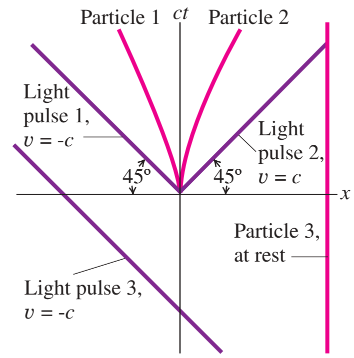
Topic 14 | Spacetime Diagrams

Graphical representations can make kinematic concepts less abstract and also give useful information. For example, not only does a v-t graph for one-dimensional motion show the velocity at any instant, but its slopes give accelerations, and areas under it give displacements. For relativity the transformations $y' = y$ and $z' = z$ are easy to understand, so we'll just consider the $\pm x$ directions.

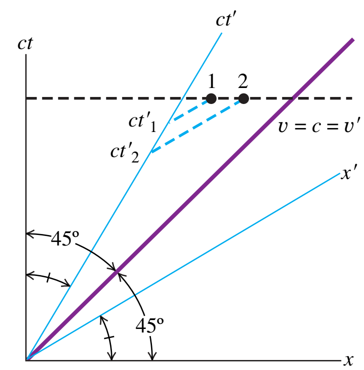
The usual convention in relativity is to graph ct on the vertical axis and x on the horizontal axis. Such a graph provides us with a **spacetime diagram.** We use ct rather than t so that both scales can have the same unit and the same scale. The path of a particle forms a line, called its **worldline,** as the particle moves in one-dimensional motion. At any point, the slope of the worldline is $d(ct)/dx = (c\,dt)/(v\,dt) = c/v$. Thus a light pulse with $v = \pm c$ has a slope of ± 1 on a spacetime diagram, giving angles of 45° with the $\pm x$-axes. Since material particles have speeds less than c, all worldlines for material particles are steeper than those 45° angles. That is, nothing known has a worldline with a slope between −1 and 1. The worldline of a particle at rest is vertical and so has infinite slope. Figure T14.1 shows six worldlines, three of light pulses and three of particles. Can you show that these six worldlines agree with the statements made in this paragraph about their slopes?

How does the S' reference frame appear on our ct-x spacetime diagram? Recall that we *always* set $x' = 0$ at $x = 0$ when $t' = 0 = t$ and let S' move at a speed u in the $+x$-direction. But $x' = 0$ all along the ct'-axis, so $x' = 0$ and the ct'-axis have a worldline of slope c/u on our ct-x spacetime diagram. For example, if $u = 0.600c$, the ct'-axis is at an angle of arctan(1/0.600) = 59.0° from the x-axis or 90.0° − 59.0° = 31.0° from the ct-axis.

Surprisingly enough, the x'-axis is not drawn perpendicular to the ct'-axis on our ct-x spacetime diagram. Since $ct' = 0$ (so $t' = 0$) all along the x'-axis, the Lorentz transformation equation for t' gives $(t - ux/c^2) = 0$ or $ct = (u/c)x$ for the x'-axis. Thus the x'-axis is drawn with a slope of u/c on our ct-x spacetime diagram. For $u = 0.600c$ the x'-axis is at an angle of arctan(0.600) = 31.0° from the x-axis. That is, the x'-axis makes the same angle with the x-axis as the ct'-axis makes with the ct-axis. Figure T14.2 shows that the worldline of a light pulse leaving $x' = 0 = x$ at $t' = 0 = t$ with a velocity $+c$ bisects the angle between either set of axes.



T14.1 A spacetime diagram showing worldlines of three light pulses and three particles. Particles 1 and 2 leave $x = 0$ at $t = 0$, accelerating from rest in opposite directions.



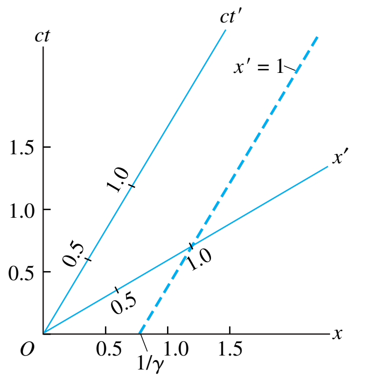
T14.2 The ct'- and x'-axes drawn on our ct-x spacetime diagram. Notice the two sets of equal angles.

Example T14.1 Simultaneity on a spacetime diagram

Stanley measures events 1 and 2 to occur simultaneously in S at positions x_1 and x_2, where $x_2 > x_1$. Use our spacetime diagram to show that Mavis, who moves in the positive x-direction relative to Stanley, measures event 2 to occur before event 1.

SOLUTION

Events that are simultaneous in S have the same time t, so in Fig. T14.2 we draw a dashed line parallel to the x-axis (constant t). We put a dot on that line for event 1, and farther from the ct-axis we put another dot for event 2 (because $x_2 > x_1$). How do we read a value of a point on a graph? We draw a line through that point parallel to one axis and measure where it intercepts the other axis. Thus to measure the times of the two events in Mavis's S' frame, in Fig. T14.2 we draw dashed lines parallel to the x'-axis that intercept the ct'-axis at ct_1' and ct_2'. We see that $ct_2' < ct_1'$, so $t_2' < t_1'$. The events are not simultaneous in S', and Mavis measures event 2 to occur before event 1.



T14.3 A spacetime diagram for $u = 0.600c$. The dashed line $x' = 1$ intercepts the x-axis at $x = 1/\gamma = 0.800$. The scale of the S'-axes is greater than that of the S-axes.

On our spacetime diagram, the scale for the S' axes is not the same as the scale for the S axes. For example, consider the dashed line $x' = 1$ in Fig. T14.3, which must be drawn parallel to the ct'-axis. (We have left off the unit for generality; it could be $x' = 1$ meter, $x' = 1$ light year, or whatever is convenient.) This dashed line intercepts the x-axis at $ct = 0$. Substituting $t = 0$ in the Lorentz transformation $x' = \gamma(x - ut)$ gives $x = 1/\gamma$ for the $x' = 1$ line. In Fig. T14.3, $u = 0.60c$ and this intercept is at $x = 0.80$. We can see that the symmetry of our spacetime diagram gives us the same scaling ratio for the ct'- and ct-axes. To summarize, in comparison to the ct- and x-axes, the ct'- and x'-axes are rotated through an angle arctan u/c toward the common $v = c = v'$ line at 45° and are stretched in scale so that the $x' = 1$ line intercepts the x-axis at $x = 1/\gamma$.

Let's finish this discussion with a simple example of length contraction. Mavis, at rest in frame S', holds a meter stick with its left end at $x' = 0$ and its right end at $x' = 1$ m. Thus in Fig. T14.3 the units are meters. At any time t' measured in frame S', the left end is at $x' = 0$ (on the ct'-axis) and the right end is on the $x' = 1$ dashed line. In frame S the positions of both ends of the meter stick are measured at the same time t, then subtracted to find the length. For instance, at $t = 0$ (on the x-axis) we see from Fig. T14.3 that the left end of her stick is at $x = 0$ and the right end is at $x = (1\text{ m})/\gamma$. Thus in S the meter stick has a contracted length of $(1\text{ m})/\gamma$.