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[15] The space of test functions defined here can be extended to include smooth functions that decrease sufficiently rapidly to zero at infinity; this is not necessary for our purposes, but is useful in dealing with Fourier transforms. See, e.g., Ref. 12.
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$$\boldsymbol{\mu}_m = \pm\beta\,\overline{\mathbf{g}}\cdot\mathbf{J}$$

with $\overline{\mathbf{g}}$ not proportional to $\overline{\mathbf{U}}$, then the hyperfine dyadic can become asymmetric, containing a first-rank tensorial part (Ref. 29). We shall not herein deal with this extra complexity.
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The case of the identically accelerated twins

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A variation on the "twin paradox" of special relativity is presented wherein twins undergo the same acceleration for the same length of time, yet they age differently. Although this problem is simple to solve, it gets to the heart of the behavior of clocks in special relativity and, hopefully, will help to dispel the notion students develop that the acceleration experienced by a relativistic traveler is directly related to the rate at which that traveler ages.

I. INTRODUCTION

Perhaps the most intriguing aspects of the special theory of relativity are those concerning the nature of time and few problems baffle the beginning student more than the paradox of the identical twins.[1] In the standard version of the twin paradox, twin No. 1 remains at home while twin No. 2 travels away at high velocity. Subsequently, No. 2 turns around, speeds home, and finds that No. 1 has aged more. This is due, of course, to time dilation, i.e., No. 1 sees No.

2's clock running slow. An apparent paradox arises if one applies the time dilation factor from the second twin's perspective who sees No. 1's clock running slow and concludes that twin No. 1 should be the younger. The resolution is that the time dilation formula of special relativity holds only in inertial, i.e., nonaccelerating, frames of reference. Twin No. 1 remains in an inertial frame and correctly applies the time dilation factor while twin No. 2 had to be accelerated in order to turn around and return home and, therefore, may not use the simple time dilation formula.

Although the preceding analysis resolves the paradox, students often inquire as to "why" the accelerated twin ages less and "when" the extra aging of the home twin occurs. These questions are not well defined in the scientific sense but have promoted a variety of analyses (many can be found in the pages of this Journal[2]) which for the most part have been useful additions to the pedagogy of special relativity theory. However, many of these analyses are quite complicated and it is doubtful that the beginning relativity student gains a great deal of insight from them particularly concerning the significance of acceleration. It has often been pointed out that while the acceleration of one twin is the key to resolution of the paradox, it is wrong to suppose that reduced aging is a direct result of acceleration. The age difference of the twins is proportional to the length of the trip while the period of acceleration is determined only by how long it takes to turn around and is independent of the length of the trip and, hence, the final age difference of the twins.

The above argument notwithstanding, many students of relativity still harbor the feeling that it is the acceleration that in some way causes the traveling twin to age more slowly. The following simple "twin paradox" is offered both to dispel this notion and at the same time emphasize the fundamental principle that underlies most if not all of the apparent paradoxes in special relativity. In the "*case of the identically accelerated twins*," twins who undergo identical accelerations for the same length of time, nevertheless, age differently. Although I am sure that the problem of identically accelerated twins is well known in one form or another to many,[3] I have been amazed over the years at how many of my colleagues have initially professed disbelief at the outcome. What the student gains from studying this problem is a grasp of the significance of the problem of clock synchronization in special relativity and a simple example of how acceleration from one inertial frame to another renders two initially synchronized clocks unsynchronized. The latter is a convenient point of departure for a discussion of the behavior of stationary clocks in a uniform gravitational field.

II. THE CASE OF THE IDENTICALLY ACCELERATED TWINS

Suppose two twins, Dick and Jane, own identical spaceships each containing the same amount of fuel. Jane's ship is initially positioned a distance x_0 to the right of Dick's, as shown in Fig. 1. Mom and Dad remain at home. The twins synchronized their watches (according to special relativity all observers in this inertial frame agree the clocks are synchronized) and at precisely 12:00 noon start their engines and accelerate off to the right. After both ships have expended all their fuel, they will coast at velocities v_D and v_J, respectively.

Since their ships are identical and the initial supplies of fuel are the same, the final velocities of the twins will be the same, $v_D = v_J$, and they will once again be in the same inertial frame. Upon comparing ships' logs, they find that their trips commenced at the same time (initial synchronization of clocks) and that they experienced the same acceleration (identical spaceships) for the same intervals of time (identical amounts of fuel). They are astonished, however, to find that Jane has aged more than Dick! Since their ships' logs contain identical entries, we know that the two were the same age when they completed their journeys. Therefore, Jane evidently arrived at the new inertial frame before her brother. They also discover that their ships are further apart than when they started. See Fig. 2.

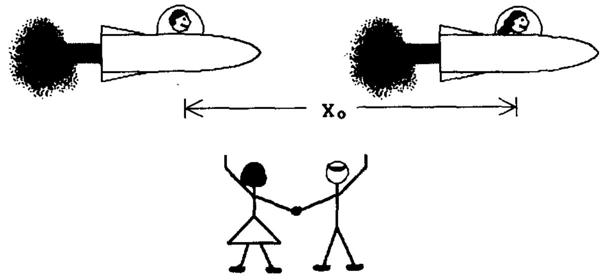

Fig. 1. The twins as they begin their journey.

Both of these results are easily derived from the observations of their parents who remained at home. According to Mom and Dad, the clocks and, hence, the ages of the twins remain the same throughout their journey. This must be so since they undergo identical accelerations and, therefore, their velocities are always the same. It also follows that, according to Mom and Dad, the distance between the two ships is always x_0. The age difference and separation of the twins in their own frame are then easily determined by the Lorentz transformations relating the two frames, i.e.,

$$x' = \gamma(x - vt) \tag{1}$$

and

$$t' = \gamma(t - vx/c^2), \tag{2}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$; $v = v_D = v_J$; c is the speed of light; x and t are the length and time coordinates of the parents' frame; and x' and t' are the coordinates of the final inertial frame of the twins.

Now consider two events in the frame of the twins: the times at which their rocket engines shut off and they arrive in their new inertial frame. Suppose these occur on the birthdays of the twins (remember their ships' logs are identical). From Eq. (2), the times of these events as seen in the inertial frame of the twins are related to the times in the

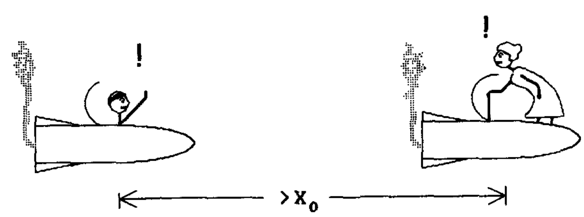

Fig. 2. The twins after having arrived at their new inertial frame.

frame of the parents by

$$t'_D = \gamma(t_D - vx_D/c^2),$$
$$t'_J = \gamma(t_J - vx_J/c^2),$$

where the subscripts indicate the birthdays of Dick and Jane, respectively. Therefore,

$$t'_D - t'_J = \gamma(t_D - t_J) - v(x_D - x_J)/c^2. \quad (3)$$

In Mom and Dad's frame, the birthdays occur simultaneously, i.e., $t_D - t_J = 0$, and the twins are always separated by a distance x_0. Then Eq. (3) yields

$$t'_D - t'_J = \gamma v x_0/c^2. \quad (4)$$

That is, according to the twins, Jane's birthday occurs $\gamma v x_0/c^2$ *before* Dick's; consequently, Jane is older. A similar calculation shows that the twins discover the distance separating the two ships after the journey is γx_0.

The above results may seem paradoxical. The two twins underwent identical accelerations for identical times and yet aged differently. Of course, there is no paradox. The situations of the twins are not exactly the same. Jane started the trip a distance x_0 from Dick in the direction of the subsequent acceleration. Had the two accelerated to the left it would have been Dick who aged more. If the acceleration had been perpendicular to their separation, the problem would be symmetrical and the twins would have aged the same.

III. DISCUSSION

At the root of the twin paradox is the problem of synchronization of clocks. Two clocks separated by a proper distance x_0 that are synchronized in their rest frame appear unsynchronized by an amount $\gamma x_0 v/c^2$ to an observer moving with velocity v in the direction of the separation of the clocks. In the above example, according to an observer in the primed frame, the clocks of the two twins before the trip were out of synchronization by $\gamma x_0 v/c^2$. After their trip, the twins are also in the primed frame and find, indeed, that one twin is older by just this amount. According to special relativity, initially synchronized clocks that accelerate from one inertial frame to another will lose their synchronization (if the acceleration is in the direction of their separation).

One can explain the ordinary twin paradox in these terms. According to twin No. 2 (see Sec. I above), twin No. 1 ages less rapidly by a factor $1/\gamma$ during the entire trip. However, because of the acceleration at turnaround, there is a change in synchronization between the two twins' clocks. This change more than compensates for the apparent slowdown in twin No. 1's aging and at the end of the trip twin No. 1 is the older of the two.

The change in synchronization of accelerated clocks also provides important insight into the behavior of clocks in a uniform gravitational field. According to the principle of equivalence,[4] physics in a uniform gravitational field is the same as physics in an accelerated frame of reference. From the above discussion, two accelerated clocks that are separated along the direction of acceleration do not remain in synchronization; rather, the forward clock runs fast. Similarly, two clocks at rest in a uniform gravitational field are in a sense forever being accelerated into new frames and, therefore, the "forward" clock, i.e., the clock at the higher gravitational potential, runs faster. This is precisely the cause of gravitational redshift in a uniform gravitational field.[5]

Perhaps the most important lesson of the "*case of the identically accelerated twins*" is that statements in special relativity about the *rates* of clocks that are in motion relative to each other always involve a comparison of clocks that are spatially separated and thus constitute a nonlocal system. The twins in the above paradox had identical local experiences (same accelerations for same time intervals) but not identical global experiences (one twin was in front of the other). It is this global asymmetry and not any local asymmetry, such as different accelerations, which lies at the heart of the paradox of the twins.

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[4]See, for example, Ref. 1, p. 17.

[5]Reference 4, p. 117.