x is the extension of the spring, then the equation of motion is

$$m\ddot{x}+cx = F.$$

Assume that F depends on the slip velocity, $v_0-\dot{x}$; a typical relation is shown in Fig. 1.15(*a*). We will replace this function by a simpler one having a discontinuity at the origin (Fig. 1.15(*b*)):

$$F = F_0 \operatorname{sgn}(v_0-\dot{x})$$

where F_0 is a positive constant (see Fig. 1.15(*b*)) and the sgn (signum) function is defined by

$$\operatorname{sgn}(u) = \begin{cases} 1, & u > 0, \\ 0, & u = 0, \\ -1, & u < 0. \end{cases}$$

The equation of motion becomes

$$m\ddot{x}+cx = F_0 \operatorname{sgn}(v_0-\dot{x}).$$

The term on the right is equal to F_0 when $v_0 > \dot{x}$, and $-F_0$ when $v_0 < \dot{x}$, and we obtain the following solutions for the phase paths in these regions:

$$y = \dot{x} > v_0: \quad my^2+(x+F_0/c)^2c = \text{constant},$$
$$y = \dot{x} < v_0: \quad my^2+(x-F_0/c)^2c = \text{constant}.$$

These are families of ellipses, the first having its centre at $(-F_0/c, 0)$ and the second at $(F_0/c, 0)$. Figure 1.16 shows the corresponding phase diagram, plotted as $m^{\frac{1}{2}}y$ against $c^{\frac{1}{2}}y$ to give circular paths.

There is a single equilibrium point, at $(F_0/c, 0)$, which is a centre. Points on y (or $\dot{x}$) $= v_0$ are not covered by the differential equation since this is where F is discontinuous, so the behaviour must be deduced from other physical arguments. On encountering the state $\dot{x} = v_0$ for

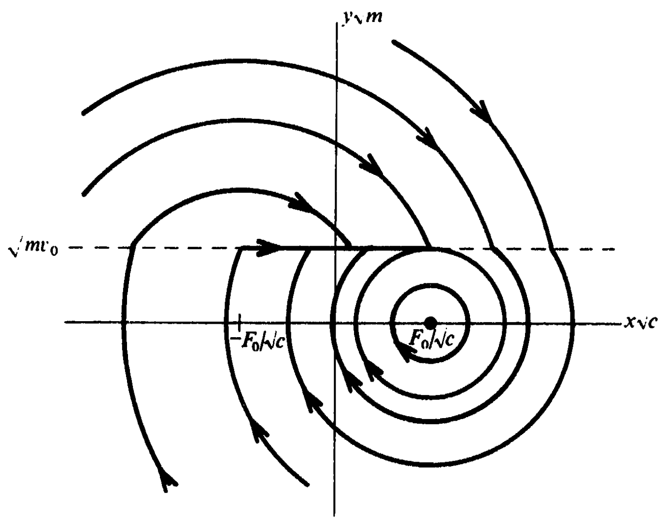


FIG. 1.16.

$|x| < F_0/c$, the block will move with the belt until the maximum available friction, F_0, is insufficient to resist the increasing spring tension. This is when $x = F_0/c$; the block then goes into an oscillation represented by the closed path through $(F_0/c, v_0)$. In fact, for any initial conditions lying outside this ellipse, the system ultimately settles into this oscillation. A computed phase diagram corresponding to a frictional force as in Fig. 1.15(*a*) is displayed in Exercise 50, Chapter 3.

(ii) *The brake*

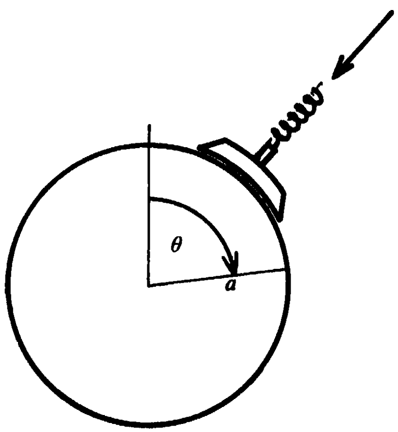


FIG. 1.17. A brake

Consider a simple brake shoe applied to the hub of a wheel as shown in Fig. 1.17. The friction force will depend on the pressure and the angular velocity of the wheel, $\dot{\theta}$. We assume again a simplified dry-friction relation corresponding to constant pressure

$$F = -F_0 \operatorname{sgn}(\dot{\theta})$$

so if the wheel is otherwise freely spinning its equation of motion is

$$I\ddot{\theta} = -F_0 a \operatorname{sgn}(\dot{\theta}),$$

where I is the moment of inertia of the wheel and a the radius of the brake drum. The phase paths are found by rewriting the differential equation

$$I\dot{\theta}\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta} = -F_0 a \operatorname{sgn}(\dot{\theta}),$$

whence for $\dot{\theta} > 0$

$$\tfrac{1}{2}I\dot{\theta}^2 = -F_0 a\theta + C$$

and for $\dot{\theta} < 0$

$$\tfrac{1}{2}I\dot{\theta}^2 = F_0 a\theta + C.$$

These represent two families of parabolas as shown in Fig. 1.18. $(\theta, 0)$ is an equilibrium point for every θ.

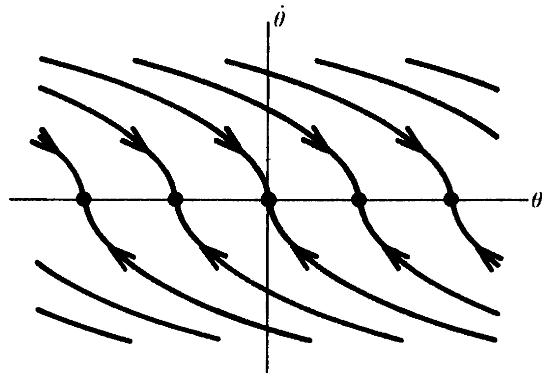


FIG. 1.18.

(iii) *The pendulum clock: a limit cycle*

Figure 1.19 shows the main features of the pendulum clock. The 'escape wheel' is a toothed wheel, which drives the hands of the clock through a succession of gears. It has a spindle around which is wound a wire with a weight at its free end. The escape wheel is arrested by the 'anchor' which has two teeth. The anchor is attached to the shaft of the

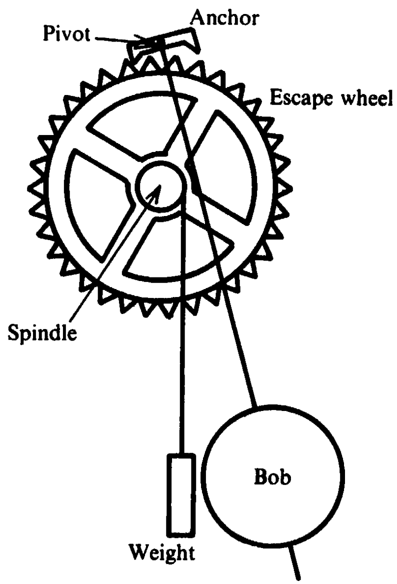


FIG. 1.19. A weight-driven clock mechanism

pendulum and rocks with it, controlling the rotation of the escape wheel. Its teeth are so designed that as the pendulum reaches its maximum amplitude on one side one tooth of the escape wheel is released, and the escape wheel is then stopped again by the other tooth on the anchor. Every time this happens the anchor receives a small impulse or pressure. It is this impulse which maintains the oscillation of the pendulum, which would otherwise die away. The loss of potential energy due to the weight's descent is therefore fed periodically into the pendulum via the anchor mechanism.

It can be shown that the system will settle into steady oscillations of fixed amplitude independently of sporadic disturbance and of initial conditions. If the pendulum is swinging with too great an amplitude, its loss of energy per cycle due to friction is large, and the impulse supplied by the escapement is insufficient to offset this. The amplitude consequently decreases. If the amplitude is too small, the frictional loss is small; the impulses will over-compensate and the amplitude will build up. A balanced state is therefore approached, which appears in the $\theta, \dot{\theta}$ plane (Fig. 1.20) as an isolated closed curve $\mathscr{C}$. Such an isolated periodic oscillation, or *limit cycle* (see Example 1.6) can occur only in systems described by nonlinear equations, and the following simple model shows

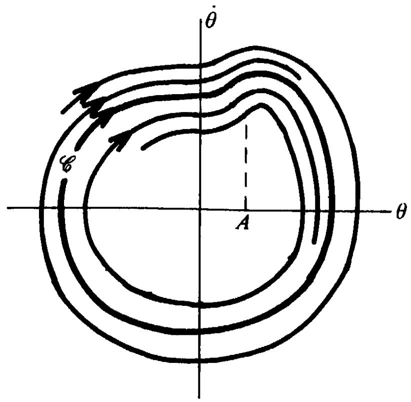


FIG. 1.20. Phase diagram for a clock, impulse applied near A

where the nonlinearity is located. The motion can be approximated by the equation

$$I\ddot{\theta}+k\dot{\theta}+c\theta = f(\theta), \tag{1.41}$$

where I is the moment of inertia of the pendulum, k is a small damping constant, c is another constant determined by gravity, θ is the angular displacement, and $f(\theta)$ is the moment, supplied once per cycle by the escapement mechanism. Since f is periodic in θ, it must be a nonlinear function of θ.

Such an oscillation, generated by an energy source whose input is not regulated externally, but which *automatically* synchronizes with the existing oscillation, is called *self-excited*. Here the build-up is limited by the friction.

1.7. Parameter-dependent conservative systems

Suppose $x(t)$ satisfies

$$\ddot{x} = f(x, \lambda)$$

where λ is a parameter. The equilibrium points of the system are given by $f(x, \lambda) = 0$, and in general their location will depend on the parameter λ. In mechanical terms, for a particle of unit mass with displacement x, $f(x, \lambda)$ represents the force experienced by the particle. Suppose there exists a function $\mathscr{V}(x, \lambda)$ such that $f(x, \lambda) = -\partial\mathscr{V}/\partial x$ for each value of λ; then $\mathscr{V}(x, \lambda)$ is the potential energy of the system and equilibrium points correspond to stationary values of the potential energy. As indicated in Section 1.3, we expect a minimum of potential energy to correspond to a stable equilibrium point, and other stationary values (the maximum and point of inflexion) to be unstable. In fact, $\mathscr{V}$ is a minimum at $x = x_1$ if $\partial\mathscr{V}/\partial x$ changes from negative to positive on passing through x_1; this implies that $f(x, \lambda)$ changes sign from positive to negative as x increases through $x = x_1$.

There exists a simple method of displaying the stability of equilibrium points for parameter-dependent systems in which both the number and stability of equilibrium points may vary with λ. We assume $f(x, \lambda)$ to be continuous in both x and λ. Plot the curve $f(x, \lambda) = 0$ in the λ, x plane; this curve represents the equilibrium points. Shade the domains in which $f(x, \lambda) > 0$ as shown in Fig. 1.21. If a segment of the curve has shading below it, the corresponding equilibrium points are stable, since for fixed λ, f changes from positive to negative as x increases.

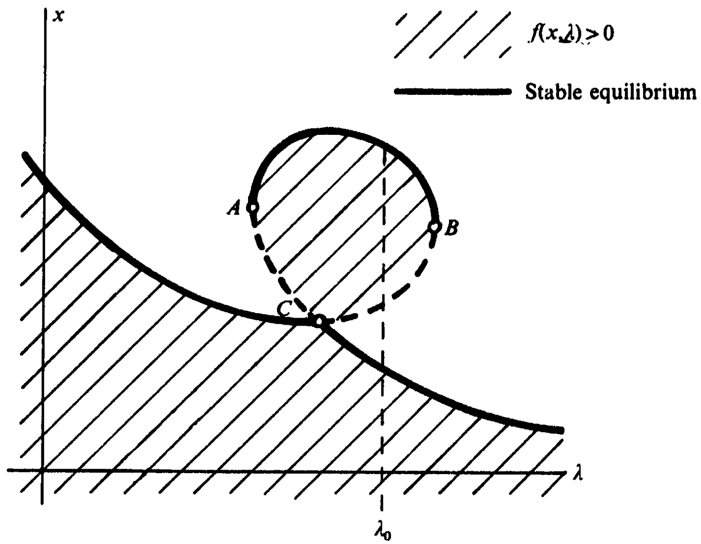


FIG. 1.21. Stability curves for the equilibrium points of $\ddot{x} = f(x, \lambda)$

For example, the solid line between A and B corresponds to stable equilibrium points. A and B are unstable: C is also unstable since f is positive on both sides of C. The nature of equilibrium points can easily be read from the figure; when $\lambda = \lambda_0$ as shown, the system has three equilibrium points, two of which are stable. A, B, and C are known as *bifurcation points*. As λ varies through such points the equilibrium point may split into two or more, or several equilibrium points may appear or merge into a single one. More information on bifurcation can be found in Chapter 12.

Example 1.7 *A bead slides on a smooth circular wire of radius a which is*

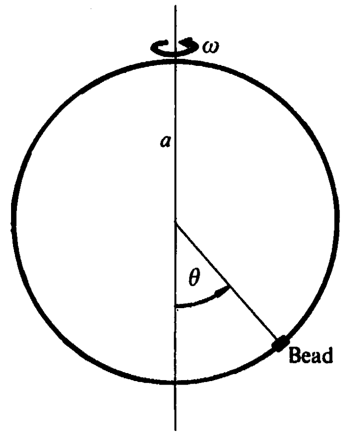


FIG. 1.22.

constrained to rotate about a vertical diameter with constant angular velocity ω. Analyse the stability of the bead.

The bead has a velocity component $a\dot{\theta}$ tangential to the wire and a component $a\omega \sin\theta$ perpendicular to the wire, where θ is the inclination of the radius to the bead to the downward vertical as shown in Fig. 1.22. The kinetic energy $\mathscr{T}$ and potential energy $\mathscr{V}$ are given by

$$\mathscr{T} = \tfrac{1}{2}ma^2(\dot{\theta}^2 + \omega^2 \sin^2\theta), \quad \mathscr{V} = -mga\cos\theta.$$

Since the system is subject to a moving constraint (that is, the angular velocity of the wire is imposed), the usual energy equation does not hold. Lagrange's equation for the system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{T}}{\partial\dot{\theta}}\right) - \frac{\partial\mathscr{T}}{\partial\theta} = -\frac{\partial\mathscr{V}}{\partial\theta},$$

which gives

$$a\ddot{\theta} = a\omega^2 \sin\theta\cos\theta - g\sin\theta.$$

Set $a\omega^2/g = \lambda$. Then

$$a\ddot{\theta} = a\dot{\theta}\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta} = g\sin\theta(\lambda\cos\theta - 1)$$

which, after integration, becomes

$$\tfrac{1}{2}a\dot{\theta}^2 = g(1 - \tfrac{1}{2}\lambda\cos\theta)\cos\theta + C,$$

the equation of the phase paths.

In the previous theory

$$f(\theta, \lambda) = g\sin\theta(\lambda\cos\theta - 1)/a$$

and the equilibrium points are given by $f(\theta, \lambda) = 0$, which is satisfied when

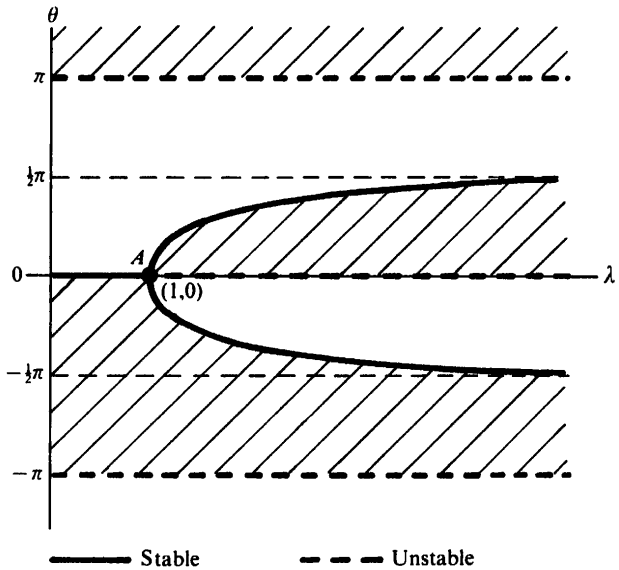


FIG. 1.23.

$\sin\theta = 0$ or $\cos = \lambda^{-1}$. From the periodicity of the problem, $\theta = \pi$ and $\theta = -\pi$ correspond to the same state of the system.

The regions where $f < 0$ and $f > 0$ are separated by curves where $f = 0$, and can be located, therefore, by checking the sign at particular points; for example, $f(\frac{1}{2}\pi, 1) = -g/a < 0$. Figure 1.23 shows the stable and unstable equilibrium positions of the bead. A is a stable bifurcation point.

Phase diagrams for the system may be constructed as in Section 1.3 for fixed values of λ. Two possibilities are shown in Fig. 1.24. Note that they confirm the stability predictions of Fig. 1.23.

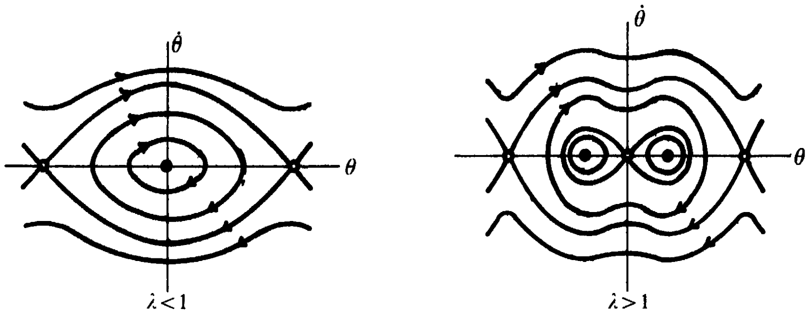


FIG. 1.24.

Exercises

1. Locate the equilibrium points and sketch the phase diagrams in their neighbourhood for the following equations:
   (i) $\ddot{x} - k\dot{x} = 0$.
   (ii) $\ddot{x} - 8x\dot{x} = 0$.
   (iii) $\ddot{x} = k\,(|x| > 1)$, $\quad \ddot{x} = 0\,(|x| < 1)$.
   (iv) $\ddot{x} + 3\dot{x} + 2x = 0$.
   (v) $\ddot{x} - 4\dot{x} + 40x = 0$.
   (vi) $\ddot{x} + 3|\dot{x}| + 2x = 0$.
   (vii) $\ddot{x} + k\,\text{sgn}\,(\dot{x}) + c\,\text{sgn}\,(x) = 0$, $\quad c > k$. Show that the path starting at $(x_0, 0)$ reaches $((c-k)^2 x_0/(c+k)^2, 0)$ after one circuit of the origin. Deduce that the origin is a spiral point.
   (viii) $\ddot{x} + x\,\text{sgn}\,(x) = 0$.

2. Sketch the phase diagram for the equation $\ddot{x} = -x - \alpha x^3$, considering all values of α. Check the stability of the equilibrium points by the method of Section 1.7.

3. A certain dynamical system is governed by the equation $\ddot{x} + \dot{x}^2 + x = 0$. Show that the origin is a centre in the phase plane, and that the open and closed paths are separated by the path $2y^2 = 1 - 2x$.

4. Sketch the phase diagrams for the equation $\ddot{x} + e^x = a$, for $a < 0$, $a = 0$, and $a > 0$.

5. Sketch the phase diagram for the equation $\ddot{x} - e^x = a$, for $a < 0$, $a = 0$, and $a > 0$.

6. The potential energy $\mathcal{V}(x)$ of a conservative system is continuous, and is strictly increasing for $x < -1$, zero for $|x| \leqslant 1$, and strictly decreasing for $x > 1$. Locate the equilibrium points and sketch the phase diagram for the system.

7. Figure 1.25 shows a pendulum striking an inclined wall. Sketch the phase diagram, for α positive and α negative, when (i) there is no loss of energy at impact, (ii) the magnitude of the velocity is halved on impact.

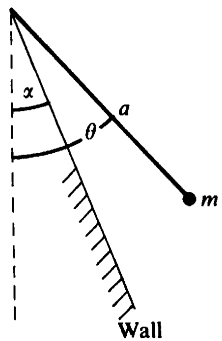


FIG. 1.25.

8. Show that the time elapsed, T, along a phase path $\mathscr{C}$ of the system $\dot{x} = y$, $\dot{y} = f(x, y)$ is given, in a form alternative to (1.11), by

$$T = \int_{\mathscr{C}} (y^2+f^2)^{-1/2}\,\mathrm{d}s,$$

where $\mathrm{d}s$ is an element of distance along $\mathscr{C}$.

By writing $\delta s \simeq (y^2+f^2)^{1/2}\delta t$, indicate, very roughly, equal time intervals along the phase paths of the system $\dot{x} = y, \dot{y} = 2x$.

9. On the phase diagram for the equation $\ddot{x}+x = 0$, the phase paths are circles. Use (1.11) in the form $\delta t \simeq \delta x/y$ to indicate, roughly, equal time steps along several phase paths.

10. Repeat Exercise 9 for the equation $\ddot{x}+9x = 0$, in which the phase paths are ellipses.

11. The pendulum equation, $\ddot{x}+\omega^2 \sin x = 0$, can be approximated for moderate amplitudes by the equation $\ddot{x}+\omega^2(x-\frac{1}{6}x^3) = 0$. Sketch the phase diagram for the latter equation, and explain the differences between it and Fig. 1.2.

12. The displacement, x, of a spring-mounted mass under the action of Coulomb dry friction is assumed to satisfy

$$m\ddot{x}+cx = -F_0\,\mathrm{sgn}\,(\dot{x}),$$

where m, c, and F_0 are positive constants (Section 1.6). The motion starts at $t = 0$, with $x = x_0 > 3F_0/c$ and $\dot{x} = 0$. Subsequently, whenever $x = -\alpha$, where $2F_0/c - x_0 < -\alpha < 0$ and $\dot{x} > 0$, a trigger operates, to increase suddenly the forward velocity so that the kinetic energy increases by a constant amount E. Show that if $E > 8F_0^2/c$, a periodic motion is approached, and show that the largest value of x in the periodic motion is equal to $F_0/c+E/4F_0$.

13. In Exercise 12, suppose that the energy is increased by E at $x = -\alpha$ for both $\dot{x} < 0$ and $\dot{x} > 0$; that is, there are two injections of energy per cycle. Show that periodic motion is possible if $E > 6F_0^2/c$, and find the amplitude of the oscillation.

14. The 'friction pendulum' consists of a pendulum attached to a sleeve, which embraces a close-fitting cylinder (Fig. 1.26). The cylinder is turned at a constant rate Ω. The sleeve is subject to Coulomb dry friction through the couple $G = -F_0\,\mathrm{sgn}\,(\dot{\theta}-\Omega)$. Write down the equation of motion, find the equilibrium states, and sketch the phase diagram.

15. By plotting the 'potential energy' of the nonlinear conservative system $\ddot{x} = x^4-x^2$, construct the phase diagram of the system. A particular path has the initial conditions $x = \frac{1}{2}$, $\dot{x} = 0$ at $t = 0$. Is the subsequent motion periodic?

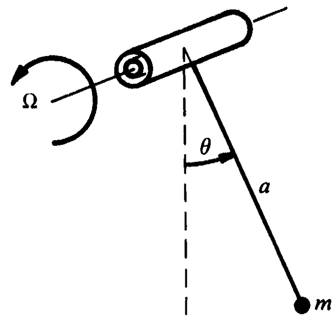


FIG. 1.26.

16. The system $\ddot{x}+x = -F_0\,\mathrm{sgn}\,(\dot{x})$, $F_0 > 0$, has the initial conditions $x = x_0 > 0$, $\dot{x} = 0$. Show that the phase path will spiral exactly n times before entering equilibrium (Section 1.6) if $(4n-1)F_0 < x_0 < (4n+1)F_0$.

17. A pendulum of length a has a bob of mass m which is subject to a horizontal force $m\omega^2 a \sin\theta$, where θ is the inclination to the downward vertical. Show that the equation of motion is $\ddot{\theta} = \omega^2(\cos\theta-\lambda)\sin\theta$, where $\lambda = g/\omega^2 a$. Investigate the stability of the equilibrium states by the method of Section 1.7 for parameter-dependent systems. Sketch the phase diagrams for various λ.

18. Investigate the stability of the equilibrium points of the parameter-dependent system $\ddot{x} = (x-\lambda)(x^2-\lambda)$.

19. If a bead slides on a smooth parabolic wire rotating with constant angular velocity ω about a vertical axis, then the distance x of the particle from the axis of rotation satisfies $(1+x^2)\ddot{x}+(g-\omega^2+\dot{x}^2)x = 0$. Analyse the motion of the bead in the phase plane.

20. A particle is attached to a fixed point 0 on a smooth horizontal plane by an elastic string. When unstretched, the length of the string is a. The equation of motion of the particle, which is constrained to move on a straight line through 0, is

$$\ddot{x} = -x+a\,\mathrm{sgn}\,(x), \quad |x| > a \quad \text{(when the string is stretched)},$$

$$\ddot{x} = 0, \quad |x| < a \quad \text{(when the string is slack)},$$

x being the displacement from 0. Find the equilibrium points and the equations of the phase paths, and sketch the phase diagram.

21. The equation of motion of a conservative system is $\ddot{x}+g(x) = 0$, where $g(0) = 0$; $g(x) < 0$ for $x < 0$ and $g(x) > 0$ for $x > 0$; and

$$\int_0^x g(u)\,\mathrm{d}u \to \infty \quad \text{as} \quad x \to \pm\infty. \qquad (a)$$

Show that the motion is always periodic.

By considering $g(x) = xe^{-x^2}$, show that if (*a*) does not hold, the motions are not all necessarily periodic.

22. The wave function $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \alpha\frac{\partial u}{\partial x} + \beta u^3 + \gamma\frac{\partial u}{\partial t} = 0,$$

where α, β, and γ are positive constants. Show that there exist permanent wave solutions of the form $u(x,t) = U(x-ct)$ for any c, where $U(\zeta)$ satisfies

$$\frac{\mathrm{d}^2U}{\mathrm{d}\zeta^2} + (\alpha-\gamma c)\frac{\mathrm{d}U}{\mathrm{d}\zeta} + \beta U^3 = 0.$$

Using Exercise 21, show that when $c = \alpha/\gamma$, all such waves are periodic.

23. The linear oscillator $\ddot{x}+\dot{x}+x = 0$ is set in motion with initial conditions $x = 0, \dot{x} = v$, at $t = 0$. After the first and each subsequent cycle the kinetic energy is instantaneously increased by a constant, E, in such a manner as to increase $\dot{x}$. Show that if $E = \frac{1}{2}v^2(1-e^{4\pi/\sqrt{3}})$, a periodic motion occurs. Find the maximum value of x in a cycle.

24. Show how solutions of Exercise 23 having arbitrary initial conditions spiral on to the periodic solution (which is a limit cycle). Sketch the phase diagram.

25. The kinetic energy, $\mathscr{T}$, and the potential energy, $\mathscr{V}$, of a system with one degree of freedom are given by

$$\mathscr{T} = T_0(x)+\dot{x}T_1(x)+\dot{x}^2T_2(x), \quad \mathscr{V} = \mathscr{V}(x).$$

Use Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{T}}{\partial\dot{x}}\right) - \frac{\partial\mathscr{T}}{\partial x} = -\frac{\partial\mathscr{V}}{\partial x}$$

to obtain the equation of motion of the system. Show that the equilibrium points are stationary points of $T_0(x)-\mathscr{V}(x)$, and that the phase paths are given by the energy equation

$$T_2(x)\dot{x}^2 - T_0(x)+\mathscr{V}(x) = \text{constant}.$$

26. Sketch the phase diagram for the equation $\ddot{x} = -f(x+\dot{x})$, where

$$f(u) = \begin{cases} f_0, & u \geqslant c, \\ f_0u/c, & |u| \leqslant c, \\ -f_0, & u \leqslant -c, \end{cases}$$

where f_0, c are constants, $f_0 > 0$, and $c > 0$. How does the system behave as $c \to 0$?

27. Sketch the phase diagram for the equation $\ddot{x} = u$, where

$$u = -\operatorname{sgn}(\sqrt{2}|x|^{1/2}\operatorname{sgn}(x)+\dot{x}).$$

(u is an elementary control variable which can switch between $+1$ and -1. The curve $\sqrt{2}|x|^{1/2}\operatorname{sgn}(x)+y = 0$ is called the switching curve.)

28. The relativistic equation for an oscillator is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{m_0\dot{x}}{\sqrt{[1-(\dot{x}/c)^2]}}\right\} + kx = 0$$

where m_0, c, and k are positive constants. Show that the phase paths are given by

$$\frac{m_0c^2}{\sqrt{[1-(y/c)^2]}} + \tfrac{1}{2}kx^2 = \text{constant}.$$

If $y = 0$ when $x = a$, show that the period, T, of an oscillation is given by

$$T = \frac{4}{c\sqrt{\varepsilon}}\int_0^a \frac{[1+\varepsilon(a^2-x^2)]\,\mathrm{d}x}{\sqrt{(a^2-x^2)}\sqrt{[2+\varepsilon(a^2-x^2)]}}, \quad \varepsilon = k/2m_0c^2.$$

The constant ε is small; by expanding the integrand in powers of ε show that

$$T \approx 2\pi\sqrt{(m_0/k)}\,(1+\tfrac{3}{8}\varepsilon a^2).$$

29. A mass m is attached to the mid-point of an elastic string of length $2a$ and stiffness λ. There is no gravity acting, and the tension is zero in the equilibrium

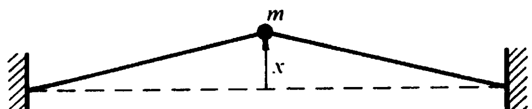


FIG. 1.27.

position. Obtain the equation of motion for transverse oscillations and sketch the phase paths.

30. Analyse, from the point of view of Section 1.5, the equation of motion for the dry-friction problem of Section 1.6, and reconcile this with the phase diagram Fig. 1.16.

31. Show that, in the phase diagram for a pendulum clock (Section 1.6), there is a net loss of energy along paths far enough from the origin, and a net gain along paths close enough to the origin. (Put eqn (1.41) into the form (1.31) and use (1.35).)

32. A pendulum with a magnetic bob oscillates in a vertical plane over a magnet, which repels the bob according to the inverse square law, so that the equation of motion is (Fig. 1.28)

$$ma^2\ddot{\theta} = -mga\sin\theta + Fh\sin\phi,$$

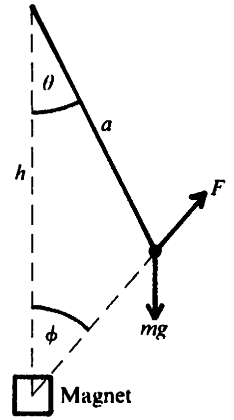


FIG. 1.28.

where $h > a$ and $F = c/(a^2+h^2-2ah\cos\theta)$ and c is a constant. Find the equilibrium positions of the bob, and classify them as centres and saddle-points according to the parameters of the problem. Describe the motion of the pendulum.

33. A pendulum with equation $\ddot{x}+\sin x = 0$ oscillates with amplitude a. Show that its period, T, is equal to $4K(\beta)$, where $\beta = \sin^2\frac{1}{2}a$ and

$$K(\beta) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1-\beta\sin^2\phi)}}.$$

$K(\beta)$ has the power series representation

$$K(\beta) = \tfrac{1}{2}\pi[1+(\tfrac{1}{2})^2\beta+(\tfrac{1.3}{2.4})^2\beta^2+\ldots], \quad |\beta| < 1.$$

Deduce that, for small amplitudes,

$$T = 2\pi(1+\tfrac{1}{16}a^2+\tfrac{11}{3072}a^4)+O(a^6).$$

34. Repeat Exercise 33 with the equation $\ddot{x}+x-\varepsilon x^3 = 0$, and show that

$$T = \frac{4\sqrt{2}}{\sqrt{(2-\varepsilon a^2)}}K(\beta), \quad \beta = \frac{\varepsilon a^2}{2-\varepsilon a^2},$$

and that

$$T = 2\pi(1+\tfrac{3}{8}\varepsilon a^2+\tfrac{57}{256}\varepsilon^2 a^4)+O(a^6), \quad \varepsilon a^2 < 2.$$

35. Show that equations of the form $\ddot{x}+g(x)\dot{x}^2+h(x) = 0$ are effectively conservative. (Find a transformation of x which puts the equation into the usual conservative form. Cf. eqn (1.14).)

36. Compare the phase diagrams of the following systems and the second-order equations in x to which they give rise:
(i) $\dot{x} = 1, \dot{y} = 0$, (ii) $\dot{x} = y, \dot{y} = 0$, (iii) $\dot{x} = xy, \dot{y} = 0$.

37. Show that the phase plane for the equation

$$\ddot{x}-\varepsilon x\dot{x}+x = 0$$

has a centre at the origin, by first calculating the phase paths.

38. Show that the equation $\ddot{x}+x+\varepsilon x^3 = 0$ with $x(0) = a, \dot{x}(0) = 0$ has phase paths given by

$$\dot{x}^2+x^2+\tfrac{1}{2}\varepsilon x^4 = (1+\tfrac{1}{2}\varepsilon a^2)a^2.$$

Show that the origin is a centre. Are all phase paths closed, and hence all solutions periodic?

39. Locate the equilibrium points of the equation

$$\ddot{x}+\lambda+x^3-x = 0,$$

in the x, λ plane. Show that the phase paths are given by

$$\tfrac{1}{2}\dot{x}^2+\lambda x+\tfrac{1}{4}x^4-\tfrac{1}{2}x^2 = \text{constant}.$$

Sketch, in the $x, \dot{x}, \lambda$ space, the surface which represents the set of phase diagrams for which the constant (the energy) is zero. Suppose the system is oscillating with zero energy and, by means of two controllers, λ is slowly reduced and the energy is kept zero. What happens as λ passes through the value $-\frac{1}{3}\sqrt{(\frac{4}{3})}$?

40. Burgers' equation

$$\frac{\partial\phi}{\partial t}+\phi\frac{\partial\phi}{\partial x} = c\frac{\partial^2\phi}{\partial x^2}$$

shows diffusion and nonlinear effects. Find the equation for permanent waves by putting $\phi = U(x-ct)$, where c is the constant wave speed. Find the equilibrium points and the phase paths for the resulting equation and interpret the phase diagram.

2 First-order systems in two variables and linearization

CHAPTER 1 describes the application of phase-plane methods to the equation $\ddot{x} = f(x, \dot{x})$ through the equivalent first-order system $\dot{x} = y$, $\dot{y} = f(x, y)$. This approach permits a useful line of argument based on a mechanical interpretation of the original equation. Frequently, however, the appropriate formulation of mechanical, biological, and geometrical problems is not through a second-order equation at all, but directly as a more general type of first-order system of the form $\dot{x} = X(x, y)$, $\dot{y} = Y(x, y)$. The appearance of these equations is an invitation to construct a phase plane with x, y coordinates in which solutions are represented by curves $(x(t), y(t))$ where $x(t), y(t)$ are the solutions. The constant solutions are represented by equilibrium points obtained by solving the equations $X(x, y) = 0$, $Y(x, y) = 0$, and these may now occur anywhere in the plane. Near the equilibrium points we may make a linear approximation to $X(x, y)$, $Y(x, y)$, solve the simpler equations obtained, and so determine the local character of the paths. This enables the stability of the equilibrium states to be settled and is a starting-point for global investigations of the solutions.

2.1. The general phase plane

Consider the general autonomous first-order system

$$\dot{x} = X(x, y), \quad \dot{y} = Y(x, y) \tag{2.1}$$

of which the type considered in Chapter 1,

$$\dot{x} = y, \quad \dot{y} = f(x, y), \tag{2.2}$$

is a special case. As in Section 1.2, a system is called *autonomous* when the time variable does not appear in the right-hand side of (2.1). We shall give examples later of how such systems arise.

The solutions $x(t), y(t)$ of (2.1) may be represented on a plane with cartesian axes x, y. Then as t increases $(x(t), y(t))$ traces out a directed curve in the plane called a *phase path*.

The appropriate form for the initial conditions of (2.1) is

$$x = x_0, \quad y = y_0 \quad \text{at} \quad t = t_0$$

where x_0 and y_0 are the *initial values* at time t_0; by the existence and uniqueness theorem (Appendix A) there is one and only one solution satisfying this condition when (x_0, y_0) is an 'ordinary point'. This does not at once mean that there is one and only one phase path through the point (x_0, y_0) on the phase diagram, because this same point could serve as the initial conditions for other starting times. Therefore it might seem that other phase paths through the same point could result: the phase diagram would then be a tangle of criss-crossed curves. We shall see that this will not be so by forming the differential equation for the phase paths. Since $\dot{y}/\dot{x} = \mathrm{d}y/\mathrm{d}x$ on a path the required equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{Y(x, y)}{X(x, y)}. \tag{2.3}$$

There are two classes of exceptional points where solution curves of (2.3) (the phase paths) may meet. First, there may be points where Y/X has some manifest singularity so that the uniqueness theorem fails: these are *singular points*. Of more immediate interest, there are usually points where

$$X(x, y) = 0, \quad Y(x, y) = 0, \tag{2.4}$$

which are the *equilibrium points*. If a solution of (2.4) is x_1, y_1, then $x(t) = x_1$, $y(t) = y_1$ are *constant solutions* of (2.1), and are degenerate phase paths. The terms *fixed point* and *critical point* are also used. There is one and only one solution curve passing through any ordinary point of (2.3), which may have singular points where (2.1) does not. Therefore there is one and only one phase path containing the ordinary point (x_0, y_0), no matter at what time the initial condition (x_0, y_0) is prescribed. Thus, infinitely many solutions of (2.1), differing only by time displacements, map onto a single phase path.

Equation (2.3) does not give any indication of the direction to be associated with a phase path, and this must be settled by reference to (2.1): the signs of X and Y at a point determine the direction through the point, and generally the directions at all other points can be settled by continuity.

The diagram depicting the phase paths is called the *phase diagram*. The point (x, y) is called a *state* of the system, and the phase diagram depicts the evolution of the states of the system from arbitrary initial states.

Example 2.1 *Compare the phase diagrams of the systems*
(i) $\dot{x} = y, \dot{y} = -x$; (ii) $\dot{x} = xy, \dot{y} = -x^2$.

The equation for the paths is the same for both, namely

$$\frac{dy}{dx} = -\frac{x}{y}$$

(strictly, for $y \neq 0$, and for $x \neq 0$ in the second case), giving a family of circles in both cases. However, in case (i) there is an equilibrium point only at the origin, but in case (ii) every point on the y axis is an equilibrium point. The directions, too, are different. By considering the signs of $\dot{x}, \dot{y}$ in the various quadrants the phase diagram of Fig. 2.1 is produced.

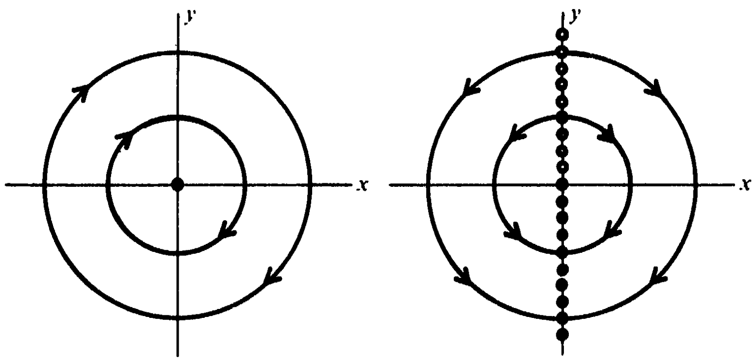


FIG. 2.1.

A second-order differential equation can be reduced to the general form (2.1) in an arbitrary number of ways and this occasionally has advantages akin to changing the variable to simplify a differential equation. For example the reduction (2.2) applied to $x\ddot{x} - \dot{x}^2 - x^3 = 0$ leads to the system

$$\dot{x} = y, \quad \dot{y} = y^2/x + x^2. \tag{2.5}$$

Suppose we define, instead of y, another variable y_1 by $y_1(t) = \dot{x}(t)/x(t)$, that is, by

$$\dot{x} = xy_1. \tag{2.6}$$

Then from (2.6), $\ddot{x} = x\dot{y}_1 + \dot{x}y_1 = x\dot{y}_1 + xy_1^2$ (using (2.6) again). But from the differential equation, $\ddot{x} = \dot{x}^2/x + x^2 = xy_1^2 + x^2$. Therefore

$$\dot{y}_1 = x. \tag{2.7}$$

The pair of equations (2.6) and (2.7) afford a representation alternative to (2.5). The phase diagram x, y_1 will, of course, be different in appearance from the x, y diagram.

The time T elapsing along a segment $\mathscr{C}$ of a phase path connecting two states (see Fig. 1.4) is given by

$$T = \int_{\mathscr{C}} dt = \int_{\mathscr{C}} \left(\frac{dx}{dt}\right)^{-1} \left(\frac{dx}{dt}\right) dt = \int_{\mathscr{C}} \frac{dx}{X(x, y)}. \tag{2.8}$$

Alternatively, let ds be a length element of $\mathscr{C}$. Then $ds^2 = dx^2 + dy^2$ on the path, and

$$T = \int_{\mathscr{C}} \left(\frac{ds}{dt}\right)^{-1} \left(\frac{ds}{dt}\right) dt = \int_{\mathscr{C}} \frac{ds}{(X^2 + Y^2)^{1/2}}. \tag{2.9}$$

The integrals above depend only on X and Y and the geometry of the phase path; therefore the time scale is implicit in the phase diagram.

2.2. Some population models

In the following examples systems of the type (2.1) arise naturally. Further examples from biology can be found in Pielou (1969) and Rosen (1973).

Example 2.2 *A predator–prey problem* (Volterra's model)

In a lake there are two species of fish: A, which lives on plants of which there is a plentiful supply, and B (the predator) which subsists by eating A (the prey). We shall construct a crude model for the interaction of A and B.

Let $x(t)$ be the population of A and $y(t)$ that of B. We assume that A is relatively long-lived and rapidly breeding if left alone. Then in time δt there is a population increase given by

$$ax\delta t, \quad a > 0$$

due to births and 'natural' deaths, and a 'negative increase'

$$-cxy\delta t, \quad c > 0$$

owing to A's being eaten by B (the number being eaten in this time being assumed proportional to the number of encounters between types A and B). The net population increase of A, δx, is given by

$$\delta x = ax\delta t - cxy\delta t,$$

so that in the limit $\delta t \to 0$

$$\dot{x} = ax - cxy. \tag{2.10}$$

We now assume that, in the absence of prey, the starvation rate of B predominates over the birth rate, but that the compensating growth of B is again proportional to the number of encounters with A. This gives

$$\dot{y} = -by + dxy \tag{2.11}$$

with $b > 0$, $d > 0$. Equations (2.10) and (2.11) are a pair of simultaneous nonlinear equations of the form (2.2).

We now plot the phase diagram in the x, y plane. Only the quadrant

$$x \geqslant 0, \quad y \geqslant 0$$

is of interest. The equilibrium points are where

$$X(x, y) \equiv ax - cxy = 0, \quad Y(x, y) \equiv -by + dxy = 0;$$

that is at $(0, 0)$ and $(b/d, a/c)$. The phase paths are given by $\mathrm{d}y/\mathrm{d}x = Y/X$, or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(-b+dx)y}{(a-cy)x},$$

which is a separable equation leading to

$$\int \frac{(a-cy)}{y}\mathrm{d}y = \int \frac{(-b+dx)}{x}\mathrm{d}x,$$

or

$$a \log_e y + b \log_e x - cy - dx = C \tag{2.12}$$

where C is an arbitrary constant, the parameter of the family. Writing (2.12) in the form $(a \log_e y - cy) + (b \log_e x - dx) = C$, the result of Exercise 25 shows that this is a system of closed curves centred on the equilibrium point $(b/d, a/c)$.

Figure 2.2 shows a detailed plot of the phase paths for a particular case. The direction on the paths, indicating the change of state as t increases, can be obtained by, say, finding the sign of $\dot{x}$ on $x = b/d$. In fact, the direction at a single point, even on $x = 0$ or $y = 0$, determines the directions at all points by continuity.

Since the paths are closed, the fluctuations of $x(t)$ and $y(t)$, starting from any initial population, are predicted as periodic, the maximum population of A being about a quarter of a period behind the maximum population of B. As A gets eaten, causing B to thrive, the population x of A is reduced, causing eventually a drop in that of B. The shortage of predators then leads to a resurgence of A and the cycle starts again. A sudden change in state due to external causes, such as a bad season for the plants, puts the state on to another closed curve, but no tendency to an equilibrium population, nor for the population to disappear, is predicted. If we expect such a tendency, then we must construct a different model (see Exercises 12 and 13).

In Section 2.4 we shall show how the nature of equilibrium points, and hence the general character of the solutions (periodic, tending to equilibrium, unstable, etc.), can be determined without solving the equations explicitly.

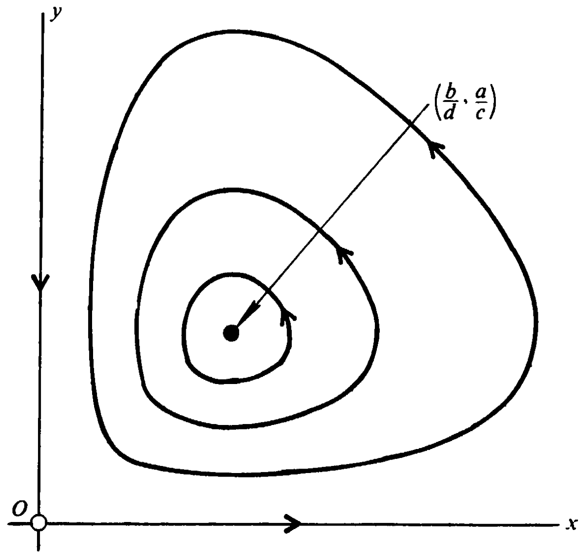


FIG. 2.2. Typical phase diagram for the predator–prey problem

Example 2.3 *A general epidemic model*

Consider the spread of a non-fatal disease in a population which is assumed to have constant size over the period of the epidemic. At time t suppose the population consists of

$x(t)$ susceptibles: those so far uninfected and therefore liable to infection;
$y(t)$ infectives: those who have the disease and are still at large;
$z(t)$ who are isolated, or who have recovered and are therefore immune.

Assume there is a steady contact rate between susceptibles and infectives and that a constant proportion of these contacts result in transmission. Then in time δt, δx of the susceptibles become infective, where

$$\delta x = -\beta xy\delta t,$$

and β is constant.

If γ is the rate at which current infectives become isolated, then

$$\delta y = \beta xy \delta t - \gamma y \delta t.$$

The number of new isolates δz is given by

$$\delta z = \gamma y \delta t.$$

Now let $\delta t \to 0$. Then the system

$$\dot{x} = -\beta xy, \quad \dot{y} = \beta xy - \gamma y, \quad \dot{z} = \gamma y, \tag{2.13}$$

with suitable initial conditions, determines the progress of the disease. Note that the result of adding the equations is

$$\frac{\mathrm{d}}{\mathrm{d}t}(x+y+z) = 0;$$

that is to say, the assumption of a constant population is built in to the model. The analysis of this problem in the phase plane is left as an exercise (Exercise 29). We shall instead look in detail at a more complicated situation:

Example 2.4 *Recurrent epidemic*

Suppose that the problem is as before, except that the stock of susceptibles $x(t)$ is being added to at a constant rate μ per unit time. This condition could be the result of fresh births in the presence of a childhood disease such as measles in the absence of vaccination. In order to balance the population in the simplest way we shall assume that deaths occur naturally and only among the immune, that is, among the $z(t)$ older people most of whom have had the disease. For a constant population the equations become

$$\dot{x} = -\beta xy + \mu, \tag{2.14}$$

$$\dot{y} = \beta xy - \gamma y, \tag{2.15}$$

$$\dot{z} = \gamma y - \mu \tag{2.16}$$

(note that $(\mathrm{d}/\mathrm{d}t)(x+y+z) = 0$: the population is steady).

Consider the variation of x and y, the active participants, represented on the x, y phase plane. We need only (2.14) and (2.15), which show an equilibrium point $(\gamma/\beta, \mu/\gamma)$.

Instead of trying to solve the equation for the phase paths we shall try to get an idea of what the phase diagram is like by forming linear approximations to the right-hand sides of (2.14), (2.15) in the neighbourhood of the equilibrium point. Near the equilibrium point we write

$$x = \gamma/\beta + \xi, \quad y = \mu/\gamma + \eta \tag{2.17}$$

(ξ, η small) so that $\dot{x} = \dot{\xi}$ and $\dot{y} = \dot{\eta}$. Retaining only the linear terms in the expansion of the right sides of (2.14), (2.15), we obtain

$$\dot{\xi} = -\frac{\beta\mu}{\gamma}\xi - \gamma\eta, \tag{2.18}$$

$$\dot{\eta} = \frac{\beta\mu}{\gamma}\xi. \tag{2.19}$$

We are said to have *linearized* (2.14) and (2.15) near the equilibrium point. Elimination of ξ gives

$$\gamma\ddot{\eta} + (\beta\mu)\dot{\eta} + (\beta\mu\gamma)\eta = 0. \tag{2.20}$$

This is the equation for the damped linear oscillator (Section 1.4), and we may compare (2.20) with eqn (1.23) (or the system (1.26)) of Chapter 1, but it is necessary to remember that it only holds as an approximation close to the equilibrium point of (2.14) and (2.15). When the 'damping' is light ($\beta\mu/\gamma^2 < 4$) the phase path is a spiral. Figure 2.3 shows some phase paths for a particular case. All starting conditions lead to the stable equilibrium point E: this point is called the 'endemic state' for the disease.

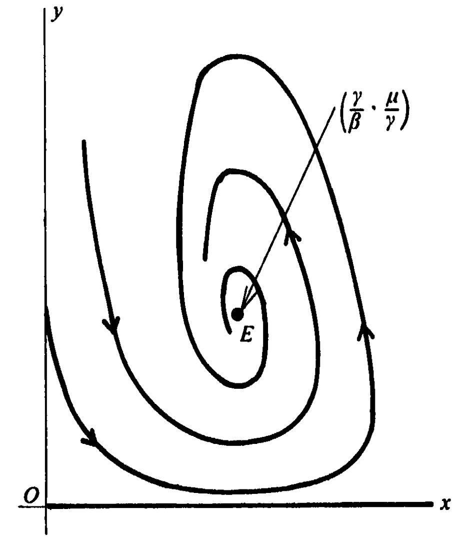


FIG. 2.3. Typical phase diagram for the recurrent epidemic

2.3. Linear approximation at equilibrium points

Approximation to a nonlinear system by linearizing it at an equilibrium point, as in the last example, is a most important and generally useful technique. If the geometrical nature of the equilibrium points can be settled in this way the general character of the phase diagram is often clear. Consider the system

$$\dot{x} = X(x, y), \quad \dot{y} = Y(x, y). \tag{2.21}$$

Suppose that the equilibrium point to be studied has been moved to the origin by a translation of axes, if necessary, so that $X(0,0) = Y(0,0) = 0$. We can therefore write, by a Taylor expansion,

$$X(x, y) = ax + by + P(x, y), \quad Y(x, y) = cx + dy + Q(x, y),$$

where $P(x, y) = O(r^2)$ and $Q(x, y) = O(r^2)$ as $r = \sqrt{(x^2+y^2)} \to 0$, and

$$a = \frac{\partial X}{\partial x}(0,0), \quad b = \frac{\partial X}{\partial y}(0,0), \quad c = \frac{\partial Y}{\partial x}(0,0), \quad d = \frac{\partial Y}{\partial y}(0,0). \tag{2.22}$$

The *linear approximation* to (2.21) in the neighbourhood of the origin is defined as the system

$$\dot{x} = ax+by, \quad \dot{y} = cx+dy. \tag{2.23}$$

We expect that the solutions of (2.23) will be geometrically similar to those of (2.21) near the origin, an expectation fulfilled in most cases (but see Exercise 7: a centre may be an exception).

2.4. The general solution of a linear system

It is known that there are non-trivial solutions of (2.23) of the form

$$x = re^{\lambda t}, \quad y = se^{\lambda t} \tag{2.24}$$

where r and s are related constants, and λ is another constant. All these constants may be complex. To find values for λ, put (2.24) into (2.23); we obtain

$$\begin{aligned} (a-\lambda)r+bs &= 0, \\ cr+(d-\lambda)s &= 0. \end{aligned} \tag{2.25}$$

Non-trivial solutions exist if and only if

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0,$$

or

$$\lambda^2-(a+d)\lambda+(ad-bc) = 0, \tag{2.26}$$

which is called the characteristic equation. When this equation has two different roots, λ_1, λ_2, two linearly independent families of solutions are generated by (2.24), corresponding to $\lambda = \lambda_1$ and $\lambda = \lambda_2$ respectively. We shall consider only this case here.

By substituting first λ_1 and then λ_2 into the equations (2.25)we find the permissible corresponding non-zero pairs r, s. Since the equations are homogeneous, if r, s is one solution, all solutions are given by Cr, Cs, where C is any constant. Let $r = r_1$, $s = s_1$ be any one fixed solution corresponding to $\lambda = \lambda_1$, and similarly for λ_2. Then, because of the linearity of the system (2.23) the general solution is

$$\begin{aligned} x(t) &= C_1 r_1 e^{\lambda_1 t}+C_2 r_2 e^{\lambda_2 t}, \\ y(t) &= C_1 s_1 e^{\lambda_1 t}+C_2 s_2 e^{\lambda_2 t}, \end{aligned} \tag{2.27}$$

where C_1 and C_2 are arbitrary constants. Note that when λ_1, λ_2 are complex, r_1, s_1, r_2, s_2 are complex; therefore, if the solutions x, y are to be real, we must allow C_1, C_2 to be complex in general.

Example 2.5 *Find the general solution of the system*

$$\dot{x} = x+y, \quad \dot{y} = -5x-3y.$$

The characteristic equation (2.26) is

$$\lambda^2+2\lambda+2 = 0$$

so that

$$\lambda_1 = -1+i, \quad \lambda_2 = -1-i.$$

Equations (2.25) are, for λ_1

$$(2-i)r_1+s_1 = 0, \quad -5r_1-(2+i)s_1 = 0.$$

(Note that these equations are equivalent.) A particular solution is

$$r_1 = 1, \quad s_1 = -2+i.$$

Since $\lambda_2 = \bar{\lambda}_1$ a solution of the equations corresponding to λ_2 is $r_2 = \bar{r}_1, s_2 = \bar{s}_1$ or

$$r_2 = 1, \quad s_2 = -2-i.$$

In the form (2.27) the general solution is therefore

$$\begin{aligned} x(t) &= C_1 e^{(-1+i)t}+C_2 e^{(-1-i)t}, \\ y(t) &= C_1(-2+i)e^{(-1+i)t}+C_2(-2-i)e^{(-1-i)t}. \end{aligned}$$

If we choose $C_2 = \bar{C}_1$ with C_1 arbitrary, all the real solutions are obtained. Put $C_1 = \frac{1}{2}c_1+\frac{1}{2}ic_2$ where c_1, c_2 are arbitrary. Then this simplifies to

$$\begin{aligned} x(t) &= e^{-t}(c_1 \cos t-c_2 \sin t), \\ y(t) &= -e^{-t}\{2c_1+c_2)\cos t+(c_1-2c_2)\sin t\}. \end{aligned}$$

2.5. Classifying equilibrium points

We are interested not so much in the solutions as in the corresponding pattern of phase paths. Write the linearized system (2.23) in the form

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{2.28}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}. \tag{2.29}$$

Look for two linearly independent solutions of the form

$$\mathbf{x} = \mathbf{s}\,\mathrm{e}^{\lambda t}, \tag{2.30}$$

where

$$\mathbf{s} = \begin{pmatrix} r \\ s \end{pmatrix} \neq \mathbf{0}. \tag{2.31}$$

Then $\dot{\mathbf{x}} = \lambda \mathbf{s}\,\mathrm{e}^{\lambda t}$, and (2.28) gives

$$(A - \lambda I)\mathbf{s} = \mathbf{0} \tag{2.32}$$

where I is the identity matrix. Non-zero $\mathbf{s}$ satisfies (2.32) if and only if

$$\det(A - \lambda I) = 0,$$

or

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0. \tag{2.33}$$

Eqn (2.33) is (2.26) again, and (2.32) is eqn (2.25) again.

Therefore λ_1, λ_2, the solutions of (2.33), are the *eigenvalues* of A; and $\mathbf{s} = \mathbf{s}_1$ and $\mathbf{s} = \mathbf{s}_2$, the solutions of (2.32), are a pair of *eigenvectors* corresponding respectively to λ_1 and λ_2. Provided $\lambda_1 \neq \lambda_2$, the general solution of (2.28) is

$$\mathbf{x} = C_1 \mathbf{s}_1 \mathrm{e}^{\lambda_1 t} + C_2 \mathbf{s}_2 \mathrm{e}^{\lambda_2 t}, \tag{2.34}$$

which is eqn (2.27).

Now reconsider the problem of finding the phase paths. In general their equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{cx + dy}{ax + by}. \tag{2.35}$$

This can always be solved but the reader can soon convince himself that, except in special cases, the results are too complicated to give a simple discussion of the geometrical nature of the solutions.

We can obtain linear transformations which reduce eqn (2.35) to manageable form. A nonsingular linear transformation from x, y to u, v:

$$\mathbf{u} = S\mathbf{x}, \quad \mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} \tag{2.36}$$

does not change the nature of the singular points; for example spirals in the x, y plane remain spirals in the u, v plane, and so on. Writing

$$\mathbf{x} = S^{-1}\mathbf{u}, \quad \dot{\mathbf{x}} = S^{-1}\dot{\mathbf{u}},$$

(2.28) becomes $S^{-1}\dot{\mathbf{u}} = AS^{-1}\mathbf{u}$, or

$$\dot{\mathbf{u}} = SAS^{-1}\mathbf{u} = B\mathbf{u}, \tag{2.37}$$

say. It is known from algebraic theory (Ayres 1962) that S can be chosen so that B takes one of several *canonical forms* which in general are simpler than A, the particular form depending on the nature of the eigenvalues of A. In the new coordinates u, v, the equations are therefore simpler although the topological character of the transformed equilibrium point at the origin is not affected. We do not need to calculate the S for this purpose: we only need the fact that the relevant S exists.

The principal cases, neglecting certain degenerate ones, are the following:

(i) λ_1, λ_2 *real, different, and non-zero*

We can choose S so that

$$\dot{u} = \lambda_1 u, \quad \dot{v} = \lambda_2 v,$$

where also $\lambda_2 > \lambda_1$. Then the equation for the phase paths is

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{\lambda_2}{\lambda_1}\frac{v}{u}$$

whose solutions are given by

$$v = C|u|^{\lambda_2/\lambda_1} \tag{2.38}$$

where C is arbitrary. $u = 0$ is also a phase path.

Since we arranged the eigenvalues so that $\lambda_2 > \lambda_1$ only two possible patterns emerge, depending on whether λ_1 and λ_2 have the same or opposite signs (Fig. 2.4). In the former case, of a *node*, its stability and hence the direction of the 'arrows' are determined from (2.34): if λ_1, $\lambda_2 > 0$, then all solutions are exponentially increasing and the node is

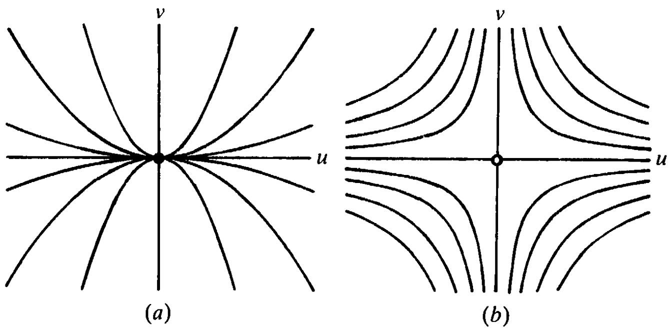


FIG. 2.4. λ_1, λ_2 real, non-zero, $\lambda_2 > \lambda_1$: (*a*) λ_1, λ_2 have the same sign (node); (*b*) λ_1, λ_2 have opposite signs (saddle)

unstable, all arrows pointing away from the origin. If $\lambda_1, \lambda_2 < 0$, then the node is *stable*.

The detailed analysis of a particular case proceeds as in the following example.

Example 2.6 *Sketch the phase paths of the system*

$$\dot{x} = x - 2y, \quad \dot{y} = 3x - 4y. \tag{2.39}$$

The eigenvalues of the system are given by

$$0 = \begin{vmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{vmatrix} \equiv \lambda^2 + 3\lambda + 2$$

so that $\lambda_1 = -2$, $\lambda_2 = -1$. Since they are real and have the same sign a node is implied in the u, v plane.

The paths in the x, y plane must be obtainable from Fig. 2.4(*a*) by a linear transformation from (u, v) to (x, y). There may be changes of scale, shears, rotations, and reflections, but straight lines remain straight and parallel lines remain parallel. The stability property is unaffected, and since $\lambda_1, \lambda_2 < 0$, the node is stable.

To obtain a more precise idea of the shape of the node, we look for those solutions of (2.39) which correspond to the straight lines $u = 0$, $v = 0$ in Fig. 2.4(*a*). Since the transformation preserves them as straight lines they take the form $y = kx$, and substitution into (2.35) gives

$$k = \frac{3-4k}{1-2k}$$

from which $k = 1$ and $3/2$. (These directions are those of the eigenvectors of A, since $\dot{\mathbf{x}} = A\mathbf{x} = \lambda\mathbf{x}$ along the eigenvectors.)

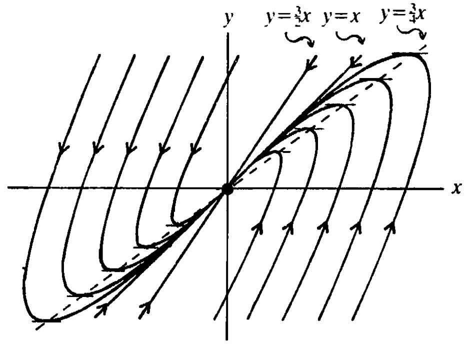


FIG. 2.5.

The second feature of Fig. 2.4(*a*) preserved by the linear transformation is that as $t \to +\infty$ (that is, approaching the origin) *all the curves become tangential to one of the solutions just found, and as* $t \to -\infty$ *they become parallel to the other*. The associated correspondence between Figs. 2.4(*a*) and 2.5 can be established by noting that, from (2.39), the paths cross the line $y = \frac{3}{4}x$ horizontally. The only possibility (Fig. 2.5) is that all paths are tangential to $y = x$ at the origin.

The *saddle* (Fig. 2.4(*b*)), for which λ_1, λ_2 have opposite signs, is simpler to construct, as the following example illustrates.

Example 2.7 *Sketch the phase paths of the system*

$$\dot{x} = 3x + 2y, \quad \dot{y} = -2x - 2y. \tag{2.40}$$

The eigenvalues of the system are given by

$$0 = \begin{vmatrix} 3-\lambda & 2 \\ -2 & -2-\lambda \end{vmatrix} \equiv \lambda^2 - \lambda - 2$$

so that $\lambda_1 = -1$, $\lambda_2 = 2$. Since these are real and of opposite sign the equilibrium point at the origin is a saddle.

In Fig. 2.4(*a*), $u = 0$ and $v = 0$ transform into straight-line paths in the (x, y)-plane. To find them, substitute $y = kx$ into the present version of (2.35) to give

$$k = \frac{-2-2k}{3+2k}$$

with solutions $k = -1/2$ *and* -2 (Fig. 2.6). (As in the last Example, these are the directions of the eigenvectors of A.)

The directions of the arrows are settled by putting, for example, $y = 0$, $x > 0$ into (2.40), showing that $\dot{y} < 0$ on the positive x-axis.

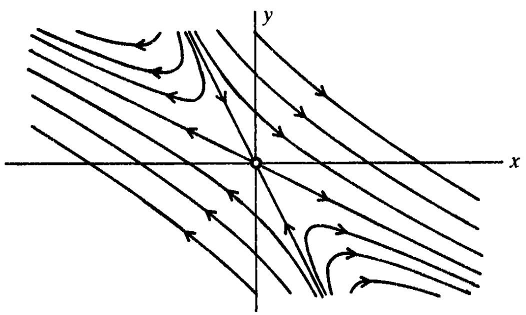


FIG. 2.6.

(ii) *Complex eigenvalues*: $\lambda_1 = \bar{\lambda}_2 = \alpha + i\beta$, $\beta \neq 0$

The transformation matrix S of (2.36) can be chosen so that the reduced eqn (2.37) takes the form

$$\dot{u} = \alpha u - \beta v, \quad \dot{v} = \beta u + \alpha v. \tag{2.41}$$

Put $z = u + iv$: then, by (2.41),

$$\dot{z} = (\alpha + i\beta)z,$$

and by writing $z = r(t)\,e^{i\theta(t)}$ where $r = |z|$ we obtain the polar equations

$$\dot{r} = \alpha r, \quad \dot{\theta} = \beta.$$

The solutions of these equations are $r = r_0 e^{\alpha t}$ and $\theta = \beta t + \theta_0$ where $r(0) = r_0$ and $\theta(0) = \theta_0$. The origin is therefore a *stable spiral* if $\alpha < 0$; an *unstable spiral* if $\alpha > 0$; and a *centre* if $\alpha = 0$.

Example 2.8 *Sketch the phase paths of the system*

$$\dot{x} = -x - 5y, \quad \dot{y} = x + 3y. \tag{2.42}$$

The eigenvalues are given by

$$0 = \begin{vmatrix} -1-\lambda & -5 \\ 1 & 3-\lambda \end{vmatrix} \equiv \lambda^2 - 2\lambda + 2$$

so that $\lambda_1, \lambda_2 = 1 \pm i$. The phase diagram is therefore an unstable spiral. The spirals 'unwind' with increasing time, but may be directed *clockwise* or *anticlockwise* in general; to determine the direction in this case, put, for example $y = 0, x > 0$ in (2.42). The second equation implies that $\dot{y} > 0$ on the positive x-axis, so that the spirals unwind in the anticlockwise direction.

(iii) *Degenerate cases*

These occur when there is a single repeated eigenvalue, and when an eigenvalue is zero. These cases will not be discussed in detail.

2.6. Constructing a phase diagram

Suppose that the given system

$$\dot{x} = X(x, y), \quad \dot{y} = Y(x, y) \tag{2.43}$$

has an equilibrium point at (x_0, y_0):

$$X(x_0, y_0) = 0, \quad Y(x_0, y_0) = 0. \tag{2.44}$$

As we remarked in Section 2.3, the pattern of phase paths close to (x_0, y_0) may be decided by linearizing the equations at this point by retaining only linear terms of the Taylor series for X and Y there. It is simplest to use the method leading up to eqn (2.22) to obtain the coefficients. If local coordinates are defined by

$$\xi = x - x_0, \quad \eta = y - y_0,$$

then, approximately,

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{2.45}$$

where the coefficients are given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} X_x(x_0, y_0) & X_y(x_0, y_0) \\ Y_x(x_0, y_0) & Y_y(x_0, y_0) \end{pmatrix}. \tag{2.46}$$

The equilibrium point is then classified using the methods of the last section. This is done for each equilibrium point in turn, and it is then possible to make a fair guess at the complete pattern of the phase paths, as in the following example.

Example 2.9 *Sketch the phase diagram for the nonlinear system*

$$\dot{x} = x - y, \quad \dot{y} = 1 - xy. \tag{2.47}$$

The equilibrium points are at $(-1, -1)$ and $(1, 1)$. The matrix for linearization, to be evaluated at each equilibrium point in turn, is

$$\begin{pmatrix} X_x & X_y \\ Y_x & Y_y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -y & -x \end{pmatrix}. \tag{2.48}$$

At $(-1, -1)$ eqns (2.46) and (2.48) give

$$\begin{pmatrix}\dot{\xi}\\ \dot{\eta}\end{pmatrix} = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \tag{2.49}$$

where $\xi = x+1$, $\eta = y+1$. The eigenvalues of the coefficient matrix are found to be $\lambda_1, \lambda_2 = 1 \pm \mathrm{i}$ implying an unstable spiral. To obtain the direction of rotation, it is sufficient to use the linear equations (2.49) (or the original equations may be used): putting $\eta = 0$, $\xi > 0$ we find $\dot{\eta} = \xi > 0$, indicating that the rotation is *anticlockwise*.

At (1, 1), we find from (2.46) and (2.48) that

$$\begin{pmatrix}\dot{\xi}\\ \dot{\eta}\end{pmatrix} = \begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \tag{2.50}$$

where $\xi = x-1$, $\eta = y-1$. The eigenvalues are given by $\lambda_1, \lambda_2 = \pm\sqrt{2}$, which implies a saddle. The directions of the 'straight-line' paths from the saddle (which become curved separatrices when away from the equilibrium point), are resolved by the technique of Example 2.7: from (2.50)

$$\frac{\dot{\eta}}{\dot{\xi}} = \frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \frac{-\xi-\eta}{\xi-\eta}. \tag{2.51}$$

We know that two solutions of this equation have the form $\eta = k\xi$ for some values of k. By substituting into (2.51) we obtain $k^2-2k-1 = 0$, so that $k = 1\pm\sqrt{2}$.

Finally the phase diagram is put together as in Fig. 2.7, where the phase paths in the neighbourhoods of the equilibrium points are now known. The process can be assisted by sketching in the direction fields on the lines $x = 0$, $x = 1$, etc., and also the curve $1-xy = 0$ on which the phase paths have zero slopes and the line $y = x$ on which the paths have infinite slopes.

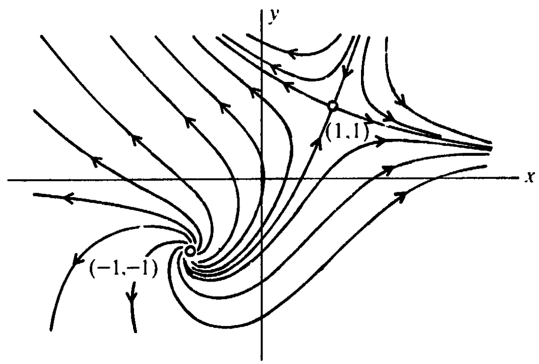


FIG. 2.7.

2.7. Transitions between types of equilibrium point

The types of equilibrium point discussed above can be classified in terms of two parameters of the system as follows. The determinantal equation giving the eigenvalues, (2.26), can be written as

$$\lambda^2 - p\lambda + q = 0$$

where

$$p = a+d, \quad q = ad-bc.$$

Hence

$$\lambda_1, \lambda_2 = \tfrac{1}{2}p \pm \tfrac{1}{2}\sqrt{\Delta}$$

where

$$\Delta = p^2 - 4q.$$

The various cases are displayed in Fig. 2.8. The degenerate cases occur on $\Delta = 0$ and $q = 0$. Note that the centre constitutes a transition between stable and unstable spirals: the existence of a centre depends on particular exact relations between the coefficients of the system and is therefore a rather fragile feature. This has the consequence that if the linear approximation of a nonlinear system predicts a centre it cannot be reliably concluded that the equilibrium point is truly a centre—it could be a stable or unstable spiral (see, for example, Exercise 7). Further reference to this matter in connection with bifurcations is made in Chapter 12.

If there exists a neighbourhood of an equilibrium point such that phase paths which start from all points in the neighbourhood

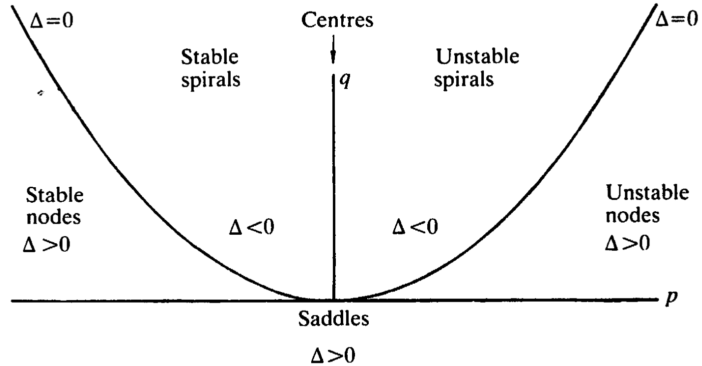


FIG. 2.8. General classification for the linear system: $\dot{x} = ax + by$, $\dot{y} = cx + dy$ with $p = a+d$, $q = ad-bc$, and $\Delta = p^2-4q$

ultimately approach the equilibrium point, the equilibrium point is known as an *attractor*. Both the stable node and stable spiral are examples of attractors. An attractor with all path directions reversed is known as a *repellor*. The unstable node and unstable spiral are both repellors (but the saddle is not).

If the eigenvalues of the linearized equation have non-zero real parts at the equilibrium point then the point is said to be *hyperbolic*. As we shall see in Chapter 10, for hyperbolic points the phase diagrams of the nonlinear equation and its linearization are locally qualitatively the same. Spirals, nodes, and saddles are hyperbolic but the centre is not.

Exercises

1. Sketch phase diagrams for the following linear systems:
(i) $\dot{x} = x-5y,\quad \dot{y} = x-y;$
(ii) $\dot{x} = x+y,\quad \dot{y} = x-2y;$
(iii) $\dot{x} = -4x+2y,\quad \dot{y} = 3x-2y;$
(iv) $\dot{x} = x+2y,\quad \dot{y} = 2x+2y;$
(v) $\dot{x} = 4x-2y,\quad \dot{y} = 3x-y;$
(vi) $\dot{x} = 2x+y,\quad \dot{y} = -x+y.$

2. The following systems either generate a single eigenvalue, or a zero eigenvalue, or in other ways vary from the types illustrated in Section 2.4. Sketch their phase diagrams.
(i) $\dot{x} = 3x-y,\quad \dot{y} = x+y;$
(ii) $\dot{x} = x-y,\quad \dot{y} = 2x-2y;$
(iii) $\dot{x} = x,\quad \dot{y} = 2x-3y;$
(iv) $\dot{x} = x,\quad \dot{y} = x+3y;$
(v) $\dot{x} = -y,\quad \dot{y} = 2x-4y;$
(vi) $\dot{x} = x,\quad \dot{y} = y;$
(vii) $\dot{x} = 0,\quad \dot{y} = x.$

3. Locate and classify the equilibrium points of the following systems. Sketch the phase diagrams: it will often be helpful to obtain isoclines and path directions at other points in the field.
(i) $\dot{x} = x-y,\quad \dot{y} = x+y-2xy;$
(ii) $\dot{x} = ye^y,\quad \dot{y} = 1-x^2;$
(iii) $\dot{x} = 1-xy,\quad \dot{y} = (x-1)y;$
(iv) $\dot{x} = (1+x-2y)x,\quad \dot{y} = (x-1)y;$
(v) $\dot{x} = x-y,\quad \dot{y} = x^2-1;$
(vi) $\dot{x} = -6y+2xy-8,\quad \dot{y} = y^2-x^2;$
(vii) $\dot{x} = 4-4x^2-y^2,\quad \dot{y} = 3xy;$
(viii) $\dot{x} = -y\sqrt{(1-x^2)},\quad \dot{y} = x\sqrt{(1-x^2)}$ for $|x| \leq 1;$
(ix) $\dot{x} = \sin y,\quad \dot{y} = -\sin x.$

4. Construct phase diagrams for the following differential equations, using the phase plane in which $y = \dot{x}$.
(i) $\ddot{x}+x-x^3 = 0;$ (ii) $\ddot{x}+x+x^3 = 0;$
(iii) $\ddot{x}+\dot{x}+x-x^3 = 0;$ (iv) $\ddot{x}+\dot{x}+x+x^3 = 0;$
(v) $\ddot{x} = \sin x\,(2\cos x-1).$

5. Confirm that the system $\dot{x} = x-5y$, $\dot{y} = x-y$ consists of a centre. By substitution into the equation for the paths or otherwise show that the family of ellipses given by

$$x^2-2xy+5y^2 = \textit{constant}$$

describes the paths. Show that the axes are inclined at about 13.3° (the major axis) and −76.7° (the minor axis) to the x direction, and that the ratio of major to minor axis length is about 8.33.

6. The family of curves which are orthogonal to the family described by the equation $(dy/dx) = f(x, y)$ is given by the solution of $(dy/dx) = -[1/f(x, y)]$. (These are called the orthogonal trajectories of the first family.) Prove that the family which is orthogonal to a centre which is associated with a linear system is a node.

7. Show that the origin is a spiral point of the system $\dot{x} = -y-x\sqrt{(x^2+y^2)}$, $\dot{y} = x-y\sqrt{(x^2+y^2)}$, but a centre for its linear approximation.

8. Show that the systems $\dot{x} = y$, $\dot{y} = -x-y^2$, and $\dot{x} = x+y_1$, $\dot{y}_1 = -2x-y_1-(x+y_1)^2$, both represent the equation $\ddot{x}+\dot{x}^2+x = 0$ in different phase planes.

9. Use eqn (2.9) in the form $\delta s \simeq \delta t\sqrt{(X^2+Y^2)}$ to mark off approximately equal time steps on some of the phase paths of $\dot{x} = xy$, $\dot{y} = xy-y^2$.

10. Obtain approximations to the phase paths described by (2.12) in the neighbourhood of the equilibrium point $x = b/d$, $y = a/c$. (Write $x = b/d+\xi$, $y = a/c+\eta$, and expand the logarithms to second-order terms in ξ and η.)

11. For the system $\dot{x} = ax+by$, $\dot{y} = cx+dy$, where $ad-bc = 0$, show that all points on the line $cx+dy = 0$ are equilibrium points.

12. The interaction between two species is governed by the deterministic model $\dot{H} = (a_1-b_1H-c_1P)H$, $\dot{P} = (-a_2+c_2H)P$, where H is the population of the host (or prey), and P is that of the parasite (or predator), all constants being positive. (Compare Example 2.2: the term $-b_1H^2$ represents interference with the host population when it gets too large.) Find the equilibrium states for the populations, and find how they vary with time from various initial populations.

13. With the same terminology as in Exercise 12, analyse the system $\dot{H} = (a_1-b_1H-c_1P)H$, $\dot{P} = (a_2-b_2P+c_2H)P$, all the constants being positive. (In this model the parasite can survive on alternative food supplies, although the

prevalence of the host encourages growth in the population.) Find the equilibrium states. Confirm that the parasite population can persist even if the host dies out.

14. Consider the host–parasite population model $\dot{H} = (a_1 - c_1 P)H$, $\dot{P} = (a_2 - c_2 P/H)P$, where the constants are positive. Analyse the system in the H, P plane.

15. In the population model $\dot{F} = -\alpha F + \beta\mu(M)F$, $\dot{M} = -\alpha M + \gamma\mu(M)F$, where $\alpha > 0, \beta > 0, \gamma > 0$, F and M are the female and male populations. In both cases the death-rates are α. The birth-rate is governed by the coefficient $\mu(M) = 1 - e^{-kM}$, $k > 0$, so that for large M the birth-rate of females is βF and that for males is γF, the rates being unequal in general. Show that if $\beta > \alpha$ then there are two equilibrium states, at $(0, 0)$ and at $([-\beta/\gamma k]\log[(\beta-\alpha)/\beta], [-1/k]\log[(\beta-\alpha)/\beta])$.

Show that the origin is stable and that the other equilibrium point is a saddle-point, according to their linear approximations. Verify that $M = \gamma F/\beta$ is a particular solution. Sketch the phase diagram and discuss the stability of the populations.

16. A rumour spreads through a closed population of constant size $N+1$. At time t the total population can be classified into three categories:

x persons who are ignorant of the rumour;

y persons who are actively spreading the rumour;

z persons who have heard the rumour but have stopped spreading it: if two persons who are spreading the rumour meet then they stop spreading it.

The contact rate between any two categories is constant, μ.

Show that the equations

$$\dot{x} = -\mu xy, \quad \dot{y} = \mu[xy - y(y-1) - yz]$$

give a deterministic model of the problem. Find the equations of the phase paths and sketch the phase diagram.

Show that, when initially $y = 1$ and $x = N$, the number of people who ultimately never hear the rumour is x_1, where

$$2N + 1 - 2x_1 + N\log(x_1/N) = 0.$$

17. The one-dimensional steady flow of a gas with viscosity and heat conduction satisfies the equations

$$\frac{\mu_0}{\rho c_1}\frac{dv}{dx} = \sqrt{(2v)}[2v - \sqrt{(2v)} + \theta]$$

$$\frac{k}{gR\rho c_1}\frac{d\theta}{dx} = \sqrt{(2v)}\left[\frac{\theta}{\gamma-1} - v + \sqrt{(2v)} - c\right]$$

where $v = u^2/2c_1^2$, $c = c_2^2/c_1^2$ and $\theta = gRT/c_1^2 = p/\rho c_1^2$. In this notation, x is measured in the direction of flow, u is the velocity, T is the temperature, ρ is the density, p the pressure, R the gas constant, k the coefficient of thermal conductivity, μ_0 the coefficient of viscosity, γ the ratio of the specific heats, and c_1, c_2 are arbitrary constants. Find the equilibrium states of the system.

18. A particle moves under a central attractive force γ/r^α per unit mass, where r, θ are the polar coordinates of the particle in its plane of motion. Show that

$$\frac{d^2u}{d\theta^2} + u = \frac{\gamma}{h^2}u^{\alpha-2},$$

where $u = r^{-1}$, h is the angular momentum about the origin per unit mass of the particle, and γ is a constant. Find the non-trivial equilibrium point in the $u, du/d\theta$ plane and classify it according to its linear approximation. What can you say about the stability of the circular orbit under this central force?

19. The relativistic equation for the central orbit of a planet is

$$\frac{d^2u}{d\theta^2} + u = \alpha + \varepsilon u^2$$

where $u = 1/r$, and r, θ are the polar coordinates of the planet in the plane of its motion. The term εu is the 'Einstein correction', and α and ε are positive constants, with ε very small. Find the equilibrium point which corresponds to a perturbation of the Newtonian orbit. Show that the equilibrium point is a centre in the $u, du/d\theta$ plane according to the linear approximation. Confirm this by using the method of Section 1.3.

20. A top is set spinning at an axial rate n about its pivotal point, which is fixed in space. The equations for its motion, in terms of the angles θ and μ are (see Fig. 2.9)

$$A\ddot{\theta} - A(\Omega + \dot{\mu})^2\sin\theta\cos\theta + Cn(\Omega + \dot{\mu})\sin\theta - Mgh\sin\theta = 0,$$

$$A\dot{\theta}^2 + A(\Omega + \dot{\mu})^2\sin^2\theta + 2Mgh\cos\theta = E;$$

where (A, A, C) are the principal moments of inertia about O, M is the mass of the top, h is the distance between the mass-centre and the pivot, and E is a constant. Show that one equilibrium state is given by $\theta = \alpha$, provided E and μ satisfy

$$A\Omega^2\cos\alpha - Cn\Omega + Mgh = 0, \quad A\Omega^2\sin^2\alpha + 2Mgh\cos\alpha = E.$$

Discuss the motion of the top in this state.

Suppose that $E = 2Mgh$, so that $\theta = 0$ is an equilibrium state. Show that, close to this state, θ satisfies

$$A\ddot{\theta} + (Cn\Omega - A\Omega^2 - Mgh)\theta = 0.$$

For what condition on Ω is the motion stable?

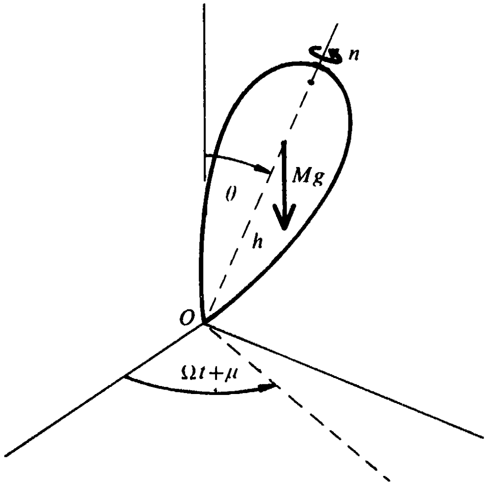


FIG. 2.9.

21. Figure 2.10 shows an 'up-and-over' garage door AB, of height $2h$, mass m, and moment of inertia I about G. The force P is applied by springs and weights, and it can be designed to have suitable characteristics. The unknown reactions S and R, and the frictional force F are as shown.

(i) If the runner B is smooth, show that it is possible to design the door so that it is in equilibrium at every angle θ, when P is a suitable constant. Sketch the phase diagram.

(ii) Suppose now that a frictional force F, with $|F/R| = \mu$, is introduced at B. What values should $P(\theta)$ have at $\theta = 0$ and $\theta = \frac{1}{2}\pi$ in order that the open and closed positions should be in equilibrium?

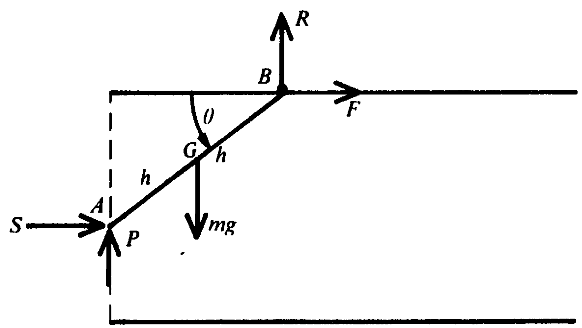


FIG. 2.10. Garage door

(iii) A larger design problem can be attempted. Construct a function $P(\theta)$ which will ensure that the open and closed positions are stable nodes (why nodes?) according to the linear approximations. Do you expect the door to have a further equilibrium position? (This problem does not, of course, have a unique solution.)

22. A disc of radius a is freely pivoted at its centre A so that it can turn in a vertical plane. An elastic string, of natural length $2a$ and stiffness λ connects a point B on the circumference of the disc to a fixed point O, distance $2a$ above A. Show that θ satisfies

$$I\ddot{\theta} = -Ta\sin\phi, \quad T = \lambda a[(5-4\cos\theta)^{1/2}-2],$$

where I is the moment of inertia of the disc about A, $O\hat{A}B = \theta$ and $A\hat{B}O = \phi$. Find the equilibrium states of the disc.

23. A man rows a boat across a river of width a occupying the strip $0 \leqslant x \leqslant a$ in the x, y plane, always rowing towards a fixed point on one bank, say $(0, 0)$. He rows at a constant speed u relative to the water, and the river flows at a constant speed v. Show that

$$\dot{x} = -ux/\sqrt{(x^2+y^2)}, \quad \dot{y} = v - uy/\sqrt{(x^2+y^2)},$$

where (x, y) are the coordinates of the boat. Show that the phase paths are given by $y+\sqrt{(x^2+y^2)} = Ax^{1-\alpha}$, where $\alpha = v/u$. Sketch the phase diagram for $\alpha < 1$ and interpret it. What kind of point is the origin? What happens to the boat if $\alpha > 1$?

24. In a simple model of a national economy, $\dot{I} = I - \alpha C$, $\dot{C} = \beta(I - C - G)$, where I is the national income, C is the rate of consumer spending, and G the rate of government expenditure; the constants α and β satisfy $1 < \alpha < \infty$, $1 \leqslant \beta < \infty$. Show that if the rate of government expenditure G_0 is constant there is an equilibrium state. Classify the equilibrium state and show that the economy oscillates when $\beta = 1$.

Consider the situation when government expenditure is related to the national income by the rule $G = G_0 + kI$, where $k > 0$. Show that there is no equilibrium state if $k \geqslant (\alpha - 1)/\alpha$. How does the economy then behave?

Discuss an economy in which $G = G_0 + kI^2$, and show that there are two equilibrium states.

25. Let $f(x)$ and $g(y)$ have local minima at $x = a$ and $y = b$ respectively. Show that $f(x)+g(y)$ has a minimum at (a, b). Deduce that there exists a neighbourhood of (a, b) in which all solutions of the family of equations

$$f(x)+g(y) = \text{constant}$$

represent closed curves surrounding (a, b).

Show that $(0,0)$ is a centre for the system $\dot{x} = y^5$, $\dot{y} = -x^3$, and that all paths are closed curves.

26. For the predator–prey problem in Section 2.2, show by using Exercise 25 that all solutions in $H > 0$, $P > 0$ are periodic.

27. Show that the phase paths of the Hamiltonian system $\dot{x} = -\partial H/\partial y$, $\dot{y} = \partial H/\partial x$ are given by $H(x, y) = $ constant. (This is a generalization of the conservative system of Section 1.3.) Equilibrium points occur at the stationary points of $H(x, y)$. If (x_0, y_0) is an equilibrium point, show that (x_0, y_0) is stable according to the linear approximation if $H(x, y)$ has a maximum or a minimum at the point. (Assume that all the second derivatives of H are non-zero at x_0, y_0.)

28. The equilibrium points of the nonlinear parameter-dependent system $\dot{x} = y$, $\dot{y} = -f(x, y, \lambda)$ lie on the curve $f(x, 0, \lambda) = 0$ in the x, λ plane. Show that an equilibrium point x_1, λ_1 is stable and that all neighbouring solutions tend to this point (according to the linear approximation) if $f_x(x_1, 0, \lambda_1) < 0$ and $f_y(x_1, 0, \lambda_1) < 0$.

Interpret stability in terms of the surface $f(x, y, \lambda) = 0$ in the x, y, λ space and the regions in which $f(x, y, \lambda) > 0$.

29. Find the equations for the phase paths for the general epidemic described (Section 2.2) by the system

$$\dot{x} = -\beta xy, \quad \dot{y} = \beta xy - \gamma y, \quad \dot{z} = \gamma y.$$

Sketch the phase diagram in the x, y plane. Confirm that the number of infectives reaches its maximum when $x = \gamma/\beta$.

30. Two species x and y are competing for a common limited food supply. Their growth equations are given by

$$\dot{x} = xP(x, y), \quad \dot{y} = yQ(x, y).$$

Express in mathematical terms the following constraints on the populations:
(i) if either species increases, the growth rate of the other goes down;
(ii) if one species is absent, the other shows limited growth characteristics;
(iii) if either population is very large, then neither species can multiply.

Assuming that P and Q have the necessary properties, sketch a first-quadrant phase diagram in which also $P = 0$ and $Q = 0$ do not intersect, and $P = 0$ is 'below' $Q = 0$. Indicate the signs of $\dot{x}$ and $\dot{y}$ in the regions into which the quadrant is now divided. Deduce the character of the phase paths by qualitative arguments.

31. Sketch the phase diagram for the competing species x and y for which

$$\dot{x} = (1 - x^2 - y^2)x, \quad \dot{y} = (1.1 - x - y)y.$$

32. A space satellite is in free flight on the line joining, and between, a planet

(mass m_1) and its moon (mass m_2), which are at a fixed distance a apart. Show that

$$-\frac{\gamma m_1}{x^2} + \frac{\gamma m_2}{(a-x)^2} = \ddot{x}$$

where x is the distance of the satellite from the planet and γ is the gravitational constant. Show that the equilibrium point is unstable according to the linear approximation.

33. The system

$$\dot{V}_1 = -\sigma V_1 + f(E - V_2), \quad \dot{V}_2 = -\sigma V_2 + f(E - V_1), \quad \sigma > 0, \quad E > 0$$

represents (Andronov and Chaikin 1949) a model of a triggered sweeping circuit for an oscilloscope. The conditions on $f(u)$ are: $f(u)$ continuous, $-\infty < u < \infty$, $f(-u) = -f(u)$, $f(u)$ tends to a limit as $u \to \infty$, and $f'(u)$ is monotonic decreasing (see Figure 3.19).

Show by a geometrical argument that there is always at least one equilibrium point, (V_0, V_0) say, and that when $f'(E - V_0) < \sigma$ it is the only one; and deduce by taking the linear approximation that it is a stable node. (Note that $f'(E - v) = -df(E - v)/dv$.)

Show that when $f'(E - V_0) > \sigma$ there are two others, at $(V', (1/\sigma)f(E - V'))$ and $((1/\sigma)f(E - V'), V')$ respectively for some V'. Show that these are stable nodes, and that the one at (V_0, V_0) is a saddle point.

34. The response of a certain biological oscillator, (x, y), $x \geqslant 0$, $y \geqslant 0$, to a stimulus measured by a constant b satisfies the system

$$\dot{x} = x - ay + b, \quad \dot{y} = x - cy \ \text{ for } x \geqslant 0, \quad y \geqslant 0;$$

$$\dot{y} = -cy \ \text{ for } x = 0.$$

Show that when $c < 1$ and $4a > (1 + c)^2$ then there exists a limit cycle, part of which lies on the y axis, whose period is independent of b. Sketch the corresponding solution function. (Varying the amount of stimulus translates the response in time without affecting the period.)

35. Figure 2.11 represents a circuit for activating an electric arc A which has the voltage-current characteristic shown. Show that $L\dot{I} = V - V_a(I)$, $RC\dot{V} = -RI - V + E$. By forming the linear approximating equations near the equilibrium points find the conditions on E, L, C, R, and V'_a for stable working.

36. The equation for the current in the circuit of Fig. 2.12(a) is

$$LC\ddot{x} + RC\dot{x} + x = I.$$

Neglect the grid current, and assume that I depends only on the relative grid potential e_g: $I = I_s$ (saturation current) for $e_g > 0$ and $I = 0$ for $e_g < 0$. Assume also that $M > 0$, so that $e_g \gtrless 0$ according as $\dot{x} \gtrless 0$. Find the nature of the phase

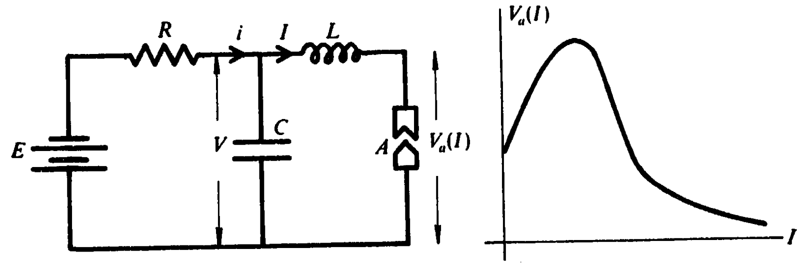


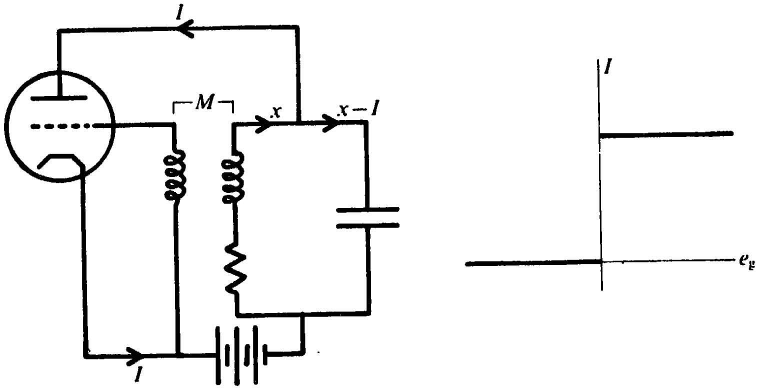
FIG. 2.11.

paths. By considering their successive intersections with the x axis show that a limit cycle is approached from all initial conditions.

37. For the circuit in Figure 2.12(a) assume that the relation between I and e_g is as in Fig. 2.13: $I = f(e_g + ke_p)$, where e_g and e_p are the relative grid and plate potentials, $k > 0$ is a constant, and in the neighbourhood of the point of inflection, $f(u) = I_0 + au - bu^3$, where $a > 0$, $b > 0$. Deduce the equation for x when the operating point is the point of inflection. Find when the origin is a stable or an unstable point of equilibrium. (A form of Rayleigh's equation (1.40) is obtained, implying an unstable or a stable limit cycle respectively.)

38. Figure 2.14(a) represents two identical D.C. generators connected in parallel, with inductance and resistance L, r. R is the resistance of the load. Show that the equations for the currents are

$$L\frac{di_1}{dt} = -(r+R)i_1 - Ri_2 + E(i_1), \quad L\frac{di_2}{dt} = -Ri_1 - (r+R)i_2 + E(i_2).$$



(a) (b)

FIG. 2.12.

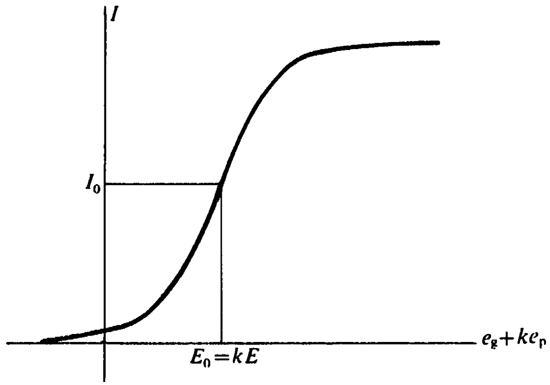


FIG. 2.13.

Assuming that $E(i)$ has the characteristics indicated by Fig. 2.14(b) show that
(i) when $E'(0) < r$ the state $i_1 = i_2 = 0$ is stable and is otherwise unstable;
(ii) when $E'(0) > r$ there is a stable state $i_1 = -i_2$ (no current flows to R);
(iii) when $E'(0) > r + 2R$ there is a state with $i_1 = i_2$, which may be unstable.

39. Show that the Emden–Fowler equation of astrophysics

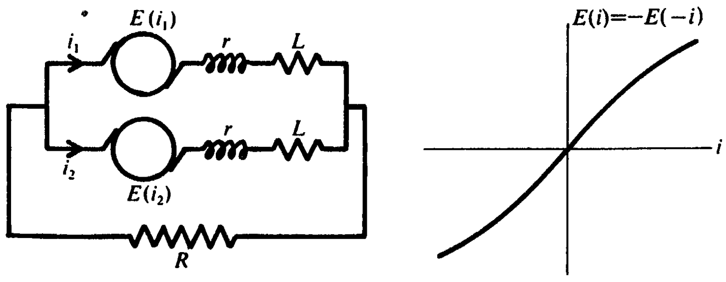
$$(\xi^2\eta')' + \xi^\lambda\eta^n = 0$$

is equivalent to the predator–prey model

$$\dot{x} = -x(1+x+y), \quad \dot{y} = y(\lambda+1+nx+y)$$

after the change of variable

$$x = \xi\eta'/\eta, \quad y = \xi^{\lambda-1}\eta^n/\eta', \quad t = \log|\xi|.$$



(a) (b)

FIG. 2.14.

40. Show that Blasius' equation

$$\eta''' + \eta\eta'' = 0$$

is transformed by

$$x = \eta\eta'/\eta'', \quad y = \eta'^2/\eta\eta'', \quad t = \log|\eta'|$$

into

$$\dot{x} = x(1+x+y), \quad \dot{y} = y(2+x-y).$$

41. Consider the family of linear systems

$$\dot{x} = X\cos\alpha - Y\sin\alpha, \quad \dot{y} = X\sin\alpha + Y\cos\alpha$$

where

$$X = ax+by, \quad Y = cx+dy,$$

and a, b, c, d are constants and α is a parameter. Show that the equilibrium point at the origin passes through the sequence stable node, stable spiral, centre, unstable spiral, unstable node, as α varies over range π.

42. Show that, given $X(x, y)$, the system equivalent to the equation $\ddot{x} + h(x, \dot{x}) = 0$ is

$$\dot{x} = X(x, y), \quad \dot{y} = -\left\{h(x, X) + X\frac{\partial X}{\partial x}\right\} \Big/ \frac{\partial X}{\partial y}.$$

43. The following system models two species with populations N_1 and N_2 competing for a common food supply:

$$\dot{N}_1 = \{a_1 - d_1(bN_1 + cN_2)\}N_1,$$
$$\dot{N}_2 = \{a_2 - d_2(bN_1 + cN_2)\}N_2.$$

Classify the equilibrium points of the system. Show that if $a_1 d_2 > a_2 d_1$ then the species N_2 dies out and the species N_1 approaches a limiting size (Volterra's Exclusion Principle).

3 Geometrical and computational aspects of the phase diagram

IN THIS CHAPTER we discuss several topics which are useful in sketching phase diagrams. The 'index' of an equilibrium point provides supporting information on its nature and complexity which is particularly useful in strongly nonlinear cases where the linear approximation is zero. Secondly, the phase diagram does not give a complete picture of the solutions; it is not sufficiently specific about the behaviour of paths 'at infinity' beyond the boundaries of any diagram; but we show various projections which include 'points at infinity' and give the required overall view. Thirdly, a difficult question is to determine whether there are any limit cycles and roughly where they are; this question is treated again in Chapter 11, but here we give some elementary conditions for their existence or non-existence. Finally, having obtained all the information our methods allow about the geometrical layout of the phase diagram we may want to compute a number of typical paths, and some suggestions for carrying this out are made in Section 3.5.

3.1. The index of a point

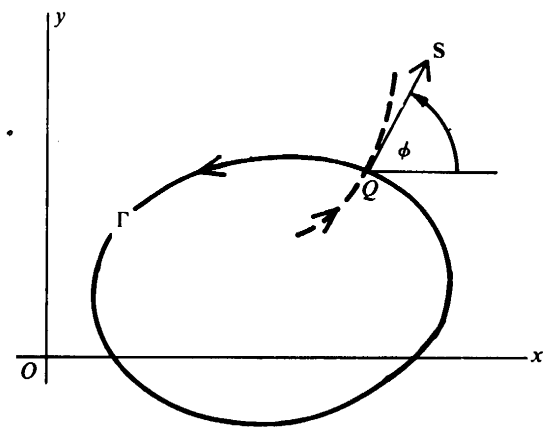


FIG. 3.1.