m_0 being the initial mass of the rocket. Since mass is not conserved, what happens to the mass that is lost?

24. In hyperbolic motion starting from the origin at rest, find the time t_0 such that if a photon is emitted from the origin after t_0 it will never catch up with the particle.

25. A particle of rest mass m, charge q, and initial velocity $\mathbf{v}_0$ enters a uniform electric field $\mathbf{E}$ perpendicular to $\mathbf{v}_0$. Find the subsequent trajectory of the particle and show that it reduces to a parabola as the limit c becomes infinite.

26. Show that the relativistic motion of a particle in an attractive inverse square law of force is a precessing ellipse. Compute the precession of the perihelion of Mercury resulting from this effect. (The answer, about 7″ per century, is much smaller than the actual precession of 40″ per century which can be accounted for correctly only by general relativity.)

27. Starting from the equation of motion (7–89), derive the relativistic analog of the virial theorem, which states that for motions bounded in space and such that the velocities involved do not approach c indefinitely close, then

$$\overline{L_0} + \overline{T} = -\overline{\mathbf{F}\cdot\mathbf{r}},$$

where L_0 is the form the Lagrangian takes in the absence of external forces. Note that although neither L_0 nor T corresponds exactly to the kinetic energy in nonrelativistic mechanics, their sum, $L + T$, plays the same role as twice the kinetic energy in the nonrelativistic virial theorem, Eq. (3–26).

28. A generalized potential suitable for use in a covariant Lagrangian for a single particle is

$$\mathscr{U} = -A_{\lambda\nu}(x_\mu)u_\lambda u_\nu,$$

where $A_{\lambda\nu}$ stands for a symmetric world tensor of the second rank and u_ν are the components of the world velocity. If the Lagrangian is made up of Eq. (7–164) minus $\mathscr{U}$, obtain the Lagrange equations of motion. What is the Minkowski force? Give the components of the force as observed in some Lorentz frame.

29. Show that if Λ satisfies the Lagrange equations, it identically satisfies Eq. (7–161) on the basis of the homogeneity of Λ, by explicitly forming the total derivative with respect to θ that occurs in the equation.

30. Covariant Lagrange equations for a single particle in terms of the proper time have been constructed incorporating the constraint of Eq. (7–75) by a method of Lagrange multipliers. The Lagrangian Λ (assumed not to depend explicitly on τ) is replaced in the variational principle by

$$\Lambda' = \Lambda + \frac{\lambda(\tau)}{2}(c^2 + u_\mu u_\mu).$$

Show that the Euler–Lagrange equation for λ gives Eq. (7–75). The Euler–Lagrange equations for x_μ involve the derivative of λ with respect to τ. Show that these can be integrated to give an expression for λ leading to the Lagrange equations:

$$\frac{d}{d\tau}\left[\frac{\partial\Lambda}{\partial u_\nu}\left(\delta_{\mu\nu} + \frac{u_\mu u_\nu}{c^2}\right) - \frac{\Lambda u_\mu}{c^2}\right] - \frac{\partial\Lambda}{\partial x_\mu} = 0.$$

CHAPTER 8
The Hamilton Equations of Motion

The Lagrangian formulation of mechanics was developed largely in the first two chapters, and most of the subsequent discussion has been in the nature of application, but still within the framework of the Lagrangian procedure. In this chapter we resume the formal development of mechanics, turning our attention to an alternative statement of the structure of the theory known as the Hamiltonian formulation. Nothing new is added to the physics involved; we simply gain another (and more powerful) method of working with the physical principles already established. The Hamiltonian methods are not particularly superior to Lagrangian techniques for the direct solution of mechanical problems. Rather, the usefulness of the Hamiltonian viewpoint lies in providing a framework for theoretical extensions in many areas of physics. Within classical mechanics it forms the basis for further developments, such as Hamilton–Jacobi Theory and perturbation approaches. Outside classical mechanics, the Hamiltonian formulation provides much of the language with which present day statistical mechanics and quantum mechanics is constructed. We shall assume in the following chapters that the mechanical systems are holonomic and that the forces are monogenic, that is, derived either from a potential dependent on position only, or from velocity-dependent generalized potentials of the type discussed in Section 1–5.

8–1 LEGENDRE TRANSFORMATIONS AND THE HAMILTON EQUATIONS OF MOTION

In the Lagrangian formulation (nonrelativistic) a system with n degrees of freedom possesses n equations of motion of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{8–1}$$

As the equations are of second order, the motion of the system is determined for all time only when $2n$ initial values are specified, e.g., the n q_i's and n $\dot{q}_i$'s at a particular time t_1, or the n q_i's at *two* times, t_1 and t_2. We represent the state of the system by a point in an n-dimensional *configuration space* whose coordinates are

the n generalized coordinates q_i and follow the motion of the system point in time as it traverses its path in configuration space. Physically, in the Lagrangian viewpoint a system with n independent degrees of freedom is a problem in n independent variables $q_i(t)$, and $\dot{q}_i$ appears only as a shorthand for the time derivative of q_i.

The Hamiltonian formulation is based on a fundamentally different picture. We seek to describe the motion in terms of *first-order* equations of motion. Since the number of initial conditions determining the motion must of course still be $2n$, there must be $2n$ independent first order equations expressed in terms of $2n$ *independent variables*. Hence the $2n$ equations of the motion describe the behavior of the system point in a *phase space* whose coordinates are the $2n$ independent variables. In thus doubling our set of independent quantities, it is natural (though not inevitable) to choose half of them to be the n generalized coordinates q_i. As we shall see, the formulation is nearly symmetric if we choose the other half of the set to be the generalized or *conjugate momenta* p_i already introduced by the definition (cf. Eq. 2–44):

$$p_i = \frac{\partial L(q_j, \dot{q}_j, t)}{\partial \dot{q}_i}. \tag{8–2}$$

The quantities (q, p) are known as the *canonical variables.**

From the mathematical viewpoint it can, however, be claimed that the q's and $\dot{q}$'s have been treated as distinct variables. In Lagrange's equations, Eq. (8–1), the partial derivative of L with respect to q_i means a derivative taken with all other q's and all $\dot{q}$'s constant. Similarly, in the partial derivatives with respect to $\dot{q}$, the q's are kept constant. Treated strictly as a mathematical problem, the transition from Lagrangian to Hamiltonian formulation corresponds to changing the variables in our mechanical functions from $(q, \dot{q}, t)$ to (q, p, t), where p is related to q and $\dot{q}$ by Eqs. (8–2). The procedure for switching variables in this manner is provided by the *Legendre transformation*,† which is tailored for just this type of change of variable.

Consider a function of only two variables $f(x, y)$, so that a differential of f has the form

$$df = u\,dx + v\,dy, \tag{8–3}$$

where

$$u = \frac{\partial f}{\partial x}, \qquad v = \frac{\partial f}{\partial y}. \tag{8–4}$$

* Unless otherwise specified, in this and subsequent chapters the symbol p will be used only for the conjugate or canonical momentum. When the forces are velocity dependent the canonical momentum will differ from the corresponding mechanical momentum (cf. Eq. 2–47).

† For a geometrical interpretation of the Legendre transformation and the role it plays in the theory of differential equations see R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. II, pp. 32–39, 1962.

We wish now to change the basis of description from x, y to a new distinct set of variables u, y, so that differential quantities are expressed in terms of the differentials du and dy. Let g be a function of u and y defined by the equation

$$g = f - ux. \tag{8–5}$$

A differential of g is then given as

$$dg = df - u\,dx - x\,du,$$

or, by (8–3), as

$$dg = v\,dy - x\,du,$$

which is exactly in the form desired. The quantities x and v are now functions of the variables u and y given by the relations

$$x = -\frac{\partial g}{\partial u}, \qquad v = \frac{\partial g}{\partial y}, \tag{8–6}$$

which are in effect the converse of Eqs. (8–4).

The Legendre transformation so defined is used frequently in thermodynamics. For example, the enthalpy X is a function of the entropy S and the pressure P with the properties that

$$\frac{\partial X}{\partial S} = T, \qquad \frac{\partial X}{\partial P} = V,$$

so that

$$dX = T\,dS + V\,dP,$$

where T and V are temperature and volume, respectively. The enthalpy is useful in considering isentropic and isobaric processes, but often one has to deal rather with isothermal and isobaric processes. In such case one wants a thermodynamic function of T and P alone. The Legendre transformation shows that the desired function may be defined as

$$G = X - TS$$

with

$$dG = -S\,dT + V\,dP, \tag{8–7}$$

where G is the well-known Gibbs function, or free energy, whose properties are correctly given by Eq. (8–7).

The transformation from $(q, \dot{q}, t)$ to (q, p, t) differs from the type considered in Eqs. (8–3) to (8–5) only in that more than one variable is to be transformed. In place of the Lagrangian one deals with a function defined in analogy to Eq. (8–5), except for a minus sign:

$$H(q, p, t) = \dot{q}_i p_i - L(q, \dot{q}, t) \tag{8–8}$$

(where, of course, the summation convention has been employed). Here H is known as the *Hamiltonian*. Considered as a function of q, p, and t only, the

differential of H is given by

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt, \tag{8-9}$$

but from the defining equation (8–8) we can also write

$$dH = \dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt. \tag{8-10}$$

The terms in $d\dot{q}_i$ in Eq. (8–10) cancel in consequence of the definition of generalized momentum, and from Lagrange's equation it follows that

$$\frac{\partial L}{\partial q_i} = \dot{p}_i.$$

Equation (8–10) therefore reduces to the simple form

$$dH = \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt. \tag{8-11}$$

Comparison with (8–9) furnishes the following set of $2n + 1$ relations, in analogy with Eqs. (8–6):

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \tag{8-12}$$

$$-\dot{p}_i = \frac{\partial H}{\partial q_i},$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}. \tag{8-13}$$

Equations (8–12) are known as the *canonical equations of Hamilton*; they constitute the desired set of $2n$ first order equations of motion replacing the Lagrange equations.*

The first half of Hamilton's equations give the $\dot{q}_i$'s as functions of (q, p, t). They form therefore the inverse of the constitutive equations (8–2), which define the momenta p_i as functions of $(q, \dot{q}, t)$. It may therefore be said that they provide no new information. In terms of solving mechanical problems by means of the

*Canonical is used here presumably in the sense of designating a simple, general set of standard equations. It appears that the term was first introduced by C. G. J. Jacobi in 1837 (*Comptes rendus de l'Academie des Sciences de Paris*, 5, p. 61) but in a slightly different context referring to an application of Hamilton's equations of motion to perturbation theory. Although the term rapidly gained common usage, the reason for its introduction apparently remained obscure even to contemporaries. By 1879, only 45 years after Hamilton explicitly introduced his equations, Thomson (Lord Kelvin) and Tait were moved by the adjective 'canonical' to exclaim: "Why it has been so called would be hard to say." (*Treatise on Natural Philosophy*, 1879, Vol. 1, p. 307.)

canonical equations, the statement is correct. But within the framework of the Hamiltonian picture, where $H(q, p, t)$ is some given function obtained no matter how, the two halves of the set of Hamiltonian equations are equally independent and meaningful. The first half says how $\dot{q}$ depends on q, p, and t; the second says the same thing for $\dot{p}$.

Of course the Hamiltonian H is constructed in the same manner, and has identically the same value, as h, the energy function defined in Eq. (2–53). But they are functions of different variables: like the Lagrangian, h is a function of $q, \dot{q}$ (and possibly t), while H must always be expressed as a function of q, p (and possibly t). It is to emphasize this difference in functional behavior that different symbols have been given to the quantities even though they have the same numerical values.

Nominally, the Hamiltonian for each problem must be constructed via the Lagrangian formulation. The formal procedure calls for a lengthy sequence of steps:

1. With a chosen set of generalized coordinates, q_i, the Lagrangian $L(q_i, \dot{q}_i, t)$ is constructed.
2. The conjugate momenta are defined as functions of q_i, $\dot{q}_i$, and t by Eqs. (8–2).
3. Equation (8–8) is used to form the Hamiltonian. At this stage one has h instead of H, or rather some mixed function of q_i, $\dot{q}_i$, p_i, and t.
4. Equations (8–2) are then inverted to obtain $\dot{q}_i$ as functions of (q, p, t). Possible difficulties in the inversion will be discussed below.
5. The results of the previous step are then applied to eliminate $\dot{q}$ from H so as to express it solely as a function of (q, p, t).

Now we are ready to use the Hamiltonian in the canonical equations of motion.

For many physical systems it is possible to shorten this drawn-out sequence quite appreciably. As has been described in Section 2–6, in many problems the Lagrangian is the sum of functions each homogeneous in the generalized velocities of degree 0, 1, and 2, respectively. In that case H by the prescription of Eq. (8–8) is given by (cf. Eq. 2–57)

$$H = L_2 - L_0, \tag{8-14}$$

where L_0 is the part of the Lagrangian independent of the generalized velocities and L_2 is the part that is homogeneous in $\dot{q}_i$ in the second degree. Further, if the equations defining the generalized coordinates don't depend on time explicitly then $L_2 = T$, and if the forces are derivable from a conservative potential V then $L_0 = -V$. When both these conditions are satisfied, the Hamiltonian is then *automatically* the total energy:

$$H = T + V = E. \tag{8-15}$$

If either Eqs. (8–14) or (8–15) holds, then much of the algebra in step 3 above is eliminated.

One can at times go further. In large classes of problems it happens that L_2 is a quadratic function of the generalized velocities and L_1 is a linear function of the same variables. The algebraic manipulations required in steps 2 through 5 can then be carried out, at least formally, once and for all. To show this let us form the $\dot{q}_i$'s into a single column matrix $\dot{\mathbf{q}}$. Under the given assumptions the Lagrangian can be written as

$$L(q, \dot{q}, t) = L_0(q, t) + \tilde{\mathbf{q}}\mathbf{a} + \frac{1}{2}\tilde{\dot{\mathbf{q}}}\mathbf{T}\dot{\mathbf{q}}, \tag{8-16}$$

where the single row matrix has been written explicitly as the transpose of a single column matrix in view of operations to be performed subsequently. Here $\mathbf{a}$ is a column matrix and $\mathbf{T}$ is a square $n \times n$ matrix (much like the corresponding matrix introduced in Chapter 6). The elements of both are in general functions of q and t. The conjugate momenta, considered as a row matrix $\mathbf{p}$, is then, by Eq. (8–2), given as

$$\mathbf{p} = \mathbf{T}\dot{\mathbf{q}} + \mathbf{a}, \tag{8-17}$$

which can be inverted (step 4) as

$$\dot{\mathbf{q}} = \mathbf{T}^{-1}(\mathbf{p} - \mathbf{a}). \tag{8-18}$$

This step presupposes $\mathbf{T}^{-1}$ exists, which it normally does by virtue of the positive definite property of the kinetic energy. By the prescription of Eq. (8–14) the Hamiltonian, identical with the energy function h, is given by

$$h = \frac{1}{2}\tilde{\dot{\mathbf{q}}}\mathbf{T}\dot{\mathbf{q}} - L_0.$$

To obtain the right functional form for H, Eq. (8–18) must be substituted for $\dot{\mathbf{q}}$ in the quadratic part of h (step 5). Now $\mathbf{T}$ is obviously a symmetric matrix, and its inverse must also be symmetric. It therefore follows that

$$\tilde{\dot{\mathbf{q}}}\mathbf{T}\dot{\mathbf{q}} = (\tilde{\mathbf{p}} - \tilde{\mathbf{a}})\mathbf{T}^{-1}\mathbf{T}\mathbf{T}^{-1}(\mathbf{p} - \mathbf{a}).$$

Hence final form for the Hamiltonian is

$$H(q, p, t) = \frac{1}{2}(\tilde{\mathbf{p}} - \tilde{\mathbf{a}})\mathbf{T}^{-1}(\mathbf{p} - \mathbf{a}) - L_0(q, t). \tag{8-19}$$

If the Lagrangian can be written in the form of Eq. (8–16), then one can immediately skip the intervening steps and write the Hamiltonian as Eq. (8–19). The inverse matrix $\mathbf{T}^{-1}$ can usually most easily be obtained straightforwardly as*

$$\mathbf{T}^{-1} = \frac{\tilde{\mathbf{T}}_c}{|\mathbf{T}|}. \tag{8-20}$$

* See almost any book on mathematical methods in physics or on matrices, e.g., Margenau and Murphy, *The Mathematics of Physics and Chemistry*, 1943 (p. 295); Hildebrand, *Methods of Applied Mathematics*, 2d ed. 1965 (p. 16); or Nering, *Linear Algebra and Matrix Theory*, 1963 (p. 83). Incidentally, $\mathbf{T}_c$ is what the mathematicians call the adjoint matrix to $\mathbf{T}$ (cf. p. 142 above).

Here $\mathbf{T}_c$ is the cofactor matrix whose elements $(\mathbf{T}_c)_{jk}$ are $(-1)^{j+k}$ times the determinant of the matrix obtained by striking out the jth row and kth column of $\mathbf{T}$. It is easy to see that if $\mathbf{T}$ is diagonal, then $\mathbf{T}^{-1}$ is also diagonal with elements that are just the reciprocals of the corresponding elements of $\mathbf{T}$.

A number of exercises in applying this formalism to various mechanical systems will be found in the problems at the end of the chapter. Two very simple examples may be considered here, particularly because they illustrate some important aspects of the technique. First consider the spatial motion of a particle in a central force field, using spherical polar coordinates (r, θ, ϕ) for the generalized coordinates. The potential energy is some $V(r)$ and the kinetic energy is

$$T = \frac{mv^2}{2} = \frac{m}{2}(\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2).$$

Clearly the Hamiltonian is the same as the total energy $T + V$, and since $\mathbf{T}$ is diagonal the form of H, by inspection, is

$$H(r, \theta, p_r, p_\theta, p_\phi) = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + V(r). \tag{8-21}$$

Note that the Hamiltonian would have a different functional form if the generalized coordinates were chosen to be the Cartesian coordinates x_i of the particle. The kinetic energy then has the form

$$T = \frac{mv^2}{2} = \frac{m\dot{x}_i\dot{x}_i}{2}$$

so that the Hamiltonian is now

$$H(x_i, p_i) = \frac{p_i p_i}{2m} + V(\sqrt{x_i x_i}). \tag{8-22}$$

It is sometimes convenient to form the canonical momenta p_i conjugate to x_i into a vector $\mathbf{p}$ such that the Hamiltonian can be written as

$$H(x_i, p_i) = \frac{\mathbf{p}\cdot\mathbf{p}}{2m} + V(\sqrt{|r|}). \tag{8-23}$$

We can of course take the components of $\mathbf{p}$ relative to any coordinate system we desire, curvilinear spherical coordinates, for example. But it is important not to confuse, say, p_θ with the θ component of $\mathbf{p}$, designated as $(\mathbf{p})_\theta$. The former is the canonical momentum conjugate to the coordinate θ; the latter is the θ component of the momentum vector conjugate to the Cartesian coordinates. Dimensionally it is clear they are quite separate quantities; p_θ is an angular momentum, $(\mathbf{p})_\theta$ is a linear momentum. *Whenever a vector is used from here on to represent canonical momenta it will refer to the momenta conjugate to Cartesian position coordinates.*

For a second example consider a single (nonrelativistic) particle moving in an electromagnetic field. By Eq. (1–66), the Lagrangian for this system is

$$L = T - V = \frac{1}{2}mv^2 - q\phi + \frac{q}{c}\mathbf{A}\cdot\mathbf{v}.$$

Using Cartesian position coordinates as generalized coordinates the Lagrangian can also be written as

$$L = \frac{m\dot{x}_i\dot{x}_i}{2} + \frac{q}{c}A_i\dot{x}_i - q\phi, \tag{8–24}$$

where the potentials ϕ and $\mathbf{A}$ are in general functions of x_i and the time (q here of course is the particle's charge, not a generalized coordinate). There is now a linear term in the generalized velocities such that the matrix $\mathbf{a}$ has the elements qA_i/c. Because of this linear term in U, the Hamiltonian is *not* $T + U$. However, it is still in this case the total energy since the "potential" energy in an electromagnetic field is determined by ϕ alone. The canonical momenta, either by Eq. (8–2) or Eq. (8–17), are

$$p_i = m\dot{x}_i + \frac{q}{c}A_i, \tag{8–25}$$

and the Hamiltonian (cf. Eq. 8–19) is

$$H = \frac{\left(p_i - \frac{q}{c}A_i\right)\left(p_i - \frac{q}{c}A_i\right)}{2m} + q\phi, \tag{8–26}$$

which is the total energy of the particle. Again the momenta p_i can be formed into a vector $\mathbf{p}$ and H written as

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 + q\phi, \tag{8–27}$$

and it must be remembered again that $\mathbf{p}$ refers only to momenta conjugate to x_i.

It will have been noticed that Hamilton's equations of motion do not treat the coordinates and momenta in a completely symmetric fashion. The equation for $\dot{p}$ has a minus sign that is absent in the equation for $\dot{q}$. Considerable ingenuity has been exercised in devising nomenclature schemes that result in entirely symmetric equations, or combine the two sets into one. Most of these schemes have only oddity value, but one has proved to be an elegant and powerful tool for manipulating the canonical equations and allied expressions.

For a system of n degrees of freedom we construct a column matrix $\boldsymbol{\eta}$ with $2n$ elements such that

$$\eta_i = q_i, \qquad \eta_{i+n} = p_i; \qquad i \leq n. \tag{8–28}$$

Similarly the column matrix $\dfrac{\partial H}{\partial \boldsymbol{\eta}}$ has the elements

$$\left(\frac{\partial H}{\partial \boldsymbol{\eta}}\right)_i = \frac{\partial H}{\partial q_i}, \qquad \left(\frac{\partial H}{\partial \boldsymbol{\eta}}\right)_{i+n} = \frac{\partial H}{\partial p_i}; \qquad i \leq n. \tag{8–29}$$

Finally, let $\mathbf{J}$ be the $2n \times 2n$ square matrix composed of the $n \times n$ zero and unit matrices according to the scheme

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}. \tag{8–30}$$

Here $\mathbf{0}$ is the $n \times n$ matrix all of whose elements are zero, and $\mathbf{1}$ is the standard $n \times n$ unit matrix. Hamilton's equations of motion can then be written in compact form as

$$\dot{\boldsymbol{\eta}} = \mathbf{J}\frac{\partial H}{\partial \boldsymbol{\eta}}. \tag{8–31}$$

This method of displaying the canonical equations of motion will be referred to as Hamilton's equations in matrix or *symplectic** notation. In subsequent chapters we shall frequently employ this matrix form of the equations. For later use, some easily verified properties of $\mathbf{J}$ may be noted. The matrix (it has no standard name) is a sort of $2n \times 2n$ version of i times the Pauli matrix $\boldsymbol{\sigma}_2$ (cf. Eq. 4–74), and its square is therefore the negative of the $2n \times 2n$ unit matrix:

$$\mathbf{J}^2 = -\mathbf{1}. \tag{8–32}$$

It is also orthogonal:

$$\tilde{\mathbf{J}}\mathbf{J} = \mathbf{1} \tag{8–33}$$

so that

$$\tilde{\mathbf{J}} = -\mathbf{J} = \mathbf{J}^{-1}. \tag{8–34}$$

From the orthogonality property it follows the square of the determinant is 1, but in fact one can prove (cf. Exercise 25) the stronger statement that

$$|\mathbf{J}| = +1. \tag{8–34′}$$

8–2 CYCLIC COORDINATES AND CONSERVATION THEOREMS

According to the definition given in Section 2–6, a cyclic coordinate q_j is one that does not appear explicitly in the Lagrangian; by virtue of Lagrange's equations

* The term *symplectic* comes from the Greek for "intertwined," particularly appropriate for Hamilton's equations where $\dot{q}$ is matched with a derivative with respect to p and $\dot{p}$ similarly with the negative of a q derivative. H. Weyl first introduced the term in 1939 in his book *The Classical Groups* (p. 165 in both the first edition, 1939, and second edition, 1946).

its conjugate momentum p_j is then a constant. But comparison of Eq. (8–9) with Eq. (8–10) has already told us that

$$\dot{p}_j = \frac{\partial L}{\partial q_j} = -\frac{\partial H}{\partial q_j}.$$

A coordinate that is cyclic will thus also be absent from the Hamiltonian.* Conversely if a generalized coordinate does not occur in H, the conjugate momentum is conserved. The momentum conservation theorems of Section 2–6 can thus be transferred to the Hamiltonian formulation with no more than a substitution of H for L. In particular the connection between the invariance or symmetry properties of the physical system and the constants of the motion can also be derived in terms of the Hamiltonian. For example, if a system is completely self-contained, with only internal forces between the particles, then the system can be moved as a rigid ensemble without affecting the forces or subsequent motion. The system is said to be invariant under a rigid displacement. Hence a generalized coordinate describing such a rigid motion will not appear explicitly in the Hamiltonian and the corresponding conjugate momentum will be conserved. If the rigid motion is a translation along some particular direction, then the conserved momentum is the corresponding Cartesian component of the total linear (canonical) momentum of the system. Since the direction is arbitrary, the total vector linear momentum is conserved. The rigid displacement may be a rotation, from whence it follows that the total angular momentum vector is conserved. Even if the system interacts with external forces, there may be a symmetry in the situation that leads to a conserved canonical momentum. Suppose the system is symmetrical about a given axis so that H is invariant under rotation about that axis. Then H obviously cannot involve the rotation angle about the axis and the particular angle variable must be a cyclic coordinate. It follows, as in Section 2–6, that the component of the angular momentum about that axis is conserved.

The considerations concerning h in Section 2–6 have already shown that if L (and in consequence of Eq. (8–13), also H) is not an explicit function of t, then H is a constant of motion. This can also be seen directly from the equations of motion (8–12) by writing the total time derivative of the Hamiltonian as

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t}.$$

In consequence of the equations of motion (8–12) the first two sums on the right cancel each other and it therefore follows that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{8–35}$$

* This conclusion also follows from the definition of Eq. (8–8), for H differs from $-L$ only by $p_i\dot{q}_i$, which does not involve q_i explicitly.

Thus if t doesn't appear explicitly in L, it will also not be present in H, and H will be constant in time.

Further, it was proved in Section 2–6 that if the equations of transformation that define the generalized coordinates (1–38),

$$\mathbf{r}_m = \mathbf{r}_m(q_1, \ldots, q_n; t),$$

do not depend explicitly on the time, and if the potential is velocity-independent then H is the total energy, $T + V$. The identification of H as a constant of the motion and as the total energy are two separate matters, and the conditions sufficient for the one are not enough for the other. It can happen that the Eqs. (1–38) do involve time explicitly but that H does not. In this case H is a constant of the motion but it is *not* the total energy. As was also emphasized in Section (2–6), the Hamiltonian is dependent both in magnitude and in functional form on the initial choice of generalized coordinates. For the Lagrangian we have a specific prescription, $L = T - V$, and a change of generalized coordinates within that prescription may change the functional appearance of L but cannot alter its magnitude. On the other hand, use of a different set of generalized coordinates in the definition for the Hamiltonian, Eq. (8–8), may lead to an entirely different quantity for the Hamiltonian. It may be that for one set of generalized coordinates H is conserved, but that for another it varies in time.

To illustrate some of these points in a simple example we may consider a somewhat artificial one-dimensional system. Suppose a point mass m is attached to a spring, of force constant k, the other end of which is fixed on a massless cart that is being moved uniformly by an external device with speed v_0 (cf. Fig. 8–1). If we take as generalized coordinate the position x of the mass particle in the stationary system, then the Lagrangian of the system is obviously

$$\begin{aligned} L(x, \dot{x}, t) &= T - V \\ &= \frac{m\dot{x}^2}{2} - \frac{k}{2}(x - v_0 t)^2. \end{aligned} \tag{8–36}$$

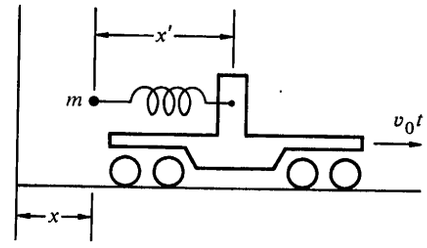

FIGURE 8–1
Example of a harmonic oscillator fixed to a uniformly moving cart.

(For simplicity the origin has been chosen so that the cart passed through it at $t = 0$.) The corresponding equation of motion is clearly

$$m\ddot{x} = -k(x - v_0 t).$$

An obvious way of solving this equation is to change the unknown to $x'(t)$ defined as

$$x' = x - v_0 t, \tag{8-37}$$

so that the equation of motion becomes

$$m\ddot{x}' = -kx'. \tag{8-38}$$

From Eq. (8–37) x' is the displacement of the particle relative to the cart; Eq. (8–38) says that to an observer on the cart the particle exhibits simple harmonic motion, as would be expected on the principle of equivalence in Galilean relativity.

Having looked at the nature of the motion, let us consider the Hamiltonian formulation. Since x is the Cartesian coordinate of the particle, and the potential does not involve generalized velocities, the Hamiltonian relative to x is the sum of the kinetic and potential energies, i.e., the total energy. In functional form the Hamiltonian is given by

$$H(x, p, t) = T + V = \frac{p^2}{2m} + \frac{k}{2}(x - v_0 t)^2. \tag{8-39}$$

The Hamiltonian *is* the total energy of the system, but since it is explicitly a function of t, it is *not* conserved. Physically this is understandable; energy must flow into and out of the "external physical device" to keep the cart moving uniformly against the reaction of the oscillating particle.*

Suppose now we formulated the Lagrangian from the start in terms of the relative coordinate x'. The same prescription gives the Lagrangian as

$$L(x', \dot{x}') = \frac{m\dot{x}'^2}{2} + m\dot{x}'v_0 + \frac{mv_0^2}{2} - \frac{kx'^2}{2}. \tag{8-40}$$

In setting up the corresponding Hamiltonian we note there is now a term linear in x', with the single component of $\mathbf{a}$ being mv_0. The new Hamiltonian is now

$$H'(x', p') = \frac{(p' - mv_0)^2}{2m} + \frac{kx'^2}{2} - \frac{mv_0^2}{2}. \tag{8-41}$$

Note that the last term is a constant involving neither x' not p'; it could, if one wished, be dropped from H' without affecting the resultant equations of motion. Now H' is *not* the total energy of the system, but it *is* conserved. Except for the last

* Put another way, the moving cart constitutes a time-dependent constraint on the particle and the force of the constraint does do work in actual (*not* virtual) displacement of the system.

term it can be easily identified as the total energy of motion of the particle relative to the moving cart. The two Hamiltonian's are different in magnitude, time-dependence, and functional behavior. But the reader can easily verify both lead to the same motion for the particle.

8–3 ROUTH'S PROCEDURE AND OSCILLATIONS ABOUT STEADY MOTION

It has been remarked that the Hamiltonian formulation is not particularly helpful in direct solution of mechanical problems. Often one can solve the $2n$ first order equations only by eliminating some of the variables, e.g., the p variables, which speedily leads back to the second order Lagrangian equations of motion. But an important exception should be noted. The Hamiltonian procedure is especially adapted to the treatment of problems involving cyclic coordinates. Consider the situation in Lagrangian formulation when some coordinate, say q_n, is cyclic. The Lagrangian as a function of q and $\dot{q}$ can then be written

$$L = L(q_1, \ldots, q_{n-1}; \dot{q}_1, \ldots, \dot{q}_n; t).$$

All the generalized velocities still occur in the Lagrangian and in general will be functions of the time. We still have to solve a problem of n degrees of freedom even though one degree of freedom corresponds to a cyclic coordinate. A cyclic coordinate in the Hamiltonian formulation, on the other hand, truly deserves its alternative description as "ignorable," for in the same situation p_n is some constant α, and H has the form

$$H = H(q_1, \ldots, q_{n-1}; p_1, \ldots, p_{n-1}; \alpha; t).$$

In effect the Hamiltonian now describes a problem involving only $n - 1$ coordinates, which may be solved completely ignoring the cyclic coordinate except as it is manifested in the constant of integration α, to be determined from the initial conditions. The behavior of the cyclic coordinate itself with time is then found by integrating the equation of motion

$$\dot{q}_n = \frac{\partial H}{\partial \alpha}.$$

The advantages of the Hamiltonian formulation in handling cyclic coordinates may be combined with the Lagrangian procedure by a method devised by Routh. Essentially, one carries out a mathematical transformation from the $q, \dot{q}$ basis to the q, p basis only for those coordinates that are cyclic, obtaining their equations of motion in the Hamiltonian form, while the remaining coordinates are governed by Lagrange equations. If the cyclic