

# Twentieth Century Physics

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# Twentieth Century Physics

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*For Sarah*



## Acknowledgments

I am not a modern physicist by trade, but a plasma physicist, so this book has borrowed heavily from many sources. Its originality consists in packaging a course for sophomore physics majors in a manner different than has been done before. I have used what I believe is the best from the standard modern physics texts, but I have also used unique treatments that I have found in unpublished notes, and journals like the *American Journal of Physics*. When I have borrowed explanations that are unique, I have cited those sources; however, standard treatments of standard concepts I consider to be in the public domain.

I learned modern physics from Matt Sands at U.C. Santa Cruz in 1984 where we used the first edition of Krane's *Modern Physics* textbook, which I really liked, especially the historical discussions. Now that I have taught the same class from that text, I still like it, but I feel that it suffers from the same problems that most modern textbooks suffer from, a too-heavy reliance on the historical approach.

I learned quantum mechanics from George Gaspari at U.C. Santa Cruz in 1986-87 and from Ernest Abers at U.C.L.A. in 1987-88. Abers has recently produced a wonderful graduate-level textbook based on his lecture notes,<sup>1</sup> and I have used some of that material that is accessible to undergraduates.

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<sup>1</sup>Abers, *Quantum Mechanics*, 2006.





## Preface

*Tomorrow is going to be wonderful because tonight I do not understand anything.* — Niels Bohr

INTRODUCTORY PHYSICS is usually taught in historical order. The first course is mechanics, which was developed in the 17th century, followed by fluids, thermodynamics and electromagnetism, which were developed in the 18th and 19th centuries. The final piece of the puzzle, modern physics (or 20th century physics), is left until last, and is also usually presented in a historical manner. It starts with special relativity, and then progresses through “old quantum theory” and basic quantum mechanics. Finally, if there is time in the typical one-semester course, a brief overview of nuclear physics, the standard model of particle physics, and possibly some cosmology is presented. This standard procedure illustrates the (now discredited) biological dictum, “ontogeny recapitulates phylogeny.”<sup>2</sup>

However, the brief emphasis placed on particle physics does not give the students a sense of the “big picture” of the standard model, which is our current best guess for how things are put together. While a fundamental understanding of the standard model requires advanced relativity and quantum mechanics, I believe that to be truly a course in “modern physics,” we must place this modern understanding in a prominent role. Also, after two or three semesters of physics, students deserve to be shown how all the physics that they have learned fits together, rather than simply viewing the Bohr model, the Schrodinger equation, and special relativity, etc., as simply more in a long list of (separate) topics. For this reason, I *start* this book with a discussion of the most fundamental particles, quarks and leptons, and then I progress outward to larger, composite, objects: nuclei, and then atoms. This is in reverse historical order, but gives the students a coherent picture of our current knowledge. Of course, some ideas from relativity and quantum mechanics are needed to understand these fundamental particles, so I have placed a basic introduction in Chapter 1, and have also introduced physical concepts as needed. Finally, in the latter part of the book, while covering relativity and quantum theory, I am able to prove some statements that I had previously only quoted. Therefore, the endpoint of our study of relativity and quanta is an explanation of our understanding at the most basic level, i.e., the important applications, rather than simply solving the 1D Schrodinger equation for various potentials, for example, with no apparent motivation.

It is true that there is much to be learned studying history, and one of the most important results of a study of physics is to understand precisely how we have come to our conclusions, and why we think they are correct. That is, how do we know what we claim to know? What are the experimental clues that lead us to believe that our current

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<sup>2</sup>An idea from developmental biology which states that the embryonic development of an organism (ontogeny) mirrors the evolutionary development of the species (phylogeny). This theory was first put forth by Ernst Haeckel (1834-1919). As Haeckel himself wrote in his book *Riddle of the Universe at the Close of the Nineteenth Century* (1899), “I established the opposite view, that this history of the embryo (ontogeny) must be completed by a second, equally valuable, and closely connected branch of thought - the history of race (phylogeny). Both of these branches of evolutionary science, are, in my opinion, in the closest causal connection; this arises from the reciprocal action of the laws of heredity and adaptation ... ‘ontogenesis is a brief and rapid recapitulation of phylogenesis, determined by the physiological functions of heredity (generation) and adaptation (maintenance).’ ”

model is the best one? And what were the previous models that experiments ruled out?<sup>3</sup> Understanding these experimental facts and the logic behind them are as important as understanding the theoretical constructs upon which we base our models. As Robert Millikan [Nobel Prize, Physics, 1923] said, “Science walks forward on two feet, namely theory and experiment ... Sometimes it is one foot which is put forward first, sometimes the other, but continuous progress is only made by the use of both – by theorizing and then testing, or by finding new relations in the process of experimenting and then bringing the theoretical foot up and pushing it on beyond, and so on in unending alternations.”<sup>4</sup> In fact, a thorough investigation into incorrect models, and the experiments that finally revised (or perhaps completely overthrew) those models, is extremely useful. Those stories are not the main thrust of this book, however, and have been relegated either to footnotes, boxed historical asides, or appendices. Several of the appendices should be studied thoroughly, as they comprise a significant fraction of the text. The main point, though, is to describe our current thinking about how the world is put together, what it is made of, and how the pieces interact. In telling that story the key historical observations and experiments will be delved into, and pointers to the appropriate appendix will be made for further study.



This book is divided approximately into three parts. First, Chapter 1 consists of a brief overview, with statements (not proof) of some of the basic principles of relativity and quantum mechanics that are needed as a foundation. Second, Chapters 2-4 are introductions to particle physics, nuclear physics, and atomic physics, which bring you up to speed on the current state-of-the-art. Finally, Chapters 5-7 develop the mathematics that explain the results stated previously: Chapter 5 is a development of special relativity; Chapter 6 is a development of “old” quantum theory, and Chapter 7 derives the full-fledged non-relativistic quantum mechanics.

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<sup>3</sup>Epistemological questions such as these tend to be swept under the rug during a study of classical physics, partially because the answers seem so self-evident among familiar surroundings. When you study physics that is further removed from everyday experience, however, such as subatomic particles, these questions come to the fore, unbidden. A careful consideration of such questions clarifies the role of classical physics and gives us a deeper understanding of the universe and its inner workings.

<sup>4</sup>Nobel Lecture, May 23, 1924, *The electron and the light-quant from the experimental point of view.*

## How to Study

There are as many study methods as there are students, but a few principles are universal, and there are a few new ones that apply specifically to modern physics. First, **do a lot of outside reading**. Unlike in your previous physics courses, which mostly covered classical physics, which is “normal” and “intuitive,” in modern physics there will be quite a few new concepts, many of them completely unfamiliar and counter-intuitive, along with lots of new jargon. One way to become familiar with the concepts and comfortable with the language is to expose yourself to as many different viewpoints as possible. Not only should you read this text carefully, but you should read other textbooks and popular accounts.

Second, true physical understanding comes through familiarity with the mathematics. So, just as in classical physics, problem solving is crucial to building physical intuition. How best to solve problems? Just as with study habits, there many problem solving methods, but Descartes developed a method 400 years ago that still works. René Descartes was one of the first to discuss the so-called “scientific method.” Such a method works as well for solving problems as it does for investigating nature — this is because they are the *same activity!* Descartes said in *Discourse on the Method*:

A multitude of laws often hampers justice, so that a state is best governed when it has only a few laws which are strictly administered; similarly, instead of the large number of laws which make up logic, I was of the opinion that the four following laws were perfectly sufficient for me, provided I took the firm and unwavering resolution to stick to them clearly at all times.

The first was never to accept anything as true if I did not clearly know it to be so; that is, carefully to avoid precipitate conclusions and preconceptions, and to include nothing more in my judgement than was presented clearly and distinctly to my mind, so that I had no reason to doubt it.

The second, to divide each of the difficulties I examined into as many parts as possible, and as might be necessary for a proper solution.

The third, to conduct my thoughts in an orderly fashion, by starting with the simplest and most easily known objects, so that I could ascend, little by little, and step by step, to more complex knowledge; and by giving some order even to those objects which appeared to have none.

And the last, always to make enumerations so complete, and review so comprehensive, that I could be sure of leaving nothing out.

The second and third parts seem to be the most helpful. Break down each problem into small, easily understood pieces. Solve each piece; then put them together to solve the entire problem. Some problems appear to be unsolvable at the beginning, but that’s because the solver tries to do it all at once. Forget about the final answer, but try to obtain information about a small part of the problem. Once you’ve succeeded there, attack another small part, then another, etc., and eventually the entire problem will be done.

Finally, in addition to outside reading, I suggested that you read this book. How? There are many ways to read this book, all of them are viable. It’s not necessary to read straight through from the beginning to the end (although I have planned this to be

a coherent strategy for someone who is getting their first glimpse at the subject). But you can jump around and read the pieces that interest you. The many appendices that supplement the main text are meant to be read in this serendipitous way. One of the best recommendations on how to read textbooks is in the book *Basic Mathematics* by Serge Lang. In the middle of this book he has a section entitled “On Reading Books,” and I quote it here in its entirety, because it is difficult to improve on.

### On Reading Books

This part of the book can really be read at any time. We put it in the middle because that’s as good as any place to start reading a book. Very few books are meant to be read from beginning to end, and there are many ways of reading a book. One of them is to start in the middle, and go simultaneously backwards and forward, looking back for the definitions of any terms you don’t understand, while going ahead to see applications and motivation, which are very hard to put coherently in a systematic development. For instance, although we must do algebra first, it is quite appealing to look simultaneously at the geometry, in which we use algebraic tools to systematize our geometric intuition.

In writing the book, the whole subject has to be organized in a totally ordered way, along lines and pages, which is not the way our brain works naturally. But it is unavoidable that some topics have to be placed before others, even though our brain would like to perceive them simultaneously. This simultaneity cannot be achieved in writing, which thus gives a distortion of the subject. It is clear, however, that I cannot substitute for you in perceiving various sections of this book together. You must do that yourself. The book can only help you, and must be organized so that any theorem or definition which you need can be easily found.

Another way of reading this book is to start at the beginning, and then skip what you find obvious or skip what you find boring, while going ahead to further sections which appeal to you more. If you meet some term you don’t understand, or if you need some previous theorem to push through the logical development of that section, you can look back to the proper reference, which now becomes more appealing to you because you need it for something which you already find appealing.

Finally, you may want to skim through the book rapidly from beginning to end, looking just at the statements of theorems, or at the discussions between theorems, to get an overall impression of the whole subject. Then you can go back to cover the material more systematically.

Any of these ways is quite valid, and which one you follow depends on your taste. When you take a course, the material will usually be covered in the same order as the book, because that is the safest way to keep going logically. Don’t let that prevent you from experimenting with other ways.

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*Science without epistemology is — in so far as it is thinkable at all — primitive and muddled. — Albert Einstein*



# Chapter 1

## Preliminaries

*The opinion seems to have got abroad that in a few years all the great physical constants will have been approximately estimated, and that the only occupation which will then be left to men of science will be to carry on these measurements to another place of decimals.* — James Clerk Maxwell, 1871

This book starts with a description of our present understanding of how the universe works. Because this description relies on physics that we will not delve into until later, I must first present some basic results of special relativity and quantum mechanics (before we actually study them in detail in Chapters 5 and 7) so that the description makes sense. You will have to take my word that I am telling the truth; I can't *prove* these results until later, but they are necessary for understanding the basics of particle physics, nuclear physics, and atomic physics, all of which we will cover in the next few chapters.

In addition, it is helpful to have some idea of the historical sequence that physics went through, so I give a brief synopsis of the state of affairs at the beginning of the period we wish to study, and also at the end (in order to proceed, it is helpful to know where we are going).

### 1.1 Historical Preview

#### Physics circa 1900

In 1895 (before the discovery of X-rays,<sup>1</sup> radioactivity,<sup>2</sup> and the electron<sup>3</sup>) there were two *forces*: the gravitational force and electromagnetic force; there were two *object properties*: mass and charge; and there was one *dynamical law* determining how objects respond to those forces: Newton's law of motion. (Well, Newton actually enumerated three laws,

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<sup>1</sup>Wilhelm Conrad Roentgen discovered X-rays on November 8, 1895, and was awarded the first Nobel Prize in Physics for 1901.

<sup>2</sup>Henri Becquerel discovered the natural radioactivity of uranium in early 1896 while investigating X-rays, and shared the Nobel Prize in Physics for 1903 with Pierre and Marie Curie.

<sup>3</sup>Joseph John Thomson discovered the electron in 1897 and was awarded the Nobel Prize in Physics for 1906. In reality, Thomson measured the charge-to-mass ratio of the electron in 1897, and it wasn't until 1899 that he was able to make an independent measurement of its charge (and hence its mass); the latter date, therefore, can be more definitively called the date of discovery.

but they act as one coherent group.) These, in principle, are all that you need to predict how objects will behave dynamically. The object properties determine the strength of the forces that act on the objects, and Newton’s dynamical laws predict the future response to those forces. Thus, the universe was envisioned as a great clock—once started it would continue to run forever. In fact, if one were able to measure (with infinite precision, of course) the positions and velocities of all objects in the universe at a specific time (i.e., the “state” of the universe), then the laws of dynamics along with a knowledge of the forces would allow one to predict their future positions and velocities. This is known as the “mechanistic worldview” or the “Newtonian worldview.”

In addition, the thermodynamic properties of matter and its interaction with light were relatively well understood. (Some of these properties are summarized in Appendix A.) So much so, in fact, that in 1875 the head of the physics department at the University of Munich advised Max Planck [Nobel Prize, Physics, 1918], the future progenitor of quantum theory, to not study physics because, as he put it, “Physics is a branch of knowledge that is just about complete. The important discoveries, all of them, have been made. It is hardly worth entering physics anymore.”



However, there was little understanding of what matter was made. No theory satisfactorily explained why a particular object was endowed with its particular values of mass and charge. Many elements (such as nitrogen and oxygen) were known, and each element had a known molar mass and volume density, but no underlying reason for these properties had been successfully proposed. As you might guess, there had been hints about the microscopic structure of matter. For instance, the atomic hypothesis had been around since Democritus (c. 400 BCE), who postulated that rather than being a continuum, matter was made up of small discrete objects called “atoms”. The word *atoms* comes from the Greek word *ατομος*, which means “that which cannot be cut,” or “uncuttable.” However, this hypothesis was nothing more than supposition until John Dalton proposed his law of multiple proportions in 1803, which states that when two elements combine to form more than one compound, the ratios of the weights are ratios of small integers.

One of the clearest sets of data was the ratio of the amounts of oxygen and nitrogen needed to make various compounds.<sup>4</sup> Experiment showed that

$$\frac{m_O}{m_N} = 0.57, 1.13, 1.71, 2.29, 2.86 \quad (1.1)$$

for the five compounds nitrous oxide ( $N_2O$ ), nitric oxide ( $NO$ ), nitrous anhydride ( $N_2O_3$ ), nitrogen dioxide ( $NO_2$ ), and nitric anhydride ( $N_2O_5$ ), respectively. The five ratios are very close to the integers 1:2:3:4:5. While this suggests that matter is made of discrete clumps, it would take another hundred years before the concept was accepted by the scientific community.<sup>5</sup>

<sup>4</sup>Friedman and Sartori, *The Classical Atom*, page 1.

<sup>5</sup>For a detailed look at the history of the atomic concept, see Boorse and Motz, *The World of the Atom*, which contains reprints from Lucretius to Einstein concerning the existence of atoms and subatomic particles.

	smaller $\rightarrow$ ( $h$ )	
faster $\downarrow$	<b>Newton</b>	quantum
( $c$ )	relativity	quantum field theory

Figure 1.1: A schematic diagram of dynamical theories. Newton’s Laws are approximately valid when velocities are small compared with the speed of light,  $c$ , and another quantity, called “action,” is large compared with Planck’s constant,  $h$ . Otherwise, quantum mechanics or special relativity is needed, or perhaps both. When both are needed, the combination results in a “quantum field theory,” such as Quantum Electrodynamics (QED) which describes electromagnetism, and Quantum Chromodynamics (QCD) which describes the strong/color force.

The discrete clumps turned out not to have exactly integer mass ratios, a fact that was first conclusively shown in 1920 by William Aston, who, along with Ernest Rutherford [Nobel Prize, Chemistry, 1908] developed an accurate mass spectrograph, and whose work included the discovery of isotopes of non-radioactive elements.



### Physics circa 2000

The current view of the fundamental nature of matter and the ways in which it interacts is certainly more detailed than in 1895, and it is tempting to believe that we have reached “the end.” However, while there are mathematical reasons that lead us to believe we might be near the “Theory of Everything,” or a “Grand Unified Theory,” past experience has at least humbled physicists of the present day and they understand that what we call “fundamental” today may turn out not to be. In fact, the situation today may be compared with that of 1895. We know of more (and smaller) particles, e.g., quarks, but, for example, we still have no idea *why* the quarks have fractional electric charge or *why* they have spin  $\frac{1}{2}$ , nor even *why* any of the particles have the masses they do.

We now know of four *forces*: the gravitational force and electromagnetic force, but also the strong nuclear force (or “color” force) and the weak nuclear force. We also can enumerate many more *properties* (or attributes) of subatomic particles: mass, charge, and color, which are related to the forces, as well as others that make sense only within the quantum description of matter, properties like spin and strangeness. Finally, we have expanded Newton’s description of how these particles interact, with the result that his *dynamical laws* have been modified both on a small scale (quantum mechanics) and at large velocities (special relativity), as shown in Figure 1.1.

The theory of *relativity* and the theory of *quanta* are the two great theoretical constructs of the early 20th century.

If you are interested in the intersection of quantum mechanics and relativity—quantum field theory—you will likely have to continue your work in graduate school because not

Relativity		
special relativity	1905	Einstein
general relativity	1915	Einstein
Old Quantum theory		
blackbody radiation	1900	Planck
photoelectric effect	1905	Einstein
hydrogen atom	1913	Bohr
Quantum Mechanics		
wave-particle duality	1925	de Broglie
wave equation	1926	Schrodinger
matrix mechanics	1926	Heisenberg, Born, Jordan
relativistic wave equation	1928	Dirac

Figure 1.2: An overview of the architects of relativity and quanta, and when their key developments were produced.

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only are advanced mathematical tools needed, but also a thorough grounding in relativity and nonrelativistic quantum mechanics. Rather than diving headlong into these mathematically difficult (and conceptually abstract) topics, I will spend the rest of this chapter describing the basics in simplified terms. In this way we can attack the conceptual differences between classical physics and modern physics first, and then show later the mathematical detail of *why* they must be this way. Also, the mathematics and physics that we will need at first is nothing more than the basics of what you have learned in your study of introductory physics so far: energy, momentum, angular momentum, etc., and straightforward algebra.

### Timeline

Relativity is, of course, the brainchild of one person, Albert Einstein [Nobel Prize, Physics, 1921], but quantum mechanics took many physicists many years to straighten out, as shown in Figure 1.2. How they were led to make the discoveries that they made was due to a long list of experiments that, for the most part, raised more questions than they answered. This list of experiments and predictions are given in Fig. 1.3. The first three experiments were essentially accidents, but the next two resulted from purposeful investigations into newly found, not understood phenomena. The next five, covering the first 15 years of the new century, were theoretical responses to the pile up of 19th century experiments that were inconsistent with 19th century physical theory. Key experiments were done during this time, however, they were continuing explorations of previous work rather than profound new advances. Most of these topics are covered in later chapters—those with an asterisk are analyzed separately in their own appendix.



1887	Hertz	photoelectric effect *
1895	Roentgen	X-rays
1896	Becquerel	radioactivity
1896	Zeeman	Zeeman effect
1897	Thomson	discovery of electron
1900	Planck	blackbody radiation *
1905	Einstein	photoelectric effect *
1905	Einstein	specific heats of solids
1908	Rydberg-Ritz	combination principle
1913	Bohr	atomic model
1914	Frank-Hertz	Franck-Hertz experiment
1922	Stern-Gerlach	Stern-Gerlach experiment *
1923	Compton	Compton effect *
1923	de Broglie	electron wavelength
1927	Davisson-Germer	electron diffraction

Figure 1.3: Listing of key experimental results and theoretical predictions that “largely shaped the physics of the twentieth century.” [Pais, *Inward Bound*, page 379.] The starred topics are discussed in detail in separate appendices.

## 1.2 Relativity

*Intuition is something one develops on the basis of experience.*

— Alfred Schild

The most important result of Einstein’s Special Theory of Relativity that we will use at this point is the equivalence of mass and energy. The “rest energy,”  $E_0$ , of an object is given by

$$E_0 = mc^2, \tag{1.2}$$

where  $m$  is the mass of the object and  $c$  is the speed of light

$$c = 299\,792\,458 \text{ m/s}. \tag{1.3}$$

This value of  $c$  is exact—it has been defined as this value—but a useful approximation is  $c \approx 3.00 \times 10^8$  m/s. In some sense, Eq. (1.2) defines how much energy is locked up in the mass of an object. In fact, it is useful to think of mass as potential energy. More importantly, this equation expanded our understanding of the rules of the natural world by replacing two “laws” that were thought to be universal (the conservation of mass and the conservation of energy) with a third law that we now believe *is* universal (the conservation of the sum of mass and energy, or the conservation of energy with mass as just another type of energy).

In 1789, the chemist Antoine Lavoisier was the first to show that matter was conserved in chemical reactions. That is, even though the compounds may change (e.g., liquid water

can be turned into gaseous hydrogen and gaseous oxygen), the quantity of matter neither increases nor decreases. He discovered this rule by carefully measuring the weights of the reagents and the products in various chemical reactions (including the gases). Today, we call this the “Law of Conservation of Mass.” In the 19th century, several physicists, James Joule among them, realized that there was another conserved quantity, energy. Joule, for instance, stirred water with a paddle, and, by carefully measuring the amount of mechanical work done by the paddle and the subsequent increase in temperature of the water, was able to demonstrate that there was a “mechanical equivalent of heat.” Subsequently, with the discovery of other types of energy (electrical energy, wave energy, etc.), the principle that the total energy in the universe is constant came to be accepted, and prompted Rudolf Clausius, thermodynamicist, to say in 1865

*“The energy of the universe is constant.”*

What Einstein said with Eq. (1.2) is that neither of those two laws are separately true, but that the *sum* of mass and energy is a constant. In reality, he discovered a new type of energy: rest energy. In fact,  $c^2$  can be thought of simply as a conversion factor between joules, units normally used to measure energy, and kilograms, units normally used to measure mass. In our analysis of atomic, nuclear, and particle physics in Chapters 2 through 4, we’ll see that a basic understanding of the physical processes involved can be obtained by keeping track of the transformation of mass to energy, and vice versa, and not worrying about the detailed dynamics. This is similar to analyzing collisions between objects by looking only at the momentum before and after the collision, but ignoring the details of the forces that caused those changes in momentum.

Although we won’t cover relativistic dynamics until Chapter 5, it is useful to know the relativistically correct expression for kinetic energy in addition to the rest energy. What happens when an object is not at rest, but is moving with respect to an observer (you, for instance)? Then, in addition to its rest energy, the observer would measure that it has some kinetic energy as well, and this kinetic energy increases as the speed of the object increases.<sup>6</sup> However, the manner in which it increases is different from  $\frac{1}{2}mv^2$ . The *total* energy of a particle (rest plus kinetic), is given by

$$E = E_0 + K = \gamma mc^2, \quad (1.4)$$

where  $\gamma$ , called the “relativistic factor,” is

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (1.5)$$

You’ll see where this comes from in Chapter 5, but we can draw some conclusions simply by looking at the mathematical form of  $\gamma$ . The most important of these is that  $v$  must be less than  $c$ . This is a fundamental property of the universe in which we live: any object with nonzero mass must travel slower than the speed of light. Objects that do not

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<sup>6</sup>Einstein’s second paper on special relativity in 1905 dealt specifically with this issue, and it was titled, “Does the Inertia of a Body Depend Upon Its Energy Content?” As you should guess, if the kinetic energy depends on velocity, so should the momentum, which is a quantitative measure of inertia.



have mass (such as photons, i.e., light) *must* travel *at* the speed of light (we will see the mathematical justification for this later). A second conclusion is that we can recover the usual form of the kinetic energy when the speed of the object is small compared with the speed of light (see Problem 2).

This is a good place to stop and think about the implications of these results. Why is  $c$  a universal speed limit? Why must photons travel at that speed? Can objects travel faster than  $c$ ? Unfortunately, science cannot answer those questions, or at least it has not been able to answer them yet. The role of science is to observe the universe and deduce, in laws that are as simple and far reaching as possible, *how* the universe works. Many laws of classical physics (e.g., Newton’s second law) seem so natural that it appears impossible for them to be any other way. This is directly due to our familiarity with them, given that they govern the macroscopic world around us. But you must realize that there is no answer to the question, “Why does  $\vec{F}$  equal  $m\vec{a}$ ?” All physics can say is, “that is the way nature works,” and we have deduced that rule and written it in simple language.

In addition to energy, momentum is the second fundamental concept in special relativity. In classical mechanics, it is customary to consider the mass  $m$  and velocity  $\vec{v}$  of a particle to be the fundamental dynamic quantities (along with the position, of course). From these you can calculate any other quantity of interest: momentum ( $\vec{p} = m\vec{v}$ ), kinetic energy ( $K = mv^2/2$ ), acceleration ( $\vec{a} = d\vec{v}/dt$ ), etc. However, it is more natural to treat energy, Eq. (1.4), and momentum

$$\vec{p} = \gamma m\vec{v}, \quad (1.6)$$

as the fundamental quantities. Newton’s second law, in the form

$$\vec{F} = \frac{d\vec{p}}{dt}, \quad (1.7)$$

still holds, and the mass, or rest energy, can be determined from Eq. (1.4). The fact that the relativistic factor  $\gamma$  appears in Eq. (1.6) implies that our “classical” definition of momentum must be modified just as we have modified the definition of energy. You can show (Problem 3) that there is still a consistent relationship between energy and momentum, although the correct, relativistic, relationship is different from what you have learned in introductory mechanics.

## 1.3 Quantum Mechanics

The implications of quantum mechanics are often much stranger than those of relativity. An important new philosophical result is that there are some questions that are not “askable” in quantum mechanics, in the sense that certain quantities cannot be measured at particular times. I will point out these questions as they arise throughout the book. For now, I will concentrate on the most fundamental difference between classical mechanics and quantum mechanics, namely the reason why it is called *quantum*.

In our classical, macroscopic world, properties of objects (both intrinsic properties such as mass, and extrinsic properties such as velocity) can take on any value among a continuous range of values—there is no restriction. In the subatomic quantum world, however, some properties of particles (though not all) are restricted to a discrete set

of values—they are “quantized.” The best known example is probably the energy of an electron that is bound in a hydrogen atom. Unlike a planet (or asteroid, or comet) orbiting the Sun, which can have any energy, the electron is restricted to occupy certain “energy levels.” Each of these levels is assigned a “quantum number,” and the electron’s energy can be calculated from that quantum number. In this particular case the quantum number is  $n$ , and it can take on the values  $n = 1, 2, 3, \dots, \infty$ . It is simply a label for the particular quantum “state” that the electron occupies. The electron’s energy when it is in state  $n$  is given by the formula

$$E_n = \frac{E_1}{n^2}, \quad (1.8)$$

where  $E_1 \approx -13.6$  eV,<sup>7</sup> and  $E_1$  is the electron energy when it occupies state  $n = 1$ . The average radius,  $r_{ave}$ , of the electron’s orbit is another physical property that can be calculated in terms of the quantum number  $n$

$$r_{ave} = a_0 n^2, \quad (1.9)$$

where  $a_0 \approx 0.0529$  nm, and is called the “Bohr radius.” This structure is ubiquitous in quantum mechanics:

A quantum number labels the state that a particular particle is in, and the physical properties of that state can be calculated from a formula that depends on that quantum number.

In the case of the hydrogen atom,  $n$  is called the “principal” quantum number.

Since we will not do any explicit quantum calculations until Chapter 7, all I can do is list the possible values for the quantum number and tell you what the formula is. The formulas, however, are related in the sense that they all depend on a constant that is intrinsically quantum in nature: Planck’s constant,  $h$ . We can see this by looking at the form of the constants  $E_1$  and  $a_0$  (which are determined from a solution of Schrodinger’s equation, the quantum equivalent of Newton’s second law)

$$E_1 = -\frac{m_e e^4}{8h^2 \epsilon_0^2}, \quad (1.10)$$

$$a_0 = \frac{\epsilon_0 h^2}{2m_e e^2}, \quad (1.11)$$

where  $m_e$  is the electron mass,  $e$  is the electron charge, and  $\epsilon_0$  is the permittivity of free space. These should be familiar to you, but  $h$  is something new. It is Planck’s constant and its measured value is

$$h = 6.626\,0693(11) \times 10^{-34} \text{ J s}, \quad (1.12)$$

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<sup>7</sup>Recall that electron volts (eV) are just another energy unit, defined as the amount of energy gained by an electron when it falls through a potential difference of one volt, and that  $1 \text{ eV} = 1.602\,176\,53(14) \times 10^{-19}$  J, or with our usual precision  $1 \text{ eV} \approx 1.60 \times 10^{-19} \text{ J}$ . Note that the energy of the electron is its total energy (kinetic plus electric potential) which is why it is negative—the convention is that that potential energy is zero when the separation of the electron and proton is infinitely large, and therefore the potential energy is always negative.

where the (11) gives the experimental uncertainty in the last two digits. Usually, three significant digits will be enough, except when we are subtracting two nearly equal numbers, which we'll do when we discuss fusion and fission and the energy those processes generate. Hence, we can take  $\boxed{h \approx 6.63 \times 10^{-34} \text{ J s}}$ . This constant, along with  $c$ , are the two numbers that in some sense define Twentieth Century Physics.

One simple way to understand this quantized behavior of the electron in the hydrogen atom—indeed, any particle confined to a finite region of space—is to view the electron as a “wave.” We will talk in detail about what this means later, but for now the situation can be thought of in the same way as the resonant harmonic structure of a string that is fixed at both ends. In that case, a frequency is resonant if an integer number of wavelengths fit along the string. In the case of the electron, if it can be described as a wave, it must have a wavelength, and while confined in the hydrogen atom it is allowed to occupy only those states where an integral number of wavelengths “fit” in the atom. Although it is not immediately clear what this resonance means for an electron, we at least have a mathematical and conceptual framework on which to hang our understanding. This type of analogy is common in modern physics: some quantum and relativistic systems behave as if they were governed by the same laws as a classical system, and so we can use our understanding of those systems to “understand” the new systems.

A second quantity that quantum mechanics says should be discrete is angular momentum. Recall that the angular momentum of an object about an axis is  $\vec{L} = \vec{r} \times \vec{p}$ , where  $\vec{r}$  is the radius vector from the axis to the object, and  $\vec{p}$  is the object's linear momentum. Classically, the angular momentum of an object can take on any value—its momentum and distance from the axis of rotation are both completely arbitrary. Quantum mechanics, however, predicts that the magnitude of  $\vec{L}$  is discrete, and has the value  $L = |\vec{L}| = \sqrt{\ell(\ell + 1)} \hbar$ , where  $\hbar = h/2\pi$ .<sup>8</sup> The quantum number in this case is  $\ell$  and can take on the values  $\ell = 0, 1, 2, \dots, \infty$ . It can easily be shown that Planck's constant  $h$  has the proper units of angular momentum (you should convince yourself that this is true before you continue reading). For example, the magnitude of the orbital angular momentum of the Earth about the Sun is about  $3 \times 10^{40}$  J s. Since the Earth's angular momentum is much larger than  $\hbar$ , this means the quantum number  $\ell$  that describes the Earth's revolution takes on the value of about  $3 \times 10^{74}$ . We will see in Chapter 7 that quantum effects are difficult to detect when objects are in states with large quantum numbers, and that the quantum dynamical laws can be approximated quite well by the classical dynamical laws.<sup>9</sup>

Finally, a third example of a system that is quantized (or discrete) is a “photon.” Since Maxwell's time, light has been thought of as an electromagnetic wave. While this appears to be true in some experiments, in some other experiments light appears to be made up of discrete packets of energy, called quanta or photons. The energy  $E$  of each photon depends on the frequency  $\nu$  of the light

$$E = h\nu, \tag{1.13}$$

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<sup>8</sup>It is up to you whether you wish to memorize  $h$  or  $\hbar$ , but a combination that appears often, and is therefore useful to know, is  $\boxed{\hbar c \approx 197 \text{ MeV fm}}$ .

<sup>9</sup>This is one instance of Niels Bohr's “Correspondence Principle.”

where the proportionality constant is the “intrinsically quantum”  $h$ . This conceptual view of light (as a discrete particle) was first proposed by Max Planck in 1900 in his study of blackbody radiation, subsequently solidified by Einstein in 1905 in his study of the photoelectric effect, and finally nailed down in 1923 by Compton’s experiments on the scattering of X-rays by graphite. It is illuminating to read Einstein’s own words on the subject from his 1905 paper describing the photoelectric effect:

It seems to me that the observation associated with black body radiation, fluorescence, the photoelectric effect, and other related phenomena associated with the emission or transformation of light are more readily understood if one assumes that the energy of light is discontinuously distributed in space. In accordance with the assumption to be considered here, the energy of a light ray spreading out from a point is not continuously distributed over an increasing space, but consists of a finite number of energy quanta which are localized at points in space, which move without dividing, and which can only be produced and absorbed as complete units.<sup>10</sup>

Equation (1.13) is one of the fundamental relations that describe the “wave-particle” duality of light: energy is a property that is usually thought to apply to particles, and frequency is a property that is usually thought to apply to waves. Light (indeed, all particles) can be thought of as having both mutually exclusive characteristics—wave and particle—and which characteristic shows itself depends on the experiment. In fact, in order to correctly interpret some experiments, *both* characteristics must be invoked (see Problem 5). The second fundamental relation that describes the wave-particle duality of all particles is de Broglie’s equation

$$p = \frac{h}{\lambda}, \quad (1.14)$$

where  $p$  is the particle’s momentum, and  $\lambda$  is the wavelength. Again, Planck’s constant acts to relate the two properties. (See Sec. 4.2.3 for de Broglie’s development.)

Equations (1.13) and (1.14) relate the particle properties ( $E$  and  $p$ ) to the wave properties ( $\nu$  and  $\lambda$ ) of an object. How are we to interpret these relations? Linus Pauling guides us to the proper interpretation:

Does light *really* consist of waves, or of particles? Is the electron *really* a particle, or is it a wave?

These questions cannot be answered by one of the two stated alternatives. Light is the name that we have given to a part of nature. The name refers to all of the properties that light has, to all of the phenomena that are observed in a system containing light. Some of the properties of light resemble those of waves, and can be described in terms of a wavelength. Other properties of light resemble those of particles, and can be described in terms of a light quantum, having a certain amount of energy,  $h\nu$ .... A beam of light is neither a sequence of waves nor a stream of particles; it is both.

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<sup>10</sup>“On a Heuristic Viewpoint Concerning the Production and Transformation of Light,” *Annalen der Physik*, **17** 132 (1905).

In the same way, an electron is neither a particle nor a wave, in the ordinary sense. In many ways the behavior of electrons is similar to that expected of small spinning particles, with mass  $m$ , electric charge  $-e$ , and certain values of angular momentum and magnetic moment. But electrons differ from ordinary particles in that they also behave as though they have a wave character, with a wavelength given by the de Broglie equation. The electron, like the photon, has to be described as having the character both of a particle and of a wave.... You might ask two other questions: Do electrons exist? What do they look like?

The answer to the first question is that electrons do exist: “electron” is the name that scientists have used in discussing certain phenomena, such as the beam in the electric-discharge tube studies by J. J. Thomson, the carrier of the unit electric charge on the oil drops in Millikan’s apparatus, the part that is added to the neutral fluorine atom to convert it into a fluoride ion. As to the second question—what does the electron look like?—we may say that some information has been obtained by studying the scattering of very-high-velocity electrons by protons and other atomic nuclei. These experiments have given much information about the size and structure of the nuclei, and have also shown that the electron behaves as a point particle, with no structure extending over a diameter as great as 0.1 fm.<sup>11</sup>

As early as 1909, Einstein was beginning to understand that this was to be the crux of any physical theory of light:

It is my opinion that the next phase of theoretical physics will bring us a theory of light that can be interpreted as a kind of fusion of the wave and the [particle] theory.

Of course, it would take until 1926 for the term “photon” to be coined, and until 1927 for Compton to receive the Nobel prize for his X-ray experiments that definitively pinned down the particle nature of light. However, we are getting ahead of the story. Now that I have stated, but not proved, some of the fundamental ideas, it is time to jump forward to the present and investigate our current understanding of how the universe is put together. For that, we start with particles in Chapter 2.

### Collateral Reading

Here are three excellent accounts of the history of physics in the twentieth century. Two of the authors, Gamow and Segre, were participants in this history, and therefore give first-hand accounts.

- George Gamow, *Thirty years that shook physics: The story of quantum theory*, Doubleday, 1966. (ERAU: QC 174.1 .G3)

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<sup>11</sup>Pauling, *General Chemistry*, pages 80-81. The current upper limit to the “diameter” of an electron is about  $10^{-7}$  fm.

- Emilio Segre, *From X-rays to quarks: Modern physicists and their discoveries*, W. H. Freeman, 1980. (ERAU: QC 7 .S44 1980)
- Barbara Lovett Cline, *Men who made a new physics*, University of Chicago Press, 1987. (ERAU: QC 15 .C4 1987)

## Problems

1. Look up the word “epistemology.” Write a paragraph on what it means and why it is important in science, in physics, and especially modern physics. (Even though you are just beginning your study of modern physics, discuss it in light of your background knowledge.)

2. Show that in the limit of small speeds ( $v \ll c$ ) the total energy ( $\gamma mc^2$ ) of a particle is approximately equal to  $E \approx mc^2 + \frac{1}{2}mv^2$ . This limit is sometimes called “nonrelativistic.” HINT: Use the binomial expansion  $(1 + \epsilon)^p \approx 1 + p\epsilon$ , where  $\epsilon \ll 1$ .

3. (a) From the definitions of energy and momentum in Eqs. (1.4) and (1.6), show that they are related in the following way

$$E^2 = p^2c^2 + (mc^2)^2.$$

Remember that  $p^2 \equiv \vec{p} \cdot \vec{p}$ . (b) Obtain the nonrelativistic limit of this result. That is, use the binomial expansion again, and apply it to the limit when  $p$  is small, that is, when  $p \ll mc$ .

4. The relativistic factor  $\gamma$  arises in nonrelativistic situations as well. For example, consider a river flowing with speed  $v$ , and a swimmer able to swim at speed  $c$  relative to the water. (a) Calculate the time  $t_u$  it takes the swimmer to swim a distance  $d$  upstream and back, where  $d$  is the distance measured relative to the stationary river bank. (b) Calculate the time  $t_a$  it takes the swimmer to swim a distance  $d$  directly across the river and back (perpendicular to the river bank). (c) Show that the ratio of the two times ( $t_u/t_a$ ) is equal to  $\gamma$ .

5. Equations (1.8) and (1.13) together represent a simple theory of the interaction of light and matter. As an example, if an electron in a hydrogen atom makes a transition from state  $n = 3$  to  $n = 2$ , it loses an amount of energy equal to  $E_3 - E_2$ . If this energy is released in the form of a photon, what is the frequency of that photon? Is it in the visible portion of the spectrum?

6. Equations (1.4) and (1.13) together represent another aspect of the interaction between light and matter. (a) If an electron and an anti-electron (both with the same mass) approach each other, one possibility is that they “annihilate” each other and, to conserve energy and momentum, they must produce two identical photons. If the electrons initially have negligible kinetic energies, calculate the frequency and wavelength of the photons. (b) Calculate the same quantities for the annihilation of a proton and anti-proton pair.

## Solutions

1. **epistemology**, *n.* The branch of philosophy that investigates the nature, limits, criteria, or validity of human knowledge; also, a particular theory of cognition. From the Greek *episteme* meaning “knowledge,” and *logos* meaning “theory.”

One question that is often asked is “Is objective knowledge possible?” A possible answer is that the senses are vague and inconsistent, so therefore abstract knowledge is superior. This is the point of view of *rationalists*, such as Descartes, Spinoza, and Leibniz. The final test of knowledge is deductive reasoning based on self-evident principles. *Empiricists*, such as Bacon and Locke, on the other hand, believe in the primacy of sense perception, and hold that all knowledge must be verifiable in experience. Locke stated, “one cannot have absolutely certain knowledge of the physical world.”

**2.** To solve this, you need the Taylor series expansion  $(1 + \epsilon)^p \approx 1 + p\epsilon$ , which is a good approximation when  $\epsilon \ll 1$ . The total energy of a particle is

$$E = \gamma mc^2 = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} mc^2 \approx mc^2 \left(1 + \frac{1}{2} \frac{v^2}{c^2}\right) = mc^2 + \frac{1}{2} mv^2.$$

Since we also know that  $E = E_0 + K$ , we see that the first order approximation to the kinetic energy is the well known  $mv^2/2$ . Can you calculate the next order term? Answer:  $K = (mv^2/2)(1 + \frac{3}{4} \frac{v^2}{c^2})$ .

**3.** (a) One method is to take the definitions of energy and momentum and form the quantity  $E^2 - p^2 c^2$

$$E^2 - p^2 c^2 = \gamma^2 (mc^2)^2 - \gamma^2 m^2 v^2 c^2.$$

Factoring out  $\gamma^2 (mc^2)^2$  gives

$$E^2 - p^2 c^2 = \gamma^2 (mc^2)^2 \left(1 - \frac{v^2}{c^2}\right) = (mc^2)^2,$$

where the last equality comes from the definition of  $\gamma$ . (b) The nonrelativistic relationship between energy and momentum can be written

$$E = \sqrt{p^2 c^2 + (mc^2)^2} = mc^2 \sqrt{1 + \left(\frac{p}{mc}\right)^2} \approx mc^2 \left[1 + \frac{1}{2} \left(\frac{p}{mc}\right)^2\right] = mc^2 + \frac{p^2}{2m}$$

Again,  $K = p^2/2m$  is the correct Newtonian expression for kinetic energy.

**4.** The Galilean transformation is needed here, which states that

$$\vec{v}_{SG} = \vec{v}_{SW} + \vec{v}_{WG},$$

or in words, “the velocity of the Swimmer relative to the Ground is equal to the velocity of the Swimmer relative to the Water plus the velocity of the Water relative to the Ground.” Note that this is a *vector* equation, so the magnitudes don’t necessarily add. In this problem, let’s let  $\vec{v}_{WG} = -v\hat{y}$  and  $|\vec{v}_{SW}| = c$ .

(a) While swimming upstream, the swimmer’s speed relative to the ground is reduced  $\vec{v}_{SG} = (+c - v)\hat{y}$  so that

$$\Delta t_u = \frac{d}{c - v}.$$

Similarly, swimming downstream the swimmer goes faster  $\vec{v}_{SG} = (-c - v)\hat{y}$  and  $\Delta t_d = d/(c + v)$ . The total time taken is

$$\Delta t = \Delta t_u + \Delta t_d = \frac{d}{c - v} + \frac{d}{c + v} = \frac{2cd}{c^2 - v^2}.$$

(b) If the swimmer wants to move *directly* across the river, they must angle slightly upstream so they don't drift downstream. In this case,  $c$  is the hypotenuse of the right triangle,  $v$  is one side, and therefore  $|\vec{v}_{SG}| = \sqrt{c^2 - v^2}$  is the speed of the swimmer relative to the ground. The time taken to swim across the river is the same as that to swim back (same speed), so that the total time taken is

$$\Delta t = \frac{2d}{\sqrt{c^2 - v^2}}.$$

(c) The ratio shows that it is quicker to swim across and back

$$\frac{\Delta t_{up}}{\Delta t_{across}} = \frac{c}{c^2 - v^2} \sqrt{c^2 - v^2} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \geq 1.$$

This, in fact, is exactly the analysis needed to interpret the Michelson-Morley experiment. The “swimmer” in that case is light, and the “river” is the ether. Michelson and Morley measured the two travel times and tried to detect a difference, which would have allowed them to determine the speed of the Earth relative to the ether. However, since their result was that the two times were identical, FitzGerald and Lorentz proposed that objects (i.e., their measuring apparatus) *contracted* in length by a factor  $\gamma$  in the direction of motion. This *ad hoc* proposal would result in the two travel times being identical. Of course, there *is* a “Lorentz contraction,” but for reasons having to do with observers in different reference frames (i.e., special relativity), rather than a physical contraction of objects.

5. From (1.8) we have that

$$E_3 - E_2 = E_1 \left( \frac{1}{9} - \frac{1}{4} \right) = \frac{5}{36} (-E_1) \approx 1.89 \text{ eV}.$$

From (1.13), the frequency of the photon is

$$\nu = \frac{E_3 - E_2}{h} \approx 4.57 \times 10^{14} \text{ Hz}.$$

The wavelength of this light, from  $c = \lambda\nu$ , is  $\lambda \approx 657 \text{ nm}$ , which is red, visible light. In spectral terminology, this is called the “Balmer  $\alpha$ ” (or  $H_\alpha$ ) line.

6. The energy before the annihilation is just  $2m_e c^2 = 2 \times 0.511 \text{ MeV}$  (the photons don't exist yet). Afterward, the electron—anti-electron pair does not exist, so the photons' energy is  $2 \times h\nu$ , which, equating the energies means that  $\nu = 1.23 \times 10^{20} \text{ Hz}$ , and  $\lambda = 2.43 \text{ pm}$  (that's a *picometer* which is  $10^{-12} \text{ m}$ ), which is a gamma ray. Gamma rays are conventionally defined as electromagnetic radiation with wavelengths less than  $100 \text{ pm}$ , or frequencies higher than  $3 \text{ EHz}$  ( $\text{E} = \text{exa} = 10^{18}$ ). For protons ( $E_0 = 938 \text{ MeV}$ ), the resulting photons have  $\nu = 2.27 \times 10^{23} \text{ Hz}$  and  $\lambda = 1.32 \text{ fm}$  (a “hard” gamma ray).



# Chapter 2

## Introduction to Particle Physics

*If I could remember the names of all the particles, I'd be a botanist.*  
— Enrico Fermi

### Matter

At its most basic level, all matter consists of combinations of 12 elementary particles, which are listed in Fig. 2.1. They can be classified into two groups, leptons and quarks: quarks interact via the color force (or strong force) but leptons do not. Both types of particles interact gravitationally (i.e., they all have mass) and via the weak force. Finally, all but the neutrinos interact electromagnetically because neutrinos are electrically neutral. The original motivation for the classification of leptons in 1947 was that the electron (the only known lepton at that time) was less massive than the proton and neutron (the only known nucleons—later determined to consist of quarks), and “lepton” is from a Greek word that means small or light. (See page 23.) Of course, after the discovery of the tau lepton in 1975 and the observation that it was almost twice as massive as a proton, the original reason no longer made sense. However, with the discovery of quarks and the fact that they are the only particles to interact via the strong force, the division into leptons and quarks is appropriate, albeit for reasons that have to do with forces rather than mass.<sup>1</sup>

Amazingly, all natural matter that we observe in the world around us consists of only three of these particles: electrons, up quarks, and down quarks. The atoms in our bodies are comprised of electrons as well as protons and neutrons, but the proton is made up of 2 up quarks and 1 down quark (commonly written ‘uud’), while the neutron is 2 down quarks and 1 up quark (commonly written ‘udd’). In this sense, the universe is very simple. There are only three particles, which combine in a myriad of ways to make up all the wonderful objects that we see: trees, rivers, oceans, mountains, planets, stars, and galaxies.

What are the intrinsic properties of these elementary particles? Two are very familiar, mass and electric charge, and three others, spin, magnetic moment, and color, are not as

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<sup>1</sup>In addition to these 12 particles, there are the so-called “exchange particles,” like the photon (denoted by the symbol  $\gamma$ ), that mediate the four forces. These particles are also called “gauge bosons,” or “intermediate vector bosons,” and they are not normally considered to be matter. I will discuss them below on page 26.

$e^-$	<b>electron</b>	Leptons
$\nu_e$	electron neutrino	
$\mu^-$	muon (mu lepton)	
$\nu_\mu$	muon neutrino	
$\tau^-$	tauon (tau lepton)	
$\nu_\tau$	tau neutrino	
u	<b>up quark</b>	Quarks
d	<b>down quark</b>	
c	charm quark	
s	strange quark	
t	top (truth) quark	
b	bottom (beauty) quark	

Figure 2.1: The twelve elementary particles that comprise all natural and man-made matter. The three particles in boldface — electron, up quark, and down quark — comprise all known natural matter. There are six leptons (three massive leptons and three massless neutrinos) and six *flavors* of quarks.

familiar. We will examine these five in detail in Sections 2.1 through 2.5. Of course, there are many others, such as strangeness, isotopic spin, lepton number, and baryon number, and we will investigate these in later chapters. The nomenclature of particle physics is very complicated, but if you remember to characterize particles based on their fundamental properties, like mass, charge, etc., it doesn't matter what they are called, you will be able to understand the physics of their interactions.

You may have noticed that I didn't mention size as an intrinsic property. The reason is that all of these elementary particles are thought to be point-like and have no size. For example, the size of an electron has been experimentally measured by Hans Dehmelt [Nobel Prize, Physics, 1989] to be less than  $10^{-22}$  meters! This simply means that the electric force that an electron feels is Coulombic (i.e.,  $\sim 1/r^2$ ) down to that distance, which means that there is no reason to think that electrons have any structure at any scale. Of course, when elementary particles combine to form protons, neutrons, atoms, and molecules, the physics of their interaction occurs on a spatial scale so that the conglomerations acquire a characteristic size and shape.



There is another characteristic of these particles that has no classical counterpart: they are identical and indistinguishable. Unlike our macroscopic world, where we can paint seemingly identical objects different colors to distinguish them (billiard balls, for example), in the microscopic world there is no way to tell two electrons apart. When a cue ball, say, collides with an eight-ball and they each move off in different directions, it is clear which ball is which after the collision. However, if two electrons collide and move off, the experimenter is not able to distinguish which electron is which after the collision. As we will see below, this fact has far-reaching implications on the allowable motions of

$e^+$	positron (anti electron)	anti Leptons
$\bar{\nu}_e$	anti electron neutrino	
$\mu^+$	anti muon (mu lepton)	
$\bar{\nu}_\mu$	anti muon neutrino	
$\tau^+$	anti tauon (tau lepton)	
$\bar{\nu}_\tau$	anti tau neutrino	
$\bar{u}$	anti up quark	anti Quarks
$\bar{d}$	anti down quark	
$\bar{c}$	anti charm quark	
$\bar{s}$	anti strange quark	
$\bar{t}$	anti top (truth) quark	
$\bar{b}$	anti bottom (beauty) quark	

Figure 2.2: The twelve elementary antiparticles.

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these particles. The most well-known implication is the Pauli exclusion principle that is applied to electrons within atomic orbitals, which I will discuss in Chapter 4.

## Antimatter

*Antimatter is as much matter as matter is matter.* — Abraham Pais<sup>2</sup>

For every particle, there is a corresponding “antiparticle,” with the same mass, but opposite electric charge, and these are listed in Fig. 2.2. The antiparticles are denoted by an overbar, or sometimes by simply changing the sign, as with the positron. Do not ascribe any mysterious properties to antimatter. As Pais implies, from an antiparticle’s point of view, *we* are made of “antimatter.” In fact, current cosmological theories suggest that in the early universe, a short time after the Big Bang, there was approximately as much matter as antimatter. As the universe cooled, equal amounts of matter and antimatter were annihilated, and what was left over was the small amount of matter that makes up the visible universe. The question of why there was an asymmetry between the amounts of matter and antimatter (i.e., why there wasn’t exactly the same amount of both kinds) is one that still has not been answered.

Why, then, does antimatter exist? No one knows, but that appears to be the way the universe is made. However, within the rules of our current structure of theoretical physics, antiparticles are a “necessary consequence of combining special relativity with quantum mechanics.”<sup>3</sup> Paul Dirac [Nobel



<sup>2</sup>Abraham Pais is perhaps one of the foremost chroniclers of the story of modern physics. His writings, listed in the Bibliography, are all the more valuable because he was a practitioner — he worked on the front lines in 1940s through the 1970s — and he knew and collaborated with several of the key players personally, e.g., Bohr, Einstein, Heisenberg.

<sup>3</sup>Martin and Shaw, *Particle Physics*, p. 2.

Prize, Physics, 1933] was the first to realize this fact when he attempted to construct a relativistic wave equation for the electron in 1928 (the Schrodinger equation was not relativistic). The mathematics implied the existence of positive electrons, which later turned out to be positrons.

## 2.1 Mass

A particle’s mass indicates how strongly it interacts via the gravitational force. The mass of the electron is

$$m_e = 9.109\,382\,6(16) \times 10^{-31} \text{ kg},$$

or, with our typical precision,  $m_e \approx 9.11 \times 10^{-31} \text{ kg}$ . Rather than using the SI unit of kilogram, a common practice is to quote particle masses in terms of their “rest energy.” Einstein’s relativistic equivalence  $E_0 = mc^2$  means that the electron’s rest energy is  $m_e c^2 \approx 8.19 \times 10^{-16} \text{ J} \approx 0.511 \text{ MeV}$ . (Sometimes, physicists omit the factor  $c^2$  because it is clear from the context that the mass is being quoted in energy units.) It is common to quote a particle’s rest energy in millions of electron volts (MeV), rather than Joules. The other massive leptons, the muon and tauon, are *identical* to the electron, except for their mass: they interact in exactly the same manner. The accepted values of the lepton masses are

$$\begin{aligned} m(e^-) &= 0.510\,998\,910(13) \text{ MeV} \\ m(\mu^-) &= 105.658\,3668(38) \text{ MeV} \\ m(\tau^-) &= 1\,776.99(29) \text{ MeV} \end{aligned}$$

The first thing to notice is the progression of larger masses with the  $\mu^-$  and  $\tau^-$  leptons. This increasing mass is characteristic of the quarks and neutrinos as well. In fact, there are three “families” (or generations) of leptons and quarks, each composed of a lepton, its corresponding neutrino, and two quarks. The following table organizes them in this way.

Leptons	Quarks	
$e^-$	u	light
$\nu_e$	d	
$\mu^-$	c	↓
$\nu_\mu$	s	↓
$\tau^-$	t	heavy
$\nu_\tau$	b	

The first family is the lightest, and each successive family is heavier than the previous. Similar to the leptons, the top and bottom quarks are the most massive, and the up and down quarks are the least massive. The neutrino masses also increase, with  $\nu_e$  the lightest and  $\nu_\tau$  the heaviest. We will ignore the neutrino masses, however, because they are very small (on the order of a few eV). In fact, experiments are only able to set upper limits on their masses, and currently they are

$$\begin{aligned} m(\nu_e) &< 2.2 \text{ eV} \\ m(\nu_\mu) &< 170 \text{ keV} \\ m(\nu_\tau) &< 15.5 \text{ MeV} \end{aligned}$$

These mass limits can be determined in two ways. The first comes from the energy analysis of  $\beta$  decay, the prototype of which is the neutron decay in Eq. (2.3). These “direct” measurements yield the upper limits given above. The second, “indirect,” method consists of analyzing cosmological data, specifically the cosmic microwave background, and determining what neutrino mass would result in a universe different from the one we observe. This method gives an upper bound to the *sum* of all three neutrino masses of about 0.3 eV. In this book I will always assume these masses to be so small as to be ignorable in our calculations.<sup>4</sup>

The quark masses are more problematic because quarks have never been observed in isolation, and therefore we can only infer their masses from theoretical arguments. That is, measurements of energy released in particle reactions must be used along with a theoretical structure, such as QCD (quantum chromodynamics), in order to predict the quarks’ “free” mass.<sup>5</sup> For example, the up quark has a “free” mass of about 3 MeV/ $c^2$ , and the down quark about 6 MeV/ $c^2$ . The other quark masses are listed in the table below.

quark	mass (GeV/ $c^2$ )
u	0.003
d	0.006
c	1.5
s	0.5
t	175 <sup>6</sup>
b	4.5

Keep in mind that the values of these masses have large error bars, and that it really only makes sense to talk about the mass of particles that can exist in isolation. Particles that can be isolated, such as protons and neutrons, have masses that *can* be experimentally measured:

$$\begin{aligned} m_p c^2 &= 938.272\,029(80) \text{ MeV} \\ m_n c^2 &= 939.565\,360(81) \text{ MeV} \end{aligned}$$

Usually, we will not need to express them so precisely, so we can use  $m_p c^2 \approx 938 \text{ MeV}$  and  $m_n c^2 \approx 940 \text{ MeV}$ . However, we shall see that the mass *difference* between them is critical, so it’s important to remember that while they are both approximately 2000

<sup>4</sup>In 1998, the SuperKamiokande neutrino experiment determined that the different types of neutrinos can change into each other, which automatically implies that they must have mass. See Dennis W. Sciama, “Consistent neutrino masses from cosmology and solar physics,” *Nature* **348**, 617-618 (13 December 1990) for an interesting discussion.

<sup>5</sup>A quark’s free mass is the mass we would theoretically expect it to have if it could be freed from the confines of the proton or neutron. However, because the quarks can’t be isolated, their free mass depends sensitively on the theoretical assumptions made about the color force. The quarks’s *constituent* masses, on the other hand, can be calculated in a straightforward manner using the concept of binding energy  $B$ , introduced below, and ignoring any potential energy due to the strong force. See Problem 14. On the other hand, the free masses of nucleons (protons and neutrons) in nuclei *can* be determined using the binding energy concept — see Sec. (3.1) — because they can be isolated.

<sup>6</sup>This has only been recently determined accurately, from a collision of a proton and anti-proton, each with about 1 TeV of kinetic energy.

times more massive than the electron, the neutron is slightly heavier than the proton. Also, it's important to note that  $m_n - m_p = 1.29$  MeV, not 2 MeV as suggested by the approximate masses. When subtracting two nearly equal numbers, retaining sufficient digits is necessary.

If you look at the free masses of the up and down quarks, it's clear that the masses of the proton and neutron are not simply the sums of the masses of their constituent particles. How can that be? The reason is because there is a significant amount of potential energy involved in assembling the proton and neutron from the quarks, and this fact highlights the need to discuss our first “modern” concept in detail, that of binding energy.

## Binding Energy

The binding energy  $B$  of a compound particle of mass  $M$  is defined as the difference between the sum of the masses  $m_i$  of the individual constituent particles and the mass of the compound particle

$$B \equiv \left( \sum_i m_i - M \right) c^2. \quad (2.1)$$

The simplest example that illustrates binding energy is the deuteron (the nucleus of deuterium,  ${}^2\text{H}$ , also known as heavy hydrogen), which consists of one proton and one neutron.<sup>7</sup> The deuteron's binding energy can be calculated from the measured rest energy of the deuteron and the (isolated) masses of the proton and neutron:

$$\begin{array}{r} m_p c^2 \quad 938.272\ 029(80) \text{ MeV} \\ +m_n c^2 \quad 939.565\ 360(81) \text{ MeV} \\ -m_D c^2 \quad 1\ 875.612\ 82(16) \text{ MeV} \\ \hline = B \quad 2.224\ 57(20) \text{ MeV} \end{array}$$

This means that if we are able to combine a free proton and a free neutron to make a deuteron, we obtain  $\approx 2.22$  MeV of energy in return<sup>8</sup> — in the language of chemistry, it's an *exothermic* reaction. Where does the released energy go? It goes into the kinetic energy of the compound particle! In fact, combining two nucleons into a single nucleon is called *fusion*, so named because two or more particles “fuse” to form one particle. A more

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<sup>7</sup>You might think that the proton or neutron would be simpler, but they are three-particle systems, not two, and more important, as previously mentioned, the binding energy is not well defined when the constituent particles cannot be isolated.

<sup>8</sup>There is another unit of mass that is commonly used when binding energy calculations are made, and that is the “atomic mass unit,” or “u.” Here, the carbon-12 atom sets the scale so that  $m({}^{12}\text{C}) \equiv 12.00$  u exactly, and the conversion to kilograms is  $1 \text{ u} = 1.660\ 538\ 86(28) \times 10^{-27} \text{ kg} \approx 1.66 \times 10^{-27} \text{ kg}$ . The atomic mass approximately measures the “atomic number” of the nucleus, i.e., the number of protons and neutrons. Working in atomic mass units, but keeping only six decimal places, the calculation of the deuteron's binding energy is

$$\begin{array}{r} m_p \quad 1.007\ 276 \text{ u} \\ +m_n \quad 1.008\ 664 \text{ u} \\ -m_D \quad 2.013\ 553 \text{ u} \\ \hline = B/c^2 \quad 0.002\ 388 \text{ u} \end{array}$$

and converting to electron volts ( $1 \text{ u} \approx 931.494 \text{ MeV}/c^2$ ) I obtain  $B \approx 2.22 \text{ MeV}$ .

complicated fusion reaction occurs in the core of the sun, where four protons fuse to form one  $\alpha$  particle (the nucleus of helium,  ${}^4\text{He}$ ).<sup>9</sup> That reaction, of course, is also exothermic, and is what powers the sun. These considerations lead us to the conclusion that mass is a form of potential energy:

Mass (and binding energy) is potential energy.

Another, more familiar, example is the case of the Earth and a 1-kg ball. If these two objects are infinitely far away from each other, they have well-defined masses,  $M_E$  and  $m = 1$  kg, that can be measured precisely. As we bring the ball to the surface of the Earth, the Earth-ball system loses potential energy. The amount lost can be calculated from our knowledge of gravitational potential energy

$$\begin{aligned}\Delta U &= \frac{GM_E m}{R_E} = 6.26 \times 10^7 \text{ J} \\ &= 3.89 \times 10^{26} \text{ eV} \\ &= 6.93 \times 10^{-10} \text{ kg } c^2,\end{aligned}\tag{2.2}$$

where I used the constants  $G = 6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2$ ,  $M_E = 5.98 \times 10^{24} \text{ kg}$ , and  $R_E = 6.37 \times 10^6 \text{ m}$ . Where did that energy ( $\approx 63 \text{ MJ}$  worth) go? It went into heat, sound, etc. The Earth eventually radiated away the heat energy, and the sound energy also travels off. This leads me to make the following claim:

CLAIM: The compound object (Earth and ball) has a smaller mass than the two separate objects combined!

The mass lost is exactly  $6.93 \times 10^{-10} \text{ kg}$ , the mass equivalent of the potential energy difference. Of course, this mass difference is extremely tiny, and cannot be measured with present day experiments, but it must exist, nonetheless. If I were to separate the Earth and ball again, it would take 63 MJ of work, and when I measured their masses, they would “recover” their original masses, because I have put energy into the system with the work that I did to separate them.

While the underlying physics of binding energy and the mass of compound objects is identical in both the classical case (Earth and ball) and the subatomic case (proton and neutron), there are some subtle differences. In the classical case, the binding energy is small compared with the rest energies of the particles involved, and we tend to think of the constituent objects retaining their identity regardless of whether they are far apart or combined. However, with subatomic particles it is often the case that the binding energy is a significant fraction of the rest energies, and the compound object is usually considered to be a different object—the constituent particles lose their identity. For example, a proton “consists” of two up quarks and a down quark: uud. However, there is another compound particle, called  $\Delta^+$ , which also consists of two up quarks and a down quark. But the mass of the  $\Delta^+$  is 1232 MeV, and it is considered to be a different particle from a proton. The

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<sup>9</sup>Note that Eq. (2.1) does not state *how* the constituent particles combine to form the compound particle; other laws of physics are needed to determine that.

mass is different because the three quarks are in a different quantum state than the proton (that is, they occupy a different energy level), which means that the proton and  $\Delta^+$  have different binding energies, and hence different masses.<sup>10</sup> On the other hand, when the 1-kg ball is on the surface of the Earth, we still consider the Earth and the ball to be separate, distinct, objects.

A final example of an interaction involving the mass-energy relationship (and anti-matter) is the decay of the neutron. A free neutron (not one that is bound in an atomic nucleus) spontaneously decays into a proton with a half life of 10.23 minutes.<sup>11</sup> The reaction equation is

$$n \rightarrow p + e^- + \bar{\nu}_e, \quad (2.3)$$

where the electron and antineutrino must be part of the decay products in order to conserve both charge and the “lepton number,” a quantity that is characteristic of the weak force. The lepton number is another quantum number that we will discuss later. Now, the neutron is NOT comprised of a proton and electron, so there is no binding energy, but we *can* calculate the energy released in this decay by computing the difference in rest energies before and after the decay

$$Q \equiv \left( \sum_{\text{initial}} m - \sum_{\text{final}} m \right) c^2. \quad (2.4)$$

The symbol  $Q$  (called “reaction energy”) is used rather than  $B$  to denote that this is not a compound particle, but that there is some energy that is released in the reaction. If  $Q > 0$  there is energy released (exothermic) and a spontaneous decay is energetically possible. However, if  $Q < 0$  then simply because of energy conservation the decay is not allowed. In the case of the neutron decay, I obtain (and you should check the math)  $Q \approx 0.782$  MeV.<sup>12</sup> What happens to this released energy? As before, it goes to the kinetic energy of the product particles, i.e., those on the right-hand-side of the reaction equation. In some sense, you can think of a neutron as being in a higher potential energy state than a proton (because it is more massive), and since objects like to lower their potential energy, the neutron wants to turn into a proton.<sup>13</sup>

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<sup>10</sup>This difference in binding energies can be traced to a difference in the spins of the quarks – see Sec. 2.3.

<sup>11</sup>Radioactive decay and the concept of half-life is discussed in detail in Chapter 3.

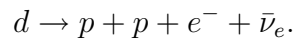
<sup>12</sup>In calculating this value, I ignored the small neutrino mass. Since the upper limit on the rest energy of the electron neutrino (and antineutrino) is about 2.2 eV, it doesn’t affect the calculation at this level of precision.

<sup>13</sup>As far as we know, the proton is a stable particle because there is no baryon that is less massive for it to decay into, although the possibility that the half-life for proton decay is so long that we haven’t noticed it yet is an active area of research. Baryon number is another quantum number that must be conserved—we will investigate this concept later. The electron is the lightest lepton and hence it, too, is stable against spontaneous decay.



Why are neutrons in nuclei stable? If all neutrons were unstable to  $\beta$  decay, then there would be no heavy atoms, no life in the universe, and we would not exist. To understand the stability of nuclei, the concepts of binding energy and reaction energy give a simple explanation.

Consider the case of a deuteron, the nucleus of deuterium (also known as  ${}^2\text{H}$ , or heavy hydrogen). It consists of a proton and a neutron, and if the neutron decayed it would emit an electron and an antineutrino, leaving two protons. Two protons (also known as a “diproton”) do not form a stable nucleus, so they immediately split into two separate protons. We therefore must look at the reaction energy of the following reaction



The deuteron has a mass of  $1875.613 \text{ MeV}/c^2$ , while the combined masses of the products is  $1877.055 \text{ MeV}/c^2$ . This means that  $Q = -1.442 \text{ MeV}$ . Thus the neutron in the deuteron cannot decay spontaneously! At least  $1.4 \text{ MeV}$  must be added to “cause” this reaction. Note that this is a different calculation than the binding energy of the deuteron.

Why is the deuteron stable but the diproton ( ${}^2\text{He}$ ) and the dineutron are not? To answer this question satisfactorily requires a knowledge of advanced quantum mechanics and nuclear physics, including spin and the Pauli exclusion principle.

## Classification according to mass, and particle names

There were three known particles in the 1930s: the electron, proton and neutron. With a mass of about  $0.5 \text{ MeV}/c^2$  the electron was the lightest, and with a mass of about  $1000 \text{ MeV}/c^2$  the nucleons (the collective name for the proton and neutron) were heavy. With the discovery in 1937 of an intermediate mass particle, about  $100 \text{ MeV}/c^2$ , in cosmic rays, the particles were given “nicknames” according to their mass. Since the electron was light, it was called a “lepton,” from the Greek word *leptos* ( $\lambda\epsilon\pi\tau\omicron\varsigma$ ) meaning “small.” And, since the nucleons were massive, they were called “baryons,” from the Greek *barys* ( $\beta\alpha\rho\nu\varsigma$ ) meaning “heavy.” The cosmic ray particle was therefore called a “meson,” from the Greek word *mesos* ( $\mu\epsilon\sigma\omicron\varsigma$ ) meaning “middle.”<sup>14</sup> It wasn’t realized until later that the intermediate mass cosmic ray particle was actually the  $\mu^{-}$  lepton, although it was originally called the  $\mu$ -meson.

Under our current naming scheme, however, a baryon has come to mean a particle that is made up of three quarks (such as a proton or neutron), a meson has come to mean a particle that is made up of a quark–anti-quark pair, and leptons are the elementary particles that do not interact via the strong force. Since any three of the six flavors of quarks can combine to form a baryon, there are 56 possible combinations, although there

<sup>14</sup>Interestingly, Hideki Yukawa, who predicted the existence of an intermediate-mass particle in 1935, initially proposed to call it a “mesotron,” in keeping with the name of the electron. However, Werner Heisenberg noted that the correct Greek word was *mesos* and it had no “tr.”

are more than 56 different baryons since it is possible for the same set of quarks to have different binding energies (see the comparison between the proton and  $\Delta^+$  above). For example, the sigma ( $\Sigma$ ) baryons are combinations of one strange quark and two up or down quarks. Their quark content and masses are

$\Sigma^+$	uus	1189.4 MeV
$\Sigma^0$	uds	1192.5 MeV
$\Sigma^-$	dds	1197.4 MeV

The pi ( $\pi$ ) mesons are composed of different combinations of up and down quarks and their anti-particles:

$\pi^+$	$u\bar{d}$	139.6 MeV
$\pi^0$	$(u\bar{u}-d\bar{d})/\sqrt{2}$	135.0 MeV
$\pi^-$	$d\bar{u}$	139.6 MeV

Note that the  $\pi^0$  meson is actually a *superposition* of quark states. This means that when an experimenter “looks” at a  $\pi^0$  meson, 50% of the time they will “see” the  $u\bar{u}$  combination, and the other 50% they will see  $d\bar{d}$ . The factor of  $\sqrt{2}$  indicates this mathematically.<sup>15</sup> This is just one of the strange features of quantum mechanics. It should look somewhat familiar, however, because it is similar to the fact that the general solution of a linear second-order differential equation is a linear combination (or superposition) of two independent solutions. In the same way, some quarks (and baryons and mesons) can be linear combinations of two (or more) independent quark states. Also note that the  $\pi^+$  and  $\pi^-$  are antiparticles of each other, and hence have the same mass, and that the  $\pi^0$  is its own antiparticle.

The  $\Sigma$  and  $\pi$  particles are just a few of the possible baryon and meson combinations that can be constructed with the six known quark flavors. A short list, along with their quark constituents, are shown in Table 2.1. At this time, no other combinations of quarks other than  $qqq$  and  $q\bar{q}$  have been observed, although there have been searches for exotic combinations such as so-called “penta-quarks,” made up of four quarks and one anti-quark:  $qqqq\bar{q}$ . In some sense this looks like a baryon and meson bound together. Either these do not exist, or their lifetimes are too short to measure.

## 2.2 Electric Charge

A particle’s charge indicates how strongly it interacts via the electromagnetic force. In addition, however, charge is quantized; that is, it appears in nature only as integer multiples of the fundamental unit of charge,  $e$ , which happens to be the charge of the electron,

$$q_e = -e = -1.602\,176\,53(14) \times 10^{-19} \text{ C},$$

<sup>15</sup>The probability of each state occurring is equal to the square of the numerical coefficient that multiplies that state.

baryon		mass (MeV/c <sup>2</sup> )	meson		mass (MeV/c <sup>2</sup> )
p	uud	938.3	$\pi^+$	ud	139.6
n	udd	940.6	$\pi^0$	$(u\bar{u}-d\bar{d})/\sqrt{2}$	135.0
$\Sigma^+$	uus	1189.4	$\pi^-$	$d\bar{u}$	139.6
$\Sigma^0, \Lambda^0$	uds	1192.5, 1115.7	$K^0, \bar{K}^0$	$d\bar{s}, \bar{d}s$	497.6
$\Sigma^-$	dds	1197.4	$K^+, K^-$	$u\bar{s}, \bar{u}s$	493.7
$\Xi^-$	dss	1321.7	$\eta$	$(u\bar{u}+d\bar{d}-2s\bar{s})/\sqrt{6}$	547.9
$\Xi^0$	uss	1314.9	$\eta'$	$(u\bar{u}+d\bar{d}+s\bar{s})/\sqrt{3}$	957.7

Table 2.1: Tables of the light (u, d, s quarks only), spin  $\frac{1}{2}$  baryons and the light, spin 0 mesons. Note that the  $\Sigma^0$  and  $\Lambda^0$  have the same quark content but different masses. The heavier one,  $\Sigma^0$ , is an electromagnetic excited state and decays in about  $7 \times 10^{-20}$  s into the lighter one,  $\Lambda^0$ . This process is identical to that which occurs when an electron in an excited state (of higher energy) in an atom decays into a lower energy level. In both cases the decay is accompanied by the emission of a photon equal to the energy difference. Here, the energy difference is indicated by the mass difference, and a gamma ray of wavelength 2.57 fm is emitted. Also note that the  $\pi^0$ ,  $\eta$ , and  $\eta'$  are all neutral mesons, but are just different linear combinations of the same set of three quark—anti-quark pairs.

or, for our purposes  $e \approx 1.60 \times 10^{-19} \text{ C}$ .<sup>16</sup> The other massive leptons (muon and tau) have the same negative charge as the electron, and the neutrinos are neutral. In fact the word neutrino was proposed by Wolfgang Pauli in 1930, and means “little neutral one” in Italian.

What about the quarks? What are their charges? The quarks come with *fractional* charges, that is, submultiples of  $e$ ! For example, the charge on the up quark is  $q_u = +\frac{2}{3}e$ , and that on the down quark is  $q_d = -\frac{1}{3}e$ . At first sight, this appears strange. How can any particle have a fractional charge? There are two ways to reconcile this with the proposed quantization of charge. First, all this really says is that the fundamental unit of charge is not  $e$ , but is  $\frac{1}{3}e$ . Charge is still quantized and all particles have integer multiples of this fundamental unit. Second, because quarks are never observed in isolation (they always appear in groups of 3 — baryons — or in a quark—anti-quark pair — mesons), the charges of particles that *can* exist in isolation must be a multiple of  $e$ . So the proton and neutron have integer charges

$$q_p = \left( +\frac{2}{3} + \frac{2}{3} - \frac{1}{3} \right) e = e$$

$$q_n = \left( +\frac{2}{3} - \frac{1}{3} - \frac{1}{3} \right) e = 0.$$

This second fact was helpful in convincing skeptics about the usefulness, and the ultimate reality, of quarks. The charges on the quarks are

u	c	t	$+\frac{2}{3}e$
d	s	b	$-\frac{1}{3}e$

<sup>16</sup>Keep in mind that  $e$  is a positive quantity, and that negative particles have charges that are integer multiples of  $-e$ .

One important fact about electric charge is that it is absolutely conserved. There is no way to transform charge into energy, as there is with mass, so the charge of compound particles is just the sum of the charges of the constituent particles. This conservation law is sometimes stated as

Electric charge is neither created nor destroyed.

Why? We don't know. All we know is that the violation has never been observed, so until then it remains a "law."

## Interactions

Gravity and electromagnetism are the two classical (non-quantum) forces. The other two "forces," the weak and strong nuclear forces are inherently quantum mechanical in nature. For this reason, you won't be able to fully understand them in detail until *after* a thorough study of quantum mechanics; however, we can discuss them now using some of the classical concepts that you already know, such as energy and momentum. This quantum nature leads to a new way of describing and understanding these forces that is completely different from our previous descriptions. Previously, physicists have thought about forces in two different ways. First, as "action-at-a-distance," propounded by Newton with his Universal Law of Gravitation.<sup>17</sup> Second, utilizing the concept of a "field," devised by Faraday (and honed by Maxwell) to explain the electric and magnetic forces. Gravity can also be described in terms of the gravitational "field," both in the Newtonian limit and in general relativity. Due to the necessity of using quantum ideas to describe the weak and strong nuclear forces, we are forced to use quantum field theory, and this third description postulates the existence of exchange particles.

For example, two electrons repel each other not because of a mysterious action-at-a-distance Coulomb force, nor even the electric field, but by exchanging photons. Just like two ice skaters who, throwing a ball back and forth, appear to repel each other, electrons exchange photons, which, due to the conservation of momentum, exert impulses on the electrons, and they appear to repel each other. The photon, therefore, is the exchange particle that "mediates" the electromagnetic force. The ice skater analogy does not work for particles that attract each other, but the concept is still valid. In his thinking that led to the proposal of the meson, the mediating particle that held the nucleons together in the nucleus, Hideki Yukawa [Nobel Prize, Physics, 1949] wrote



If one visualizes the [nuclear] force field as a game of "catch" between protons and neutrons, the crux of the problem would be the nature of the "ball" or particle.

---

<sup>17</sup>Newton had a philosophical objection to action-at-a-distance, which he expressed in a letter in 1692: "That gravity should be innate, inherent and essential to matter, so that one body may act upon another at a distance through a vacuum, without the mediation of anything else by which their action and force may be conveyed from one to another, is to me so great an absurdity that I believe no man who has in philosophical matters a competent faculty of thinking can ever fall into it."

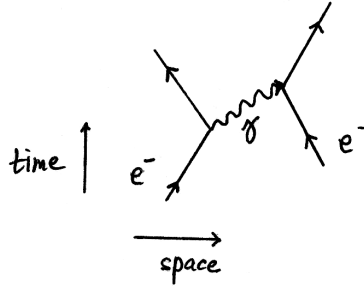


Figure 2.3: Feynman diagram depicting the electromagnetic interaction between two electrons. Only one spatial dimension is shown, and the time axis is conventionally drawn upward. Matter particles are depicted by straight lines, while the photon, an exchange particle, is shown as a wavy line and indicated by a  $\gamma$ .

This view has three aesthetically pleasing features. First, all interactions are “local,” which means that particles must be in the same location for any effect. Second, it nicely explains the  $1/r^2$  nature of the electric and gravitational forces: the “density” of mediating particles must decrease as  $1/r^2$  from the “source” particle, a simple geometrical effect. Finally, effects are not instantaneous, but take a finite time as the mediating particle traverses the intervening distance. A graphical method of describing interactions that automatically displays the first and third of these features is called a “Feynman diagram.” A Feynman diagram of the electromagnetic interaction between two electrons is shown in Fig. 2.3. This is similar to a position-time graph from elementary mechanics, where the trajectories of all particles are shown. Note that the photon ( $\gamma$ ) comes into and out of existence when it interacts with an electron, and each electron undergoes a momentum change. At each “vertex” all quantities such as charge and other quantum numbers are conserved, the only exception being energy. That is, energy is created when the photon is emitted by the first electron, and then lost when the photon is absorbed by the second electron. The time interval over which the photon exists (and during which energy conservation is violated) is short enough so that Heisenberg’s uncertainty principle is not violated<sup>18</sup> (see App. I). A photon of this type is called a “virtual” photon, so that in this third picture of interacting electrons, they do not exert a Coulomb force (at a distance), nor do they create an electric field, but they exchange virtual photons in order to exchange momentum and repel each other.

To the extent that each of the fundamental forces can be described by a quantum field theory, each force must be mediated by an exchange particle. If the photon mediates the electromagnetic force, what particles mediate the other forces? They are listed below, along with their mass, charge, spin, and color.

<sup>18</sup>Technically, this description is not correct, as quantum mechanics does not allow energy violation. However, the correct description is more mathematically sophisticated than we have space for here. If you’re interested, see Griffiths, *Introduction to Quantum Mechanics*, page 118.

		mass	charge	spin	color
gravity	graviton <sup>19</sup>	0	0	2	no
E&M	photon	0 <sup>20</sup>	0	1	no
color	gluons	0	0	1	yes
weak	$W^\pm$	80.4 GeV/ $c^2$	$\pm e$	1	no
	$Z^0$	91.2 GeV/ $c^2$	0	1	no

Our “zoo” of particles is now complete. We have 12 particles of matter, 12 of anti-matter, and 13 “gauge bosons.” (There are 8 types of gluons, which are distinguished because they also carry color.)

We will see later that when the mass of the mediating particle is zero, then the interaction is long range. This makes sense for gravity and electromagnetism, in that they both are  $1/r^2$  forces which means that although they become weaker with distance, they never go to zero. The weak force, on the other hand, is extremely short range because the  $W$  and  $Z$  bosons are very massive. This means that the weak force is very “weak” (hence the name) and particles must be very close to interact in this manner. The color force is also long range, but it turns out to become stronger with distance rather than weaker. The strong force, which is the force between baryons and mesons, is a short range force that is the residual, or “leftover,” color force between objects that are color neutral.<sup>21</sup>

## 2.3 Spin

Our third property of interest, after mass and electric charge, is spin. It is a property that does not relate specifically to one of the four fundamental forces, but is an inherently quantum property, and therefore nicely illustrates the differences between the quantum world and the classical world. A discussion of spin is a nice place to introduce the Heisenberg uncertainty principle and the Pauli exclusion principle so that you can see just how the quantum world operates and how it differs from the world you know.

Spin is also known as “intrinsic” angular momentum. The Earth, for example, has both orbital angular momentum due to its revolution about the sun, as well as “spin” angular momentum: it rotates on its axis once every 24 hours.<sup>22</sup> In the same way the elementary particles, such as electrons, have intrinsic angular momentum that appears to be due to their actual spin about an internal axis. There are two problems with this explanation, however. First, if elementary particles have no size, then in the definition of angular momentum,  $\vec{r} \times \vec{p}$ , the factor  $\vec{r}$  is zero, which means that the angular momentum must be zero. Second, in Sec. 1.3 I claimed that angular momentum is quantized, and that the allowed values for the angular momentum quantum number  $\ell$  are integers. We’ll see in Chapter 7 that this restriction comes from the fact that when you turn around  $360^\circ$ , you must see the same thing. However, measurements show that the spin angular momentum quantum number can also be a half-integer, which means that it is somehow different from

<sup>19</sup>The graviton, while postulated to exist, has not yet been observed.

<sup>20</sup>The current upper bound for the photon mass is  $1.2 \times 10^{-51}$  g. (Luo *et al.*, Phys. Rev. Lett. **90** 081801, 2003)

<sup>21</sup>See Sec. 2.5 for more details.

<sup>22</sup>The “spin” angular momentum of the Earth about its geographic axis is about  $7 \times 10^{33}$  J s.

our classical view of angular momentum. These two problems remind us that while the *mathematics* of spin is identical to that of the usual angular momentum, it is really only an analogy, and it has no classical counterpart.

In classical mechanics, the symbol  $\vec{L}$  is used for all types of angular momentum, but in particle physics  $\vec{L}$  represents the *orbital* angular momentum of one particle about another (similar to the angular momentum of the Earth orbiting about the Sun), and  $\vec{S}$  is used to denote the *intrinsic* angular momentum (i.e., spin) of the particle. So, just like  $L$ ,  $S$  is quantized, and its value depends on a quantum number. The quantum number is  $s$  and can take on the values  $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \infty$ . All twelve of the elementary particles (and their antiparticles) have  $s = \frac{1}{2}$ . It is usually stated, for example, that the electron is a “spin  $\frac{1}{2}$ ” particle. The magnitude of the angular momentum vector of a particle with spin quantum number  $s$  is given by  $S = |\vec{S}| = \sqrt{s(s+1)}\hbar$ , where  $\hbar = h/2\pi$ , and  $h$  is Planck’s constant. The manner in which  $S$  depends on  $s$  is exactly the same as the way that  $L$  depends on  $\ell$ . The physics of the two types of angular momentum,  $\vec{L}$  and  $\vec{S}$ , are identical, except for the fact that  $s$  can take on half-integer values while  $\ell$  is restricted to integer values. In the discussion that follows, I will focus on spin ( $S$ ), but the results will be equally valid for  $L$ .

While subatomic particles do have spin angular momentum, it is not correct to conclude that they are actually “spinning.” Our model of the elementary particles is one of mathematical points, and, of course, points cannot spin. One way to see that this picture of a spinning sphere must be wrong is to calculate how fast the electron and proton would be spinning if they actually had  $\frac{\sqrt{3}}{2}\hbar \approx 9 \times 10^{-35}$  J s of angular momentum. A point on the “equator” would be traveling at a speed greater than the speed of light. (See Problem 24.)

Angular momentum is a vector, of course, and to specify it completely we need a magnitude and direction. The magnitude is given by  $s$ , but what is the direction? It turns out that we don’t know! We can determine one component of the vector, usually the  $z$  component, but no others. This turns out to be a consequence of Heisenberg’s Uncertainty Principle.

### 2.3.1 The Heisenberg Uncertainty Principle and angular momentum

*The more precisely the position is determined, the less precisely the momentum is known in this instant, and vice versa. — Werner Heisenberg*

This quote embodies the most familiar form of Heisenberg’s principle, which is that there is a special relationship between momentum and position: they can’t be known simultaneously with arbitrary precision. The exact mathematical statement of this principle is

$$\Delta p_x \Delta x \geq \frac{\hbar}{2}, \quad (2.5)$$

where  $\Delta$  signifies the “uncertainty” of a particular variable. We will discuss this in more detail later in Chapters 6 and 7, and there you’ll see that there is a precise definition of  $\Delta x$ , namely, the predicted standard deviation of a series of measurements. However, for now it is sufficient to realize that this principle implies that there is a fundamental

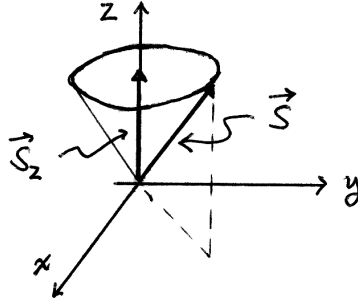


Figure 2.4: Relationship between the spin vector  $\vec{S}$  and its  $z$  component  $S_z$ .

limit to our knowledge—at a given instant we cannot know, with arbitrary precision, both the momentum and position of a particle, regardless of the resolution of our measuring apparatus. Unfortunately, the name is somewhat misleading. I feel that a better name would be the “principle of indeterminacy,” because it is not that we are ‘uncertain’ of the values of position and momentum. It is much deeper than that. The values of position and momentum simply cannot be measured simultaneously: they are ‘indeterminant.’ As Abraham Pais says,

“I have often felt that the expression ‘uncertainty relation’ is unfortunate since it has all too often invoked imagery in popular writings utterly different from what Heisenberg very clearly had in mind, to wit, that the issue is not: what don’t I know? but rather: what can’t I know? In common language, ‘I am uncertain’ does not exclude ‘I could be certain’. It might therefore have been better had the term ‘unknowability relations’ been used. Of course one neither can nor should do anything about that now.” — *Inward Bound*, page 262.

It turns out that there is also an uncertainty relationship between *angular* momentum and *angular* position (i.e., angle), and another version of Heisenberg’s principle states that we cannot know both of those quantities precisely. Mathematically,

$$\Delta S_z \Delta \phi \geq \frac{\hbar}{2}, \quad (2.6)$$

where  $\phi$  is the usual cylindrical azimuthal angle (see Fig. 2.4). That is,  $\Delta S_z$  is the “error” (or indeterminateness) in the measurement of  $S_z$ , and likewise  $\Delta \phi$  is the indeterminateness in the measurement of  $\phi$ . It turns out that we are able to know  $S_z$  exactly, so that  $\Delta S_z$  must be zero, and therefore  $\Delta \phi$  must be infinitely large. What this means is that while we know the magnitude of  $\vec{S}$  and its  $z$  component, the vector itself lies somewhere (we don’t know where) in the cone in Fig. 2.4. Or, put another way, we have no knowledge of the  $x$  or  $y$  components of  $\vec{S}$ .

The fact that we are able to know  $S_z$  exactly implies that there must be a quantum number associated with the  $z$  component of the spin vector. There is, and it is called  $m_s$ . The allowed values of  $m_s$  are  $s, s-1, \dots, -s$ . That is,  $m_s$  can range from  $s$  to  $-s$ , but each



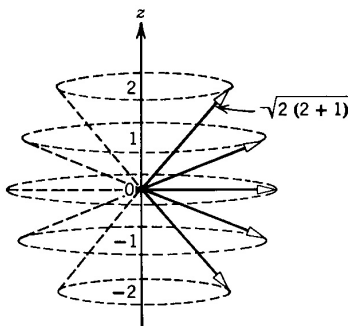


Figure 2.5: Possible orientations of the spin angular momentum vector for a particle with  $s = 2$ . Note that the angle between  $\vec{S}$  and the  $z$  axis is observable, but not the  $x$  or  $y$  components. Figure 7-12 from Eisberg and Resnick, *Quantum Physics*, page 258.

allowed value must differ from its neighbor by 1. For the case of  $s = \frac{1}{2}$ , the allowed values of  $m_s$  are  $\pm\frac{1}{2}$ . Since the  $z$  component can point in the positive or negative direction, this is the origin of the notion that electrons can be either “spin up” or “spin down.” For the case of  $s = 1$ , the allowed values of  $m_s$  are  $m_s = 1, 0, -1$ , for the case of  $s = \frac{3}{2}$ , the allowed values of  $m_s$  are  $m_s = \pm\frac{3}{2}, \pm\frac{1}{2}$ , and for the case of  $s = 2$ , the allowed values of  $m_s$  are  $m_s = 2, 1, 0, -1, -2$ , (see Fig. 2.5). As mentioned on page 8, each quantum number allows the calculation of a physical quantity. Which quantity does  $m_s$  represent? The  $z$  component of the spin vector,  $S_z$ , and it takes on the possible values  $S_z = m_s\hbar$ . Notice that the allowed values of  $m_s$  and  $s$  automatically require that the  $z$  component of the spin vector is smaller than the magnitude of the spin vector itself. This, of course, is consistent with our basic definition of a vector.

The fact that we are unable to know the actual direction of the spin vector is one of the properties of quantum mechanics that many people find “weird.” An “explanation” for this quantum weirdness is stated nicely by Eisberg and Resnick:

The fact that [the wave function] does not describe a state with a definite  $x$  and  $y$  component of angular momentum, because these quantities are not quantized, is mysterious from the point of view of classical mechanics. ... [This] is a consequence of the fact that there is an uncertainty principle relation which states that no two components of an angular momentum can be known simultaneously with complete precision. Because the  $z$  component of angular momentum has the precise value  $m_s\hbar$ , the relation requires that the values of the  $x$  and  $y$  components be indefinite. But one thing *can* be said about the values of these components: ... their average values ... both equal zero. So, although the particular value of  $S_x$  that would be obtained in any particular measurement cannot be predicted, it *can* be predicted that the average value that would be obtained in a set of measurements of  $S_x$  is zero. And similarly for  $S_y$ .<sup>23</sup>

<sup>23</sup>Eisberg and Resnick, *Quantum Physics*, page 258.

The quantum number  $s$  is a fundamental property, an unchangeable attribute, of each elementary particle, but  $m_s$  can change. That is, the magnitude of a particle's spin vector is fixed, but it is allowed to change direction.

## Nucleons and the addition of spin

Protons and neutrons both are also spin  $\frac{1}{2}$ . Why? How do the spins of the constituent quarks, combined with the possible orbital angular momentum of the quarks about each other, add to give the nucleons a total intrinsic angular momentum quantum number of  $s = \frac{1}{2}$ ? A simple model of a nucleon is to assume that the quarks have zero orbital angular momentum.<sup>24</sup> Since spin is a vector, adding two spins must be done vectorially; but we don't know which direction  $\vec{S}$  points, so the formula  $\vec{S} = \vec{S}_1 + \vec{S}_2$  is impossible to evaluate. A simple algorithm for adding the angular momenta of spin  $\frac{1}{2}$  particles is to note that although  $\vec{S}$  can point in any direction, they add *as if* they were parallel or anti-parallel.<sup>25</sup> Therefore, if  $s_1 = \frac{1}{2}$  and  $s_2 = \frac{1}{2}$ , then  $s$  can take on two possible values:

$$\begin{aligned} s &= 1 && \text{if the spins are parallel} \\ s &= 0 && \text{if the spins are anti-parallel} \end{aligned}$$

(The two cases above refer to when the  $z$  components are parallel or anti-parallel.) If we add a third spin  $s_3 = \frac{1}{2}$  to the  $s = 0$  state we end up with  $s = \frac{1}{2}$ . If we add it to the  $s = 1$  state, there are two possibilities again  $s = \frac{1}{2}, \frac{3}{2}$ . Therefore, our conclusion is:

The only possible 3-quark spins are  $\frac{1}{2}$  and  $\frac{3}{2}$ .

Recall that the proton consists of two up quarks and a down quark. But on page 21 we saw that there is another uud baryon that has  $s = \frac{3}{2}$  and a mass of 1232 MeV—it's called the  $\Delta^+$ . In addition, another particle, the  $\Delta^0$ , is udd like the neutron, and is spin  $\frac{3}{2}$  and has the same mass as the  $\Delta^+$ . These particles, while having the same quark constituents, are in a different spin state, and the energy due to the interactions of their spins causes them to be in a different energy state, and hence have a different mass. The baryons listed in Table 2.1 have  $s_1 = \frac{1}{2}$ , and there is a similar listing of the baryons that have  $s_1 = \frac{3}{2}$ .

The  $\Delta$ -particles are sometimes called “resonances,” and were discovered in 1949 by Fermi and Anderson at the University of Chicago by scattering  $\pi^\pm$  mesons off protons in an  $H_2$  target. The term resonance, rather than particle, was used because they live for such a short period of time before spontaneously decaying that their existence is inferred from their decay products rather than direct observation of the particles themselves. That is, the “scattering” can be thought of as  $\pi + p \rightarrow \Delta$  followed by a decay  $\Delta \rightarrow p + \pi$ . The  $\Delta$  state has a half-life of only about  $5 \times 10^{-24}$  s, but has all the properties of a (unstable) particle.

<sup>24</sup>Recent work at Brookhaven National Laboratory seems to point to the conclusion that the quark spins contribute only 20% of the total angular momentum of the proton. Presumably, then, the other 80% is due to orbital angular momentum. The details of this interaction is not well understood.

<sup>25</sup>In reality, it is the  $z$  components that are adding, but without delving into the mathematical formalism, it is equivalent to think of the spins as “up” or “down.”

### 2.3.2 The proton-electron model of the nucleus

Before 1932, when James Chadwick [Nobel Prize, Physics, 1935] discovered the neutron, it was not known what the constituents of nuclei were. One proposed model, the leading contender since 1913, was that it was composed of protons and electrons. This made sense because in the 1920s the only particles that were known were the electron and the proton (which at that time was defined to be the smallest nucleus known, that of hydrogen). It was known that the electron was negatively charged and not very massive. It was also known that the proton was positively charged and much more massive than the electron (about 2000 times), so that the logical conclusion was that the mass of atoms must be made up primarily of protons. These pieces of information came from Thomson's work on the electron, Rutherford's work on the atom, and Bohr's work on the hydrogen atom.



Hence, a nucleus of mass number<sup>26</sup>  $A$  must have  $A$  protons to account for its mass. But it also had to have a charge  $Ze$  (where  $Z$  is the atomic number) to neutralize the  $Z$  electrons that were *not* part of the nucleus. For this to be the case, the nucleus must include  $A - Z$  electrons, so that its charge is  $A(+e) + (A - Z)(-e) = Ze$ . These electrons reduced the charge but did not add significantly to the mass.

As we know now, there are several reasons why this model was incorrect, and one of them is spin.<sup>27</sup> In 1928, Ralph Kronig observed the spectra of  $N_2^+$  ions<sup>28</sup> and was able to determine that the spin quantum number of the nucleus of  $^{14}\text{N}$  was  $s = 1$ . The current view is that this nucleus contains 7 protons and 7 neutrons. However, the proton-electron model predicted 14 protons and 7 electrons. The difference between these models is that one has an even number of spin  $\frac{1}{2}$  particles and the other has an odd number of spin  $\frac{1}{2}$  particles. From our discussion above of how spins add, we see that a compound particle that consists of an even number of spin  $\frac{1}{2}$  particles must have an integer spin, and an odd number must have a half-integer spin. The proton-electron model, therefore, would predict a half-integer spin for the nucleus of  $^{14}\text{N}$ , which is not what is observed. The proton-neutron model, however, has no such discrepancy.

A further clue was that was the observation that cadmium has at least one isotope whose nucleus has  $s = \frac{1}{2}$ . You can show (Problem 26) that the proton-electron model predicts that the nuclei of all isotopes of cadmium should have integer spin. In fact, you can also show (Problem 27) that for a given isotope, if  $N$  is even, the two models (proton-electron and proton-neutron) give the same prediction, but when  $N$  is odd, they give opposite predictions concerning the nuclear spin. Among the first ten elements (hydrogen through neon), there are only nine stable isotopes (out of 20) that have odd  $N$ :  $^2\text{H}$ ,  $^3\text{He}$ ,

<sup>26</sup>For the definitions of  $A$ ,  $Z$ , and  $N$  see the beginning of Chapter 3.

<sup>27</sup>Another reason is size. It was a widespread view that the electron was a sphere of charge of radius  $r_e = 2.82 \times 10^{-15}$  m (the "classical" radius, see Problem 8) which is about the same size as the nucleus. This was disconcerting since it is hard to see how it is possible to fit many electrons into a box (the nucleus) that is about equal to the size of one electron.

<sup>28</sup> $N_2^+$  is singly-ionized molecular nitrogen, and Leonard Ornstein of Utrecht, the Netherlands, had observed its rotational band spectrum. The reasoning for determining the nuclear spin is fairly sophisticated; a nice description is in Pais, *Inward Bound*, page 299.

${}^6\text{Li}$ ,  ${}^9\text{Be}$ ,  ${}^{10}\text{B}$ ,  ${}^{13}\text{C}$ ,  ${}^{14}\text{N}$ ,  ${}^{17}\text{O}$ , and  ${}^{21}\text{Ne}$ . Only *two* of these have appreciable abundance:  ${}^9\text{Be}$  (100% of all beryllium) and  ${}^{14}\text{N}$  (99.6% of all nitrogen), so it's not surprising that this discrepancy was first observed with  ${}^{14}\text{N}$ . All the other, more common, isotopes:  ${}^1\text{H}$ ,  ${}^4\text{He}$ ,  ${}^7\text{Li}$ ,  ${}^{11}\text{B}$ ,  ${}^{12}\text{C}$ ,  ${}^{16}\text{O}$ ,  ${}^{18}\text{O}$ ,  ${}^{19}\text{F}$ ,  ${}^{20}\text{Ne}$ ,  ${}^{22}\text{Ne}$ , and of course the rare  ${}^{15}\text{N}$ , have an even number of neutrons and hence require a different method to distinguish between the two models.

### 2.3.3 The Pauli Exclusion Principle and classification according to spin

Just like placing subatomic particles in three categories depending on their mass (lepton, meson, or baryon), we can place all particles into one of two categories depending on their spin:

fermions	—	half-integer spin, $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$
		⇒ MUST obey the Pauli Exclusion Principle
bosons	—	integer spin, $s = 0, 1, 2, \dots$
		⇒ no exclusion principle

Fermions are named after Enrico Fermi and Paul Dirac, who developed “Fermi-Dirac statistics” to describe this type of particle, and bosons are named after Satyendra Bose and Albert Einstein, who developed “Bose-Einstein statistics.”

What is Pauli's exclusion principle? In 1924 he stated it in the following manner

In the atom there can never be two or more equivalent electrons for which ... the values of all quantum numbers coincide. If there is an electron in the atom for which these quantum numbers have definite values then the state is “occupied.”

Prior to this, from observations of atomic spectra when the atoms are placed in magnetic fields, it had been determined that each electron had three quantum numbers,  $n$ ,  $\ell$ , and  $m_\ell$ . The first,  $n$ , is a label for the shell,  $\ell$  labels the subshell, and  $m_\ell$  is the so-called “magnetic quantum number” because it would split the spectral lines only when the atom was placed in a magnetic field. It was realized that *two* electrons could be placed in each of these quantum states, and so a fourth quantum number for the electron,  $m_R$ , was proposed by Samuel Goudsmit which could take on the two values  $m_R = \pm\frac{1}{2}$ . This now doubled the number of allowed states, and Pauli's principle works. As you might guess,  $m_R$  is nothing but  $m_s$ , the  $z$  component of the electron's spin. That is, two electrons can occupy a single state, but with the caveat that one must be spin up and the other spin down. This implies, therefore, that they are really occupying *different* quantum states, since the external configuration (i.e., position) as well as the internal configuration (i.e., spin) must be included in the definition of “quantum state.”

The Pauli principle has since been generalized and made a more formal part of quantum mechanics. The new statement (and you shall see what this means in Chapter 7) is that

fermions must have an antisymmetric wave function, and bosons must have a symmetric wave function.

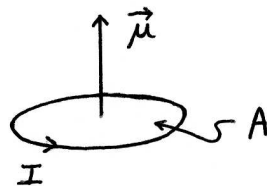


Figure 2.6: Geometry of a magnetic dipole. Current  $I$  flowing in a loop with cross sectional area  $A$  has a magnetic dipole moment of magnitude  $\mu = IA$ . The direction of  $\vec{\mu}$  is determined by the right hand rule: curl the fingers of your right hand in the direction of  $I$  and your thumb points in the direction of  $\vec{\mu}$ .

It is not important at this stage to understand what a “wave function” is, but I want to introduce early on the concept of symmetry, which underlies all of modern particle physics. You have seen the concept of symmetry before when you labeled a function either even or odd. A function that satisfies  $f(-x) = f(x)$  is an even, or symmetric, function; and one that satisfies  $g(-x) = -g(x)$  is an odd, or antisymmetric, function. In general, a symmetry can be defined as a quantity that remains unchanged when another quantity changes. For example, a sphere is unchanged when it is rotated about an axis that passes through its center. It is said to have “rotational symmetry.” The function  $f$  above is symmetric under the interchange of  $x \leftrightarrow -x$ , while  $g$  is antisymmetric. We will study symmetry more in Chapter 7, but for now I want to mention one symmetry that you are familiar with: a mirror. What is the symmetry (or antisymmetry) of an object and its image in a flat mirror? More specifically, why does a mirror reverse left and right, but not up and down? (See Problem 28.)

## 2.4 Magnetic moment

How do you experimentally determine the spin of a particle? It is difficult to measure it directly, but is relatively straightforward to infer it through a measurement of the particle’s magnetic dipole moment. What is a magnetic dipole moment? You may recall from your study of electrostatics that two electric charges, one positive and the other negative ( $\pm q$ ), separated by a distance  $d$ , is called an electric dipole, and has an electric dipole moment  $\vec{p} = q\vec{d}$ . Here, the vector  $\vec{d}$  is chosen to point from the negative charge toward the positive charge. Similarly, since the fundamental unit of magnetism is current, a current  $I$  in the shape of a loop of cross-sectional area  $A$  is a magnetic dipole with a magnetic dipole moment  $\vec{\mu} = I\vec{A}$ , where the direction of  $\vec{A}$  is determined by the right-hand-rule (see Fig. 2.6). This is the origin of the magnetic properties of matter: both an electron’s orbital motion in the atom as well as its spin contribute to a magnetic dipole moment that both is the source of a magnetic field, and also is affected by the magnetic field due to other objects.

## The Electron

What is the magnetic dipole moment (from now on called simply “magnetic moment”) of a charge  $q$  and mass  $m$  moving in uniform circular motion with speed  $v$ ? In addition to the current due to the moving charge, the mass of the moving particle means that it also has angular momentum. It turns out that its magnetic moment  $\vec{\mu}$  is directly proportional to its angular momentum  $\vec{L}$

$$\vec{\mu} = \frac{q}{2m} \vec{L}, \quad (2.7)$$

where  $q/2m$  is called the *gyromagnetic* ratio (see Problem 30). I have used the symbol  $\vec{L}$  because it is the particle’s *orbital* angular momentum that we are concerned with. Since quantum mechanics tells us that  $\vec{L}$  is quantized, Eq. (2.7) implies that  $\vec{\mu}$  must also be quantized. The  $z$  component of the magnetic moment can be expressed as

$$\mu_z = \frac{q}{2m} L_z = \frac{q}{2m} m_\ell \hbar, \quad (2.8)$$

where  $m_\ell$  is the quantum number associated with the  $z$  component of the orbital angular momentum  $\vec{L}$ .<sup>29</sup>

For an electron (orbiting in an atom, say),  $q = -e$  and  $m = m_e$ , and we have

$$\mu_z = \frac{-e}{2m_e} m_\ell \hbar = - \left( \frac{e\hbar}{2m_e} \right) m_\ell \equiv -m_\ell \mu_B, \quad (2.9)$$

where  $\mu_B \equiv e\hbar/2m_e = 927.400\,949(80) \times 10^{-26}$  J/T is called the “Bohr magneton,” and is the unit in which atomic magnetic moments are measured. As Eq. (2.9) shows, the magnetic moment due to an electron’s orbital motion must be an integer multiple of the Bohr magneton. A measurement of  $\mu_z$  therefore allows us to infer a value of  $m_\ell$ . How can we measure  $\mu_z$ ? By placing the atom in a magnetic field and observing the effect on the magnetic dipole moment. The Stern-Gerlach experiment (Appendix C) is a classic example of such a measurement.

In addition to the magnetic moment due to its orbital motion, the fact that an electron is an electric charge that is “spinning” on its axis means that it not only has intrinsic angular momentum, but an intrinsic magnetic moment. If the electron, however, were truly a spinning sphere of charge  $-e$  and mass  $m_e$ , and if its charge and mass were uniformly distributed, then it would have exactly the same intrinsic gyromagnetic ratio as a single current loop (see Problem 31). But it does not! The dimensions are, of course, correct, but the numerical factor is incorrect. The intrinsic (or spin-related) magnetic moment of an electron can be written

$$\vec{\mu}_s = g \left( \frac{-e}{2m_e} \vec{S} \right), \quad (2.10)$$

where  $g$  is a dimensionless number, called the Landé  $g$ -factor.<sup>30</sup> Experimentally, it turns out that  $g \approx 2$ . In 1921, Alfred Landé first proposed  $g = 2$ . Although it explained some

<sup>29</sup>Recall that the orbital angular momentum quantum number  $\ell$  is always an integer, which means that  $m_\ell$  is also an integer.

<sup>30</sup>The quantity  $g$  was originally a “fudge” factor, proposed by Landé in 1921 to explain spectroscopic measurements of the Zeeman effect.

spectroscopic measurements,<sup>31</sup> it was *ad hoc*, and did not make sense in light of classical electromagnetic theory. At that time there was no inkling that the electron could be anything *but* a physical object, most probably a sphere. In hindsight, however, this value of  $g$  is incontrovertible evidence that the electron is a new kind of object. In 1928, Paul Dirac developed a relativistic theory of quantum mechanics, and one of the results of that theory was that any point particle (with zero spatial extent) *must* have a  $g$ -factor equal to 2. This made physicists realize that the electron does not have any physical size, but is truly a point particle, “and there is no sense asking what is inside it.”<sup>32</sup>

Actually, the quantum theory of Dirac was only an approximation to QED, and QED, physical quantities must be calculated in a series of approximations, similar to a Taylor series in calculus. The exact value of  $g$  for the electron has been calculated to be

$$g = 2 + \left(\frac{\alpha}{\pi}\right) - 0.656957930 \left(\frac{\alpha}{\pi}\right)^2 + 2.362482912 \left(\frac{\alpha}{\pi}\right)^3 + \dots = 2.002\,319\,304\dots, \quad (2.11)$$

where  $\alpha$  is called the “fine structure constant”<sup>33</sup> and has the value

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137.035\,999\,070(98)} \approx \frac{1}{137}. \quad (2.12)$$

This theoretical prediction for  $g$  agrees with experimental results to at least 10 digits, and is *the most accurate prediction of any physical theory* at the present time. In fact, the calculation of the coefficient of  $\alpha^3$  took 20 years, and recently,<sup>34</sup> the next term in the series has been calculated, using numerical approximation methods (and several years of supercomputer time) and has the value

$$-3.8288(70) \left(\frac{\alpha}{\pi}\right)^4 \approx -1.01 \times 10^{-10}. \quad (2.13)$$

If you look at the experimental value for  $g$  listed in the CODATA sheet,  $g = 2.002\,319\,304\,3718(75)$ , it is getting close to the level of experimental uncertainty.<sup>35</sup>

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<sup>31</sup>As Samuel Goudsmit, the discoverer of spin, commented, Landé’s assumption explains “completely the extensive and complicated material of the anomalous Zeeman effect.”

<sup>32</sup>Abers, *Quantum Mechanics*, page 176.

<sup>33</sup> $\alpha$  is “the coupling constant or measure of the strength of the electromagnetic force that governs how electrically charged elementary particles and light interact.” —NIST. It determines the strength of the fine-structure splitting of the hydrogen atom’s spectral lines, a subject that we will take up in Chapter 4.

<sup>34</sup>Aoyama, et al., “Revised Value of the Eighth-Order Contribution to the Electron  $g$ -2,” *Phys. Rev. Lett.* **99** 110406 (2007).

<sup>35</sup>Often the electron magnetic moment “anomaly,”  $a_e \equiv (g - 2)/2$  is quoted rather than the value of  $g$  itself

$$a_e = \frac{1}{2} \left(\frac{\alpha}{\pi}\right) - 0.328478965 \left(\frac{\alpha}{\pi}\right)^2 + 1.181241456 \left(\frac{\alpha}{\pi}\right)^3 - 1.9144 \left(\frac{\alpha}{\pi}\right)^4 = 0.00115965218279(771),$$

This value of  $g$  for the electron is the **third** piece of evidence that leads us to believe that electrons are point particles. The **first** was its size—or lack thereof. It has no discernable structure on any scale that has been measured. The **second** was the fact that its spin quantum number  $s$  is half integer, i.e., it's a fermion. Angular momentum obtained from the classical definition  $\vec{r} \times \vec{p}$  is required to have an integer quantum number. Finally, the nail in the coffin is that  $g \neq 1$ . More important,  $g \approx 2$ , which, by Dirac's theory means it must be a point particle.

## 2.5 Color Charge

To interact gravitationally, an object must have mass, and to interact electrically it must have electric charge. There is only one kind of mass, but there are two kinds of electric charge (positive and negative). The force between quarks is called the color force, and for a particle to feel the color force, it must have “color charge.” In this case, however, there are three kinds of color charge: red, green, and blue.<sup>36</sup> Just like equal positive and negative electric charges “cancel” and result in a charge neutral object, so too a combination of all three colors results in a color neutral object that does not interact via the color force. The three colors, while they have nothing to do with the actual colors of light, were chosen because of the property that these three colors added together make white (something without color).

The color, therefore, is a new quantum number of the quarks. An up quark, for example, can either be in the red state, the green state, or the blue state. Anti-quarks come in “anti-colors,” which can be thought of the complementary color on the color wheel; i.e., anti-red is equivalent to mixing green and blue, which gives cyan. The colors magenta and yellow are the anti-colors of green and blue, respectively. Since quarks are never seen in isolation, this means that bare color is never seen, and quarks must exist only in combinations that are color neutral. The only such combinations are three quarks (qqq), each with a different color, or a quark–anti-quark pair ( $q\bar{q}$ ), with a color and its anti-color. These, of course, are just baryons and mesons.

Why do we need this extra quantum number? Let's look at three of the baryons that have been observed along with their quark constituents, the  $\Delta^{++}$  (uuu), the  $\Delta^-$  (ddd), and the  $\Omega^-$  (sss). Each of these exist only in the spin  $\frac{3}{2}$  state, and, since they are composed of identical, indistinguishable particles, they seem to violate the Pauli exclusion principle. That is, in each of these baryons there seems to be three identical particles with the same value of  $m_s$ . If, however, we postulate that there is another quantum number (color) that can take on three different values, and if each quark has a different color, then Pauli's principle is not violated. For example, the  $\Delta^-$  could be composed of a red down quark ( $d_r$ ), a blue down quark ( $d_b$ ), and a green down quark ( $d_g$ ).

Mesons also must be color neutral, and, for example, the quarks in the  $\pi^+$  meson ( $u\bar{d}$ ) must be anti colors. However, which colors does it choose? Red and cyan? Green and magenta? Blue and yellow? In fact, the mesons are composed of linear combinations

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<sup>36</sup>Originally, the three colors were red, white, and blue, but the concept of color neutrality is more pleasing using the well-known primary colors.



(superpositions) of all three possibilities. We can write

$$\pi^+ = \frac{1}{\sqrt{3}} (u_r \bar{d}_r + u_g \bar{d}_g + u_b \bar{d}_b), \quad (2.14)$$

where the  $\sqrt{3}$  is there just to make sure that we are counting only one quark. Essentially, the  $\pi^+$  meson can be thought of consisting of  $33\frac{1}{3}\%$  of each color combination.

This is exactly analogous to the logic inherent when Pauli first proposed his exclusion principle. If you read his statement on page 34, he states that *two* electrons are allowed in each state. In fact, only *one* fermion is allowed per quantum state, so that meant that there must be another quantum number for the electron in an atom (beyond the three that were known) that could take on two possible values. It turned out that this quantum number was  $m_s$ , the  $z$  component of the electron's spin, which could take on the values  $\pm\frac{1}{2}$ . That is, the electron could be in either a spin up state, or a spin down state. With color, the logic is the same: there must be another quantum number that “allows” three identical particles to be in the “same state.” In reality, since only one fermion per state is allowed, the new quantum number must take on three different values. In addition to explaining why baryons with three identical quarks exist only in a spin  $\frac{3}{2}$  state, this proposition solves another problem—it explains why we see quarks only in baryons and mesons, but in no other combination, and definitely not in isolation. This idea of color was proposed by Oscar Greenberg in 1964, and M. Y. Han and Yoichiro Nambu [Nobel Prize, Physics, 2008] in 1965.



## The Color Force

Even though the mathematics of the color interaction are quantum mechanical in nature, and also very complicated, we can understand the force between two quarks in a simple way. One of the most important properties of the color force is that, unlike the electric and gravitational forces, it becomes *stronger* with distance. Quarks can be thought of as being connected by a rubber band: as you pull them apart, the attractive force between them becomes stronger. This is a partial explanation for why quarks are not seen in isolation—you cannot separate them because the attractive force is so strong. Mathematically, we can express this force in terms of a potential energy that is a function of the distance  $r$  between two quarks

$$U(r) = -\frac{a}{r} + br, \quad (2.15)$$

where  $a = 94.6 \text{ MeV fm}$ , and  $b = 913 \text{ MeV/fm}$ .<sup>37</sup> The first term in the potential energy is electrostatic in character—i.e., it is due to a  $1/r^2$  attractive force. For comparison, the Coulomb potential energy between a proton and an electron is

$$U(r) = -\frac{e^2}{4\pi\epsilon_0} \frac{1}{r} \approx -\frac{1.44 \text{ MeV fm}}{r}, \quad (2.16)$$

<sup>37</sup>This is the so-called “Cornell potential.” See Martin and Shaw, *Particle Physics*, p. 138.

which shows that this part of the strong force is about 65 times stronger than the electrostatic force. (See Problem 34 for the calculation of a similar constant for the gravitational force.) The second term represents the “spring-like” force that gets stronger with distance. Equation (2.15) is only valid, however, when the distance is in the range  $0.2 \text{ fm} < r < 0.8 \text{ fm}$ . Why such a restricted regime? Because it was obtained experimentally by looking at the spectrum of “charmonium,”<sup>38</sup> a meson that consists of a charm quark and an anti-charm quark ( $c\bar{c}$ ), and comparing it with the spectrum of positronium (which is an “atom” made up of an electron and positron.) Since these two objects are both made up of a particle and its anti-particle, and positronium is held together by the electric force only (which is well understood), a comparison yields information concerning the color force, which holds charmonium together. It turns out that the spectrum is determined by the potential energy over the spatial range above, because this is the typical inter-quark distance.

How is this color force mediated? Through the exchange of gluons, which are the exchange particles for the color force. Although gluons are massless, just like photons, they do have color, unlike photons. That is, while photons mediate the electromagnetic force, they have zero electric charge. Gluons mediate the color force, but they *do* have a nonzero color charge. This means that a single gluon cannot be observed in isolation because it is not color neutral. There are, however, predictions of “glueballs,” which are particles made up of two gluons (of a color and its anticolor) that are color neutral. They have not yet been seen, and one possible reason is that they are difficult to distinguish from mesons.

There are two predictions of QCD that make the color force different from other forces. The first is CONFINEMENT. As  $r$  increases the potential energy increases, and at some point this energy is large enough to create new particles, e.g., a quark–anti-quark pair. So, if you try to separate two quarks to observe one in isolation, not only does the force required increase with distance, but you must do so much work that you create matter from the energy you put in. In some sense, this is like the inability to observe a single magnetic pole: cutting a magnet in half simply creates two magnets. Quarks are confined in color neutral combinations, and any attempt to separate them requires so much energy that new quarks are created that retain the color neutrality of each particle.

The second prediction is called ASYMPTOTIC FREEDOM. At small distances, less than about 0.2 fm, the interaction becomes very weak. As quarks “asymptotically” approach each other, they become “free,” with no force acting on them. Again, this is consistent with our picture of quarks happy to exist in groups of two or three (that are color-neutral, of course), but unhappy to be isolated.<sup>39</sup>

## The strong nuclear force

If nucleons are color neutral, what holds them together in the nucleus of an atom? The answer is the *residual* color force, also called the strong force, that exists because the color force between two color-neutral nucleons does not exactly cancel. The situation is similar

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<sup>38</sup>Charmonium is also called the  $J/\psi$  particle.

<sup>39</sup>The Nobel Prize in Physics for 2004 was awarded to David Gross, Frank Wilczek and David Politzer for discovering asymptotic freedom in 1973.

to the force that electrically neutral atoms exert on each other. This residual electric force exists because the electric charge in the atoms (the positive charge in the nucleus and the negative charge in the electron cloud) are not in exactly the same location. This means that they act like electric dipoles, and two electric dipoles exert a force on each other that has a shorter range than the Coulomb force between bare charges. The Coulomb force falls off as  $1/r^2$ , and you can show (Problem 36) that the force between permanent dipoles falls off as  $1/r^4$ . It becomes weaker more quickly as the dipoles move apart, and therefore they must be close together to feel a significant force. This weak residual force is also known as the “van der Waals” force, postulated by Johannes van der Waals (Nobel Prize, Physics, 1910) to obtain an equation of state for a non-ideal gas that included a liquid phase.

The exact same partial cancelation occurs with the color charge of the quarks in a nucleon. The fact that the three quarks in a nucleon are not in exactly the same location means that there will be a nonzero residual color force, which is usually called the strong nuclear force. If this is truly a “force,” then using our new description of forces it must be mediated by an exchange particle. This exchange particle is a pion, or  $\pi$  meson, and is what Yukawa envisioned as holding the nucleus together. He knew that the force must be short range, because atomic nuclei do not compress as more nucleons are added—they have a relatively constant density. As Yukawa correctly deduced, this implies that nucleons only interact with their “nearest neighbors,” and do not feel any attraction to distant nucleons on the other side of the nucleus. As I have stated on page 28, short-range forces must be mediated by massive exchange particles, and a range of 1 fm corresponds to a mass of about  $100 \text{ MeV}/c^2$ , which is very close to the mass of the pion.



One final note on terminology. Quarks are the only elementary particles that have color and interact via the color force. Baryons and mesons are the only particles that are composed of quarks. Therefore, baryons and mesons are the only particles that interact via the strong nuclear force. Collectively, baryons and mesons are called “hadrons,” meaning a particle that exerts and feels the strong force.

## 2.6 Weak interactions

Of the four fundamental forces, the weak force is the most difficult to describe in simple mathematical terms. It does, however, have one feature that none of the other forces have: it can change quarks and leptons from one flavor to another. Because the exchange particles ( $W^\pm$  and  $Z^0$ ) are so massive, the weak force acts over extremely short distances—so short, in fact that the interactions appear to be point-like, and the existence of the  $W^\pm$  and  $Z^0$  particles must be inferred from their decay products.

One example of the weak interaction is the neutron decay on page 22. One of the down quarks in the neutron is transformed into an up quark (making a proton), but in the process a  $W^-$  particle is created, which then decays into an electron and antineutrino. This reaction can most easily be described graphically by means of a “Feynman diagram,” shown in Fig. 2.7. The diagram depicts the decay process with the spatial dimension on

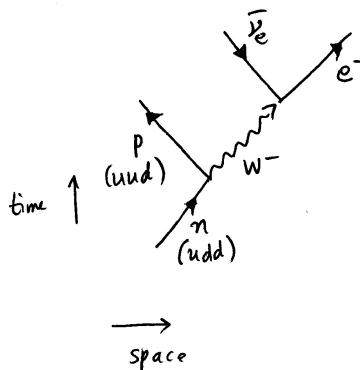


Figure 2.7: Feynman diagram of neutron decay. Note that only one d quark in the neutron decays into a u quark, and the other d quark and the one u quark are “just along for the ride.”

the horizontal axis and time running vertically. First, the neutron is transformed into a proton and a  $W^-$ :  $n \rightarrow p + W^-$ , and then the  $W^-$  decays:  $W^- \rightarrow e^- + \bar{\nu}_e$ . The net reaction is, of course, identical to Eq. (2.3). Note that at each vertex in the Feynman diagram electric charge is conserved. In addition, “lepton number” is also conserved at each vertex (lepton number is a quantum number assigned so that leptons have a lepton number of 1, and anti-leptons have a lepton number of  $-1$ ). However, the mass is *not* conserved at each vertex: the extra mass of the  $W^-$  violates the law of the conservation of mass and energy, but does so only for a short time in accordance with the Heisenberg uncertainty principle.

One other strange thing you may notice about the Feynman diagram is that the anti-neutrino is depicted with an arrow directed backward in time. This is because in quantum field theories, the mathematical description of particles moving forward in time is identical to anti-particles moving backward in time. For this reason, you can think of the anti-neutrino colliding with the  $W^-$  and forming an electron. On that weird note, we now turn to more mundane matters—nuclear physics—where we ignore the sub-nuclear particles and concentrate on protons and neutrons, and on the nuclei that they comprise.

### Collateral Reading

- “The Discovery of the Top Quark,” by Tony M. Liss and Paul L. Tipton, *Scientific American*, September 1997, pages 54-59.
- “Top-ology,” by Chris Quigg, *Physics Today*, May 1997, pages 20-26.

### Problems

7. At the beginning of the twentieth century, physicists wondered what mechanism endowed an electron with mass. One idea was that it came from the electrostatic energy needed to assemble a uniform sphere of charge  $-e$ . One way to calculate this energy is to recall from elementary electromagnetic theory that when you assemble charge in such a

way as to create an electric field, the energy density of that field is  $\frac{1}{2}\epsilon_0 E^2$  (this is usually derived for a parallel plate capacitor). Assuming that the electron is a uniform sphere of radius  $R$  and charge  $Q$ , use Gauss's Law to calculate the electric field everywhere (it will only have a component in the radial direction), and then integrate the energy density over all space to find the total energy  $\mathcal{E}$ . Your answer should be

$$\mathcal{E} = \frac{3}{5} \frac{Q^2}{4\pi\epsilon_0 R}.$$

See the box on page 98 for another perspective on this concept of “electromagnetic mass.”

**8.** In the early 1900s, the electron was thought to be a uniform sphere of charge; therefore, its size could be determined by assuming that its rest energy,  $m_e c^2$ , is equal to the electrostatic potential energy stored in the charge distribution,  $e^2/4\pi\epsilon_0 r_e$ . In this problem you are to calculate the numerical value of  $r_e$ , known as the “classical electron radius.” How does your answer compare with the experimental fact that the electron is smaller than  $10^{-22}$  m? NOTE: the correct formula for the energy of a uniformly charged sphere is  $(3/5)(e^2/4\pi\epsilon_0 r_e)$  — see Problem 7. Historically, physicists were only interested in the order of magnitude, and so they ignored the factor of  $(3/5)$  which is approximately unity.

**9.** Calculate the classical proton radius using the same method as in Problem 8. How does your answer compare with the experimental value for the proton radius?

**10.** The mass of the hydrogen *atom* has been measured as 1.007 825 032 07(10) u. Using the known masses of the proton and electron, determine the binding energy of the hydrogen atom in eV. How well does your answer agree with the known ionization potential of hydrogen? Recent measurements of the electron mass gives 0.000 548 579 9111(12) u, and for the proton 1.007 276 466 77(10) u.

**11.** Calculate (a) the binding energy  $B$  of the deuteron, and (b) the reaction energy  $Q$  of the deuteron “decaying” into two protons:  $d \rightarrow p + p + e^- + \bar{\nu}_e$ .

**12.** Calculate the binding energy  $B$  of the helion (the nucleus of  ${}^3\text{He}$ ).

**13.** The  ${}^4\text{He}$  nucleus, also called an  $\alpha$ -particle, is one of the most tightly bound nuclei. For this reason it is used as the end product for effective fusion reactions because those reactions release a large amount of energy. What happens when you try to put two  $\alpha$ -particles together to form one  ${}^8\text{Be}$  nucleus? Calculate (a) the binding energy of  ${}^8\text{Be}$  as well as (b) the reaction energy for  ${}^8\text{Be}$  to split up into two  $\alpha$ -particles. (c) Is  ${}^8\text{Be}$  stable or unstable? NOTE: The *atomic* mass of  ${}^8\text{Be}$  is 8.005305 u. The  $\alpha$ -particle mass listed in the CODATA sheet is the *nuclear* mass.

**14.** The concept of binding energy only makes sense when the force between the constituent particles vanishes as their separation distance becomes infinitely large. This is the case, for instance, with the Earth and the ball, and also for the proton and neutron (the proton and neutron interact via the strong force, which is extremely short range, and only has an effect within a few femtometers). However, the quarks are bound by a force (the “color” force) that gets stronger with distance, which explains why no free quarks have been observed. For this reason, the concept of “constituent quark mass” has developed, which simply ignores any possible binding energy, and is a calculation of the quark masses simply by measuring the observable baryon and meson masses.

(a) Your task here is to calculate the constituent masses of the up and down quarks by

using the observed masses of the proton and neutron. (you simply have to solve a system of equations with two unknowns). (b) Another idea is to calculate the constituent masses of the up, down, and strange quarks by using the observed masses of the three sigma baryons. You'll find that there is a problem with this method, however. What is it? And, what exactly *can* you calculate with this method?

**15.** Calculate the reaction energy  $Q$  for neutron decay.

**16.** Calculate the reaction energy  $Q$  for the decay of the muon  $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$ .

**17.** Confirm that the three  $\Sigma$  baryons and the three  $\pi$  mesons have the proper charge due to their quark constituents.

**18.** List all possible distinct baryons (particles consisting of combinations of three quarks) if you only have u, d, and s quarks. How many are there? Also, for each baryon, list its electric charge and strangeness.

**19.** (a) Determine a formula for the number  $N$  of distinct baryons (particles consisting of three quarks) if there are  $n$  flavors of quarks. For  $n = 3$  does your answer agree with Problem 18? For  $n = 6$ , what number do you get? (b) Do the same for mesons (one quark and one antiquark).

**20.** Convert the rest energies of the weak vector bosons,  $W^\pm$  and  $Z^0$ , to kg as well as u. What atoms have the same masses as these particles?

**21.** Calculate the magnitudes of the orbital and intrinsic (rotational or spin) angular momentum of the Earth. What direction do these vectors point?

**22.** Calculate the angular momentum (both orbital and rotational) of the Sun, Mercury, Venus, Earth, Mars, Jupiter and Saturn. In which body does most of the solar system's angular momentum reside? Can you explain this result?

**23.** What possible angles does the angular momentum vector  $\vec{L}$  make with the positive  $z$  axis when  $\ell = 2$ ?

**24.** The electron is known to have a spin (intrinsic angular momentum) quantum number of  $s = \frac{1}{2}$ . This means that the magnitude of its angular momentum is  $S = \sqrt{s(s+1)} \hbar = \sqrt{3} \hbar/2$ . Assuming that the electron is a uniform sphere of radius  $r_e$  (equal to its classical radius, Problem 8) and that its spin is due to rotation about an axis that passes through its center, calculate (a) the angular velocity  $\omega$  of the electron, and (b) the speed of a point on the equator of the electron. Does your answer make sense?

**25.** Using the same method as in Problem 24, calculate (a) the angular velocity of the proton and (b) the speed of a point on its surface assuming it is a uniform sphere whose radius is equal to its classical radius (Problem 9). (c) For part (b) you should obtain the same answer as you got in part (b) of Problem 24. Why?

**26.** In 1929, it was observed by Schuler and Benck that cadmium had at least one isotope that was spin  $\frac{1}{2}$ . Why does this contradict the predictions of the proton-electron model of the nucleus?

**27.** Obtain a general rule (based on  $A$ ,  $N$ , and  $Z$  of a nucleus) that will allow you to determine for which nuclei the proton-electron model and the current proton-neutron model give contradictory predictions for the nuclear spin.

**28.** When you look at yourself in the mirror, your image is reversed left $\leftrightarrow$ right but it is not reversed up $\leftrightarrow$ down. Why? It is not because you have two eyes, since the illusion remains when you close one eye. It is also not because you stand vertically, since even if

you lie horizontally the illusion remains.

**29.** Is it correct to think of the electron as a tiny ball of charge spinning on its axis? Is it useful?

**30.** For a point particle of mass  $m$  and charge  $q$  that is moving in a circle of radius  $r$  at constant speed  $v$ , calculate separately the magnitudes of its angular momentum  $\vec{L}$  and magnetic dipole moment  $\vec{\mu}$ . What directions do they point? What is the ratio of their magnitudes (the *gyromagnetic* ratio or *magnetomechanical* ratio)?

**31.** Consider a solid sphere of radius  $R$  with a uniform mass density  $\rho_M$  and total mass  $M$ , uniform charge density  $\rho_Q$  and total charge  $Q$ , that is spinning about an axis through its center with angular velocity  $\omega$ . (a) Calculate the magnitude of the angular momentum. (b) Calculate the magnitude of the magnetic dipole moment. For this part, one method is to first calculate the moment of a charged disk of radius  $r'$  and then add up (i.e., integrate) all the disks to make a sphere. (c) What is the ratio of the two quantities?

**32.** In the Bohr model of the hydrogen atom, in the lowest energy state the electron orbits the proton at a speed of  $2.2 \times 10^6$  m/s in a circular orbit of radius  $5.3 \times 10^{-11}$  m. (a) Calculate the current  $I$  due to the moving electron and the magnetic moment due to this orbital motion of the electron. (b) Calculate the orbital angular momentum of the electron. (c) How do these quantities compare with the intrinsic values of the spin and magnetic moment for the electron.

**33.** Prove that the gyromagnetic ratio of a sphere of mass  $M$  and charge  $Q$ , with a uniform mass density  $\rho$  and uniform *surface* charge density  $\sigma$  is characterized by  $g = 5/3$ . You will have to integrate the contributions of many infinitesimal rings of charge to obtain the magnetic moment of this object.

**34.** Equation (2.16) showed that the potential energy due to the electric force between an electron and a proton can be written  $U = -\alpha/r$  where  $\alpha = 1.44$  MeV fm. One of the terms in the potential energy for the color force is  $-a/r$  where  $a = 94.6$  MeV fm. (This shows that the color force is stronger than the electric force.) The gravitational force between two *nucleons* has a similar potential energy function. That is,  $U_G = -\alpha_G/r$ . Calculate the constant  $\alpha_G$  for gravity (in units of MeV fm) for two nucleons.

**35.** Consider the potential energy function between two quarks in a meson, Eq. (2.15). (a) Assume that the two quarks have zero potential energy. At what separation distance  $r_0$  is  $U = 0$ ? (This is approximately the stable distance between the quarks in the meson.) (b) When the quarks are at this distance, what is the magnitude of the attractive force between them? (Recall that  $F_r = -\partial U/\partial r$ .) (c) If you pull the quarks apart, you must do work on them, and their potential energy increases. How far must you pull them apart so that you have enough done enough work to create a  $\pi^0$  meson? (d) Does this suggest a possible reason why quarks are never observed in isolation?

**36.** To investigate the strong force visualized as the residual color force between nucleons, in this problem we consider two neutral atoms, which do not exert an electric force on each other, but if their charges incompletely cancel—that is, they have an electric dipole moment—then they do. Consider two electric dipoles (each with a dipole moment  $p = qa$ ) located a distance  $r$  apart. That is, two charges  $\pm q$  are located at  $x = \pm a/2$ ,  $y = 0$ , and two more charges  $\pm q$  are located at  $x = r \pm a/2$ ,  $y = 0$ . Use the binomial expansion to take the limit where  $r \gg a$ . Your result *should* be that the  $1/r^2$  terms cancel, leaving only an attractive force that depends on  $r$  like  $1/r^4$ . This residual force, which is

weaker than that between two bare charges at the same distance, is similar to the strong nuclear force, which is weaker than the color force between quarks because quarks always combine in color neutral combinations, just like the electric dipole is charge neutral.

**37.** Draw a Feynman diagram for the decay of the  $\mu^-$  lepton:  $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$ . This decay illustrates that there is a lepton number for each lepton flavor that must be conserved separately. That is,  $\mu^-$  and  $\nu_\mu$  have muon lepton numbers of  $+1$ , while  $\mu^+$  and  $\bar{\nu}_\mu$  have muon lepton numbers of  $-1$ . A similar scheme holds for electron-type leptons and tau-type leptons.

## Solutions

**7.** From a consideration of Gauss's Law in its integral form, it is straightforward to obtain

$$E_r = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2} & r \geq R \\ \frac{Q}{4\pi\epsilon_0 R^3} r & r \leq R \end{cases}$$

and then, for example, the total energy outside the sphere is given by

$$\frac{1}{2} \int_0^{2\pi} \int_0^\pi \int_R^\infty \epsilon_0 \left( \frac{Q}{4\pi\epsilon_0 r^2} \right)^2 r^2 dr \sin\theta d\theta d\phi = \frac{Q^2}{8\pi\epsilon_0} \int_R^\infty \frac{dr}{r^2} = \frac{Q^2}{8\pi\epsilon_0 R},$$

where I've used the fact that the integrand is only a function of the radius  $r$  so that the angular integrations give  $\int \int \sin\theta d\theta d\phi = 4\pi$ . A similar integration can be done for the region  $r \leq R$ .

**8.** Equating

$$m_e c^2 = \frac{e^2}{4\pi\epsilon_0 r_e},$$

and solving for  $r_e$ , the "classical electron radius" is  $r_e = e^2/4\pi\epsilon_0 m_e c^2 \approx 2.82 \times 10^{-15}$  m. This is much larger than the observed upper limit of the size of the electron,  $10^{-22}$  m. This result means that the supposition that the electron's mass is due to electrostatic energy is wrong.

**9.** Since the only dependence in the formula is on the mass, we can write

$$r_p = \frac{e^2}{4\pi\epsilon_0 m_p c^2} = \frac{r_e}{1836} = 1.54 \times 10^{-18} \text{ m},$$

where 1836 is the ratio of the proton mass to the electron mass. Whereas the classical electron radius was far too large (since it actually is a point particle), the classical proton radius is far too small. A proton has a radius on the order of 1 fm. These examples show that classical ideas are completely inadequate to describe the quantum world.

**10.** Adding the masses of the proton and electron, then subtracting the mass of the hydrogen atom, I get  $1.4611 \times 10^{-8} \text{ u} = 13.61 \text{ eV}/c^2$  (where I have used the conversion factor from the CODATA sheet:  $(1 \text{ u})c^2 = 931.494 \times 10^6 \text{ eV}$ ). This matches the known ionization energy of hydrogen, as it should.

**11.** (a) See page 20. (b) See page 23.

**12.** From the definition of binding energy, Eq. (2.1), where  $M$  is the *nuclear* mass of the helion (from the CODATA sheet), and  $\sum_i m_i$  are the constituent particles, two protons and one neutron, I get



$$\begin{array}{rcl}
 2m_p & 2 \times & 1.007\,276 \text{ u} \\
 +m_n & & 1.008\,664 \text{ u} \\
 -m_{He} & & 3.014\,932 \text{ u} \\
 \hline
 = B/c^2 & & 0.008\,285 \text{ u} = 7.72 \text{ MeV}
 \end{array}$$

Note that the atomic mass of  ${}^3\text{He}$  is 3.016029 u, which, when you subtract the electron masses (0.000549 u each), gives 3.014932 u for the nuclear mass.

**13.** (a) For the binding energy of a nucleus, we define the constituents to be protons and neutrons. If we use the proton and neutron masses, we must use the mass of the beryllium-8 nucleus in the calculation. However, since we only know the beryllium-8 *atomic* mass (from the NUBASE listing), we need to account for the electrons. One way to do this is to use the hydrogen atomic mass rather than the proton mass, and then the electron masses will cancel. In this case, I get

$$B = (4 \times 1.007825 \text{ u} + 4 \times 1.008665 \text{ u} - 8.005305 \text{ u}) c^2 = 0.060655 \text{ u } c^2 = 56.47 \text{ MeV}$$

So, the binding energy of  ${}^8\text{Be}$  is positive and you might expect it to be stable. (b) However, the reaction energy  $Q$  for the reaction  ${}^8\text{Be} \rightarrow 2{}^4\text{He}$  is (using atomic masses again) is found to be  $Q = 0.000099 \text{ u} = 92 \text{ keV}$ . (c) The fact that  $Q$  is positive proves that  ${}^8\text{Be}$  is unstable to break up into two  $\alpha$  particles.

**14.** (a) Since the masses of the proton and neutron are known, we have two equations and two unknowns

$$\begin{aligned}
 2m_u + m_d &= m_p \\
 m_u + 2m_d &= m_n,
 \end{aligned}$$

which we can write as a matrix equation

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} m_u \\ m_d \end{pmatrix} = \begin{pmatrix} m_p \\ m_n \end{pmatrix}.$$

This can be solved by Gaussian elimination and back substitution, or simply by inverting the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and solving for the quark masses

$$\begin{pmatrix} m_u \\ m_d \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} m_p \\ m_n \end{pmatrix} = \begin{pmatrix} \frac{2}{3}m_p - \frac{1}{3}m_n \\ -\frac{1}{3}m_p + \frac{2}{3}m_n \end{pmatrix}.$$

Since  $m_p = 938.27 \text{ MeV}/c^2$  and  $m_n = 939.57 \text{ MeV}/c^2$ , I obtain  $m_u = 312.32 \text{ MeV}/c^2$  and  $m_d = 313.62 \text{ MeV}/c^2$ .

(b) From the quark constituents of the  $\Sigma$  baryons, the matrix equation is

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} m_u \\ m_d \\ m_s \end{pmatrix} = \begin{pmatrix} m_+ \\ m_0 \\ m_- \end{pmatrix},$$

where  $m_{\pm}$  are the masses of the  $\Sigma^{\pm}$ , and  $m_0$  is the mass of the  $\Sigma^0$ . However, the determinant of the matrix is zero, which means that it can't be inverted (and therefore only two of the three equations are linearly independent), so that we cannot solve for all three masses. The most information we can obtain is two of the masses in terms of the third.

One interesting quantity that we *can* calculate is the mass difference between the up and down quarks. Subtracting the second equation (for  $m_0$ ) from the first (for  $m_+$ ), we get

$$m_u - m_d = m_+ - m_0 = 1189 \text{ MeV}/c^2 - 1192 \text{ MeV}/c^2 = -3 \text{ MeV}/c^2.$$

This is inconsistent with the same quantity obtained from part (a), which was -1.3 MeV. It turns out we need a more sophisticated model that takes quantum mechanics and relativity into account, i.e., a quantum field theory.

15. See page 22.

16. In the calculation of  $Q$  for muon decay, we can ignore the neutrino masses since they are much smaller than the electron and muon masses. The difference in the muon and electron masses results in a reaction energy of 105.147 MeV, signifying that the muon is unstable to decay into an electron.

17.

$$\begin{aligned} \Sigma^+ &= uus & q &= +e \\ \Sigma^0 &= uds & q &= 0 \\ \Sigma^- &= dds & q &= -e \\ \pi^+ &= u\bar{d} & q &= +e \\ \pi^0 &= u\bar{u}, d\bar{d} & q &= 0 \\ \pi^- &= d\bar{u} & q &= -e \end{aligned}$$

18. Simply by listing the possibilities (where uud is indistinguishable from udu), I find that there are 10 distinct baryons:

	charge	strangeness	particle $s = \frac{1}{2}$	particle $s = \frac{3}{2}$
uuu	+2	0		$\Delta^{++}$
uud	+1	0	p	$\Delta^+$
uus	+1	-1	$\Sigma^+$	$\Sigma^{*+}$
udd	0	0	n	$\Delta^0$
uds	0	-1	$\Sigma^0, \Lambda^0$	$\Sigma^{*0}$
uss	0	-2	$\Xi^0$	$\Xi^{*0}$
ddd	-1	0		$\Delta^-$
dds	-1	-1	$\Sigma^-$	$\Sigma^{*-}$
dss	-1	-2	$\Xi^-$	$\Xi^{*-}$
sss	-1	-3		$\Omega^-$

The  $\Sigma^0$  has a mass of 1192.5 MeV/ $c^2$ , while  $\Lambda^0$  is 1115.6 MeV/ $c^2$ . In addition, the decay

$$\Sigma^0 \rightarrow \Lambda^0 + \gamma$$

occurs, which means that  $\Sigma^0$  can be thought of as an electromagnetic excited state of  $\Lambda^0$ . Note that the three baryons with three identical quarks only exist in the spin  $\frac{3}{2}$  state. This is a consequence of the Pauli exclusion principle, applicable to identical particles.

**19.** (a) When  $n = 1$ , then  $N = 1$ , because if there is only one type of quark (say  $u$ ) then the only possible baryon is  $uuu$ . When  $n = 2$ , then  $N = 4$ , because if there are two types of quarks, say  $u$  and  $d$ , then there are four possible baryons:  $uuu$ ,  $uud$ ,  $udd$ , and  $ddd$ . Note that the order of the constituent quarks doesn't matter, only the final composition. When  $n = 3$ , you showed in problem #18 that  $N = 10$ . Using the theory of combinations and permutations, can you find a formula,  $N(n)$ , that will result in the following general answer:

$$N = \frac{n(n+1)(n+2)}{6},$$

which, when  $n = 6$ , gives  $N = 56$ . (b) For mesons, you will obtain different answers depending on whether you count anti-particles as distinct or not. I'll leave this as an exercise.

**20.** Using the conversion factors on the CODATA sheet, I get

$$\begin{aligned} W^\pm: 80.4 \text{ GeV}/c^2 &= 86.3 \text{ u} = 1.43 \times 10^{-25} \text{ kg} \\ Z^0: 91.2 \text{ GeV}/c^2 &= 97.9 \text{ u} = 1.63 \times 10^{-25} \text{ kg} \end{aligned}$$

This makes  $Z^0$  almost the same mass as  $^{98}\text{Tc}$ , the longest-lived of the radioactive isotopes, and  $W^\pm$  between the masses of  $^{85}\text{Rb}$  and  $^{88}\text{Sr}$  (rubidium and strontium).

**21.** (a) The orbital angular momentum  $L$  can be calculated from  $|\vec{r} \times \vec{p}| = mvr = M_E v R_{ES}$  where  $v = 2\pi R_{ES}/T$  and  $T = 1$  year. I get  $L = 2.65 \times 10^{40} \text{ kg m}^2/\text{s}$ . (b) For the spin, we need the integral over the entire Earth, or simply recall the other definition,  $S = I\omega$ , where  $I = \frac{2}{5}M_E R_E^2$  – since the Earth is a sphere – and  $\omega = 2\pi/T$ , but this time  $T = 1$  day. I get  $S = 7.06 \times 10^{33} \text{ kg m}^2/\text{s}$ . (c) Both of these “spins” are in the same direction, i.e., both angular momentum vectors point northward.

**23.** The vector  $\vec{L}$  and the  $z$ -component form a triangle, see Fig. 2.5, where  $\cos \theta = L_z/L = m_\ell/\sqrt{\ell(\ell+1)}$  (the factors of  $\hbar$  have canceled). Given that  $\ell = 2$  and the possible values for  $m_\ell$  are 2, 1, 0, -1, and -2, the five values for  $\theta$  are  $35.3^\circ$ ,  $65.9^\circ$ ,  $90^\circ$ ,  $114^\circ$ ,  $145^\circ$ .

**24.** (a) Since we are assuming that the electron is a sphere of radius  $r_e = 2.82 \times 10^{-15} \text{ m}$ , its angular velocity can be calculated from the definition of angular momentum:  $\omega = S/I = 3.15 \times 10^{25} \text{ s}^{-1}$ . (NOTE: Again, I assumed that the electron was a uniform sphere with moment of inertia  $I = \frac{2}{5}m_e r_e^2$ .) (b) Since a point on the “equator” is executing uniform circular motion, its speed is  $v = \omega r_e = 8.88 \times 10^{10} \text{ m/s}$ . This is almost 300 times larger than the speed of light, so, no, it does not make sense. The electron must *not* be a spinning sphere.

**25.** Using the classical proton radius of  $r_p = 1.54 \times 10^{-18} \text{ m}$ , and assuming that the proton is a sphere, I get (a) an angular frequency of  $\omega = 5.76 \times 10^{28} \text{ s}^{-1}$ . (b) Interestingly, the speed of a point on its equator is the same as for an electron,  $v = 8.88 \times 10^{10} \text{ m/s}$ . (c) Why are the speeds identical? For any sphere of charge  $Q$ , mass  $M$ , and radius  $R$ , if its spin angular momentum is a multiple of  $\hbar$ ,  $S = f\hbar$  (where  $f$  is a pure number), then the equatorial speed is

$$v = \omega R = \frac{f\hbar}{\frac{2}{5}MR^2} R = \frac{5f\hbar}{2MR}.$$

This, in general, will be different for each object. However, if the particle's rest energy is presumed to arise from its electrostatic energy

$$\frac{Q^2}{4\pi\epsilon_0 R} = Mc^2,$$

then the quantity  $MR$  depends only on the charge  $Q$ , and the equatorial speed can be expressed as a fraction of the speed of light  $c$

$$\frac{v}{c} = \left(\frac{5}{2}f\right) \frac{4\pi\epsilon_0 \hbar c}{Q^2} = \frac{5}{2} \left(\frac{S}{\hbar}\right) \left(\frac{e}{Q}\right)^2 \frac{1}{\alpha},$$

where  $\alpha$  is the fine structure constant. Since the proton and electron have the same charge  $Q$  and the same spin  $S$ , they have the same equatorial speed. Also, since  $S \approx \hbar$  and  $Q \approx e$ , the fact that  $\alpha \approx 1/137$  means that  $v \gg c$ .

**26.** Cadmium's atomic number is  $Z = 48$ , and its most abundant isotope has  $A = 112$ , which means  $N = 64$ . The p-e model predicts 112 protons and 64 electrons, for a total of 176 particles of spin- $\frac{1}{2}$  which implies an integer spin. If we look at the second most abundant isotope,  $A = 111$ , the p-e model predicts 111 protons and 63 electrons, for a total of 174 particles of spin- $\frac{1}{2}$  which *again* implies an integer spin. No matter whether  $A$  is odd or even, the p-e model predicts integer spin, which contradicts the observation of spin- $\frac{1}{2}$ .

**27.** Consider a nucleus of element with atomic number  $Z$ , and mass number  $A$ . These two are related by  $A = Z + N$ . NOTE: Only in the proton-neutron model does  $N$  denote the number of neutrons. In the proton-electron model,  $N$  is simply the difference  $A - Z$ .

PROTON-NEUTRON MODEL: This model predicts  $Z$  protons and  $N$  neutrons, for a total of  $Z + N (= A)$  spin- $\frac{1}{2}$  particles. Therefore, if  $A$  is even, the nucleus must have an integer spin, but if  $A$  is odd, the nucleus must have a half-integer spin.

PROTON-ELECTRON MODEL: This model predicts  $A$  protons, but in order to have a charge  $Ze$  there must be  $N$  electrons, for a total of  $A + N (= Z + 2N)$  spin- $\frac{1}{2}$  particles.  $2N$  is always even, of course, so that if  $Z$  is even, the nucleus must have an integer spin, but if  $Z$  is odd, the nucleus must have a half-integer spin.

CONCLUSION: If  $Z$  and  $A$  are both even or both odd (i.e.,  $N$  is even), then the two models make the same prediction. However, if one of  $Z$  and  $A$  is even and the other is odd (i.e.,  $N$  is odd), the two models make different predictions. For example,  ${}^4\text{He}$  has both even  $A$  and  $Z$ , so there is no way to distinguish the models. But for  ${}^{14}\text{N}$ ,  $A$  is even but  $Z$  is odd. The same is true for  ${}^{111}\text{Cd}$ :  $A$  is odd but  $Z$  is even, so this allowed Schuler and Benck to distinguish between the two models.

**28.** The mirror actually reverses front $\leftrightarrow$ back. Your head is still on the top of your image, and your left hand is still on that same side of the image. But because the mirror reverses front $\leftrightarrow$ back, our minds turn our body around so as to fit the image's body. This makes it *appear* to reverse left $\leftrightarrow$ right.

**29.** No, it is not correct to think of the electron as a spinning sphere of charge (see the box on page 38). However, it can be very useful, because that helps us think about its mass and angular momentum in an intuitive way. One must be careful to realize, however, that there is no classical analogue to spin, or to  $g \neq 1$ .

**30.** The angular momentum is  $L = mvr$ , and the magnetic moment is

$$\mu = IA = \frac{q}{T}(\pi r^2) = \frac{q}{2\pi r/v}(\pi r^2) = \frac{qvr}{2}.$$

so the ratio of the magnitudes is

$$\frac{\mu}{L} = \frac{q}{2m},$$

which confirms Eq. (2.7) If  $q > 0$  they point in the same direction, and if  $q < 0$  they are “anti-parallel.”

**31.** (a) The angular momentum is straightforward  $L = I\omega = \left(\frac{2}{5}MR^2\right)\omega$ . (b) The magnetic moment, on the other hand, must be obtained via integration, but it’s the same integration that is used to calculate the moment of inertia  $I$  for a sphere. Hence, you can find it in any standard physics textbook. Here’s an outline:

I’ll assume that the sphere is centered at the origin and is spinning about the  $z$  axis, and I’ll consider a volume element  $dV$  at an arbitrary location  $(r, z)$  within the sphere that has a charge  $dq$ . The infinitesimal moment of inertia  $d\mu$  that an infinitesimal charge  $dq$  contributes is  $d\mu = AdI$ , where  $A = \pi r^2$  and  $dI = \frac{\omega}{2\pi}dq$ . Here,  $r = \sqrt{x^2 + y^2}$  is the cylindrical coordinate (i.e., the horizontal distance from the  $z$  axis, not the distance from the origin). Cylindrical coordinates are natural, and the total magnetic moment is thus

$$\mu = \int d\mu = \frac{\omega}{2} \int dq r^2.$$

where the last integral is identical to the moment of inertia integral, except that you are summing over charge not mass. Since  $dq = \rho_Q dV = \rho_Q r dr d\theta dz$ , the integral becomes

$$\begin{aligned} \mu &= \frac{\omega\rho_Q}{2} \int r^3 dr d\theta dz \\ &= \frac{\omega\rho_Q}{2} 2\pi \int_0^R \left[ \int_0^{\sqrt{R^2-z^2}} r^3 dr \right] dz \\ &= \frac{1}{5}QR^2\omega. \end{aligned}$$

There is no dependence on  $\theta$  so that integration gives  $2\pi$ , and the  $r$  integration is over a disk, then the  $z$  integration adds up all the disks stacked to make a sphere. (c) The ratio is the same as before

$$\frac{\mu}{L} = \frac{Q}{2M}.$$

**32.** (a) The current is  $I = qv/2\pi r = 1.1 \times 10^{-3}$  A, and the magnetic moment is  $\mu = IA = 9.3 \times 10^{-24}$  Am<sup>2</sup>. (NOTE: the SI units of  $\mu$  can also be written as J/T.) (b) This is just like the Earth calculation,  $L = mvr = 1.1 \times 10^{-34}$  kg m<sup>2</sup>/s. (NOTE: The SI units of  $L$  can also be written as J s.) (c) The intrinsic magnetic moment is, of course, on the order of a Bohr magneton, so it turns out that the orbital  $\mu$  that we just calculated is exactly one Bohr magneton. The intrinsic spin is on the order of  $\hbar$ , which again is exactly the same as the orbital angular momentum that we just calculated!

**33.** See the solution to Problem 31 for the technique. The answer should be  $g = 5/3$ .

**34.** The gravitational potential energy between two protons is  $U = -Gm_p^2/r$ , which has the same form as the Coulomb potential energy. The numerator gives the strength of the force, and is

$$Gm_p^2 = 1.865 \times 10^{-64} \text{ J m} = 1.16 \times 10^{-36} \text{ MeV fm.}$$

So the gravitational force is *significantly* weaker than either the electric force or the color force, and we can safely ignore it for the rest of this book.

**35.** (a) Since  $U = -a/r + br$ , setting  $U(r_0) = 0$  and solving for  $r_0$  gives

$$r_0 = \sqrt{\frac{a}{b}} = 0.322 \text{ fm.}$$

(b) Evaluating the force at this distance gives

$$F_r(r_0) = - \left[ \frac{a}{r^2} + b \right]_{r_0} = -2b = -1826 \text{ MeV/fm} = -2.93 \times 10^5 \text{ N.}$$

This is a very large force! Especially acting on a small object. (c) At what value of  $r$  does  $U \equiv U_\pi = m_\pi c^2 = 135 \text{ MeV}$ ? Solving  $U_\pi = -a/r + br$  for  $r$  results in a quadratic equation, and we want the plus sign (the minus sign results in a negative value for  $r$ )

$$R = \frac{U_\pi \pm \sqrt{U_\pi^2 + 4ba}}{2b} = 0.404 \text{ fm,}$$

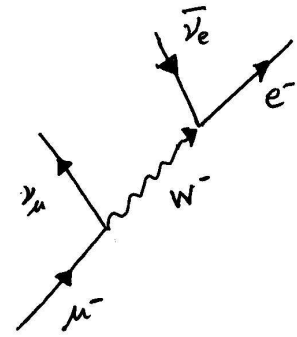
which shows that if you pull two quarks apart 0.082 fm further than their equilibrium separation distance, you have put in enough work (i.e., energy) to create a pion! (d) And a pion *will* be created in the process, with each pair of quarks combining to form a new meson, therefore not allowing you to observe any free quarks. Challenge: draw a Feynman diagram for this process.

**36.** There are four force terms between the two pairs of charges: two repulsive, and two attractive. The net *attractive* force that one dipole exerts on the other is

$$\begin{aligned} F &= \frac{q^2}{4\pi\epsilon_0} \left( -\frac{1}{r^2} + \frac{1}{(r+a)^2} + \frac{1}{(r-a)^2} - \frac{1}{r^2} \right) \\ &= \frac{q^2}{4\pi\epsilon_0 r^2} \left( -2 + \frac{1}{(1+a/r)^2} + \frac{1}{(1-a/r)^2} \right) \\ &\approx \frac{q^2}{4\pi\epsilon_0 r^2} \left( -2 + \left\{ 1 - 2\frac{a}{r} + 3\frac{a^2}{r^2} \right\} + \left\{ 1 + 2\frac{a}{r} + 3\frac{a^2}{r^2} \right\} \right) \\ &= \frac{q^2}{4\pi\epsilon_0 r^2} \left( 6\frac{a^2}{r^4} \right) = \frac{6}{4\pi\epsilon_0} \frac{p^2}{r^4}, \end{aligned}$$

where I've used the binomial expansion  $(1 + \xi)^{-2} = 1 - 2\xi + 3\xi^2 + \dots$ , and kept only the first nonzero term. This force falls off as  $1/r^4$ , which is much weaker than the bare Coulomb force. It's a "residual" Coulomb force, and is similar to the strong nuclear force, which is a residual color force.

**37.** The first thing to remember about Feynman diagrams is that at each vertex, there are several quantities that must be conserved EXACTLY. For this problem, the relevant ones are electric charge, electron lepton number, and muon lepton number. Of course, mass is NOT conserved exactly, because of the Heisenberg uncertainty principle. The figure shows a muon changing to a muon-neutrino and a  $W^-$  (exchange particle). The  $W^-$  then “decays” into an electron and an electron anti-neutrino, just as in the neutron decay shown in Fig. 2.7.







# Chapter 3

## Introduction to Nuclear Physics

*For a nucleus to be stable it must have a mass which is less than the combined masses of any pair of nuclei made by subdividing it. — Hans Bethe*

<p><b>nucleon</b> a proton or a neutron <b>nuclide</b> a specific nucleus with <math>Z</math> protons and <math>N</math> neutrons (plural: nuclides or nuclei) <b>isotopes</b> nuclides with identical <math>Z</math> but different <math>N</math> <b>isotones</b> nuclides with identical <math>N</math> but different <math>Z</math> <b>isobars</b> nuclides with identical <math>A</math> <b>isomer</b> a nuclide in an excited state</p>
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The nuclei of atoms of ordinary matter consist of protons and neutrons. The atomic number  $Z$  is the number of protons in a nucleus, and  $N$  is the number of neutrons. The sum is  $A = Z + N$ , which is called the atomic mass number. Unlike chemical (or atomic) properties, which are determined solely by  $Z$  (because  $Z$  is also the number of electrons in the atom, and the interactions between these electron *are* chemistry), the nuclear properties depend on both the proton *and* neutron number. This is because the forces through which the nucleons interact, in addition to the electromagnetic force, are the strong and weak nuclear forces. Because they consist of quarks, both protons and neutrons interact via these nuclear forces. In fact, both protons and neutrons (nucleons) interact identically via the strong nuclear force because they have the same “strong charge.”

The notation for an isotope of element X is  ${}^A_Z\text{X}_N$ , which is usually shortened to  ${}^A\text{X}$ . For example, the common isotope of helium, denoted  ${}^4\text{He}$ , consists of 2 protons and 2 neutrons. The fact that it is helium automatically means  $Z = 2$ , and the number of neutrons can be determined from the values of  $A$  and  $Z$  ( $N = A - Z = 2$ ). The less common isotope of helium is  ${}^3\text{He}$ , pronounced “helium-3,” which consists of 2 protons and 1 neutron. Our first task is to investigate the intrinsic properties of nuclei in the same way we looked at particles in Chapter 2. The relevant properties are also the same: mass, electric charge, color, spin, and magnetic moment. There is one new property, and that is size.

### 3.1 Mass

The mass of a nuclide with  $Z$  protons and  $N$  neutrons (which I will denote by  $M_{Z,N}$ ) is approximately given by

$$M_{Z,N} \approx Zm_p + Nm_n. \quad (3.1)$$

The equality is not exact because each nucleus has some binding energy (see Eq. 2.1). But the fact that it is a good approximation is one of the clues that led to the discovery of the periodic table—elements had atomic weights that were almost integer multiples of the atomic weight of hydrogen. For a nucleus, this binding energy is defined as

$$B(\text{nucleus}) \equiv (Zm_p + Nm_n - M_{Z,N}) c^2, \quad (3.2)$$

where  $M_{Z,N}$  is the mass of the nucleus with  $Z$  protons and  $N$  neutrons. As discussed on page 20, the constituent particles are taken to be nucleons, rather than quarks.

For example, the  ${}^4\text{He}$  nucleus ( $\alpha$  particle) is one of the most tightly bound nuclei, which can be seen by using the known masses of protons, neutrons, and  $\alpha$  particles in the calculation of the binding energy:

$$\begin{array}{r|l} 2 \times m_p & 2 \times 1.007\,276 \text{ u} \\ +2 \times m_n & 2 \times 1.008\,665 \text{ u} \\ -m_\alpha & 4.001\,506 \text{ u} \\ \hline = B/c^2 & 0.030\,376 \text{ u} \end{array}$$

Converting the atomic mass units (u) to MeV results in 28.295 MeV. Since  $A = 4$ , this binding energy is about 7.07 MeV *per nucleon*, or, as it is commonly denoted  $B/A = 7.07$  MeV.<sup>1</sup> Note that in this calculation, I used *nuclear* masses because that is the quantity with which  $B$  is defined in Eq. (3.2). However, for atoms with many electrons it is difficult to measure the mass of the nucleus and much easier to measure the mass of the entire atom. For this reason, in tables such as Nubase, it is the *atomic* masses that are listed. Atomic masses can be used, but the proper number of electrons must be included, as in Eq. (3.5) below. Strictly speaking this is incorrect because the binding energy of the electrons (to the atoms) must be included. However, this electron binding energy typically is much smaller than the uncertainty in the atomic masses (see page 95), and therefore for practical reasons it will result in the correct numerical answer.

Remember that the binding energy is a theoretical construct which says how much energy would be released *if we were able to break the compound nucleus apart into its constituent nucleons*. However, it is not usually possible to construct a compound particle simply by “fusing” the constituent particles. Let’s take helium as an example. How is it actually created? In the core of the Sun the nucleus of  ${}^4\text{He}$  is produced in a series of nuclear fusion reactions called the “proton-proton chain,” and the net result of these reactions is

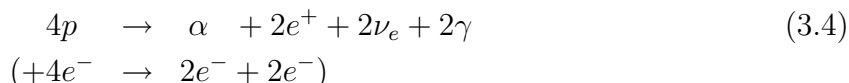



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<sup>1</sup>The binding energies of nuclei are more commonly expressed as the binding energy per nucleon,  $B/A$ , rather than just the binding energy,  $B$ , because  $B/A$  gives information on whether a given type of reaction (e.g., fission or fusion) is exothermic or endothermic.

where the “ $\alpha$ -particle” is a common name for the nucleus of  ${}^4\text{He}$ . It is four *protons* (not two protons and two neutrons) that fuse together, but in the process (which must involve the weak interaction) two of those protons are converted to neutrons, plus the requisite positrons, neutrinos, and photons.<sup>2</sup> How much energy is released in this reaction? That is, what is the  $Q$  value? Using the proton and  $\alpha$ -particle masses (i.e., the nuclear masses), and not including the neutrino masses in Eq. (3.3), I get  $Q/c^2 = 0.026501$  u, or  $Q = 24.685$  MeV. Since  $Q$  is positive, this is an exothermic reaction.

However, if we want to analyze carefully what happens to the positrons, it’s possible to use atomic masses. To see this, let’s add four electrons to each side of the reaction in (3.3)



where I have grouped the electrons with their respective nuclei, but there are still two electrons left over. The positrons that were created will annihilate with any electrons nearby and this annihilation process results in the creation of four photons. So, in reality the net reaction is: four hydrogen *atoms* are converted into one helium *atom* plus six photons (which escape the Sun and illuminate the Earth) and two neutrinos (which head off into space and rarely interact with matter)



Of course, it’s too hot and dense for neutral atoms to exist in the solar core, so again this reaction equation is a theoretical construct that allows us to properly take into account all of the energy released. Finally, therefore, we can calculate  $Q$  using atomic masses (it is simply the mass difference between one  ${}^4\text{He}$  atom and four  ${}^1\text{H}$  atoms), which is 26.731 MeV. This is not the same as the (theoretical) binding energy, but is the (practical) energy released. Where does this energy go? Most of it is taken away by the photons, but each neutrino carries 0.26 MeV away, on average, and this 0.52 MeV is lost forever as the neutrinos leave the sun. (Neutrinos can pass through about one light year of lead before having a significant probability of reacting.) Hence, the final energy that is available to illuminate and heat the Earth is about 26.21 MeV per net fusion reaction. (See Problems 40 and 41 for a calculation of how many of these nuclear fusion reactions actually occur.)

Another measure of the binding energy of a nucleus is its *mass excess*,  $\Delta$ , defined as

$$\Delta \equiv M_{Z,N} - A \times (1 \text{ u}). \quad (3.6)$$

The dimension of  $\Delta$  is mass, and therefore the dimension of  $\Delta c^2$  is energy. In standard tables, such as NUBASE, the mass excess is listed in keV (rather than the actual mass or rest energy).<sup>3</sup> In reality, then,  $\Delta c^2$  is given, and if you wish you can calculate  $M_{Z,N}$  from Eq. (3.6).

<sup>2</sup>Note that both electric charge and lepton number are conserved, as they must be.

<sup>3</sup>A third way to characterize the nuclide mass is by its “packing fraction,”  $f$ , where  $f \equiv \Delta/A$ . This was first proposed in 1915 by Harkness and Wilson [*J. Amer. Chem. Soc.* **37** 1367 (1915)] while trying to understand why isotopes had masses that differed from integral multiples of the hydrogen mass.

The stability of a given nucleus can be determined using the criteria proposed by Hans Bethe in the quote on the top of page 55: for a nucleus to be unstable, it is only necessary to find *one* pair of nuclei whose combined masses are less than the nucleus in question. That is, if the value of  $Q$  is positive for any reaction where the nucleus in question splits, then the nucleus is unstable. In particular, it is found that there are no stable nuclei with  $A = 5$  or  $A = 8$ , a fact that is extremely important in the explanation of element formation in the early universe. Let's investigate the possible isobars with  $A = 5$ :  ${}^5\text{H}$ ,  ${}^5\text{He}$ ,  ${}^5\text{Li}$ ,  ${}^5\text{Be}$ , and  ${}^5\text{B}$ . The helium and lithium are unstable to the emission of a neutron and proton, respectively,



with reaction energies of 893.8 keV and 1.966 MeV, respectively, and half-lives of 700 ys and 370 ys,<sup>4</sup> respectively.  ${}^5\text{Be}$  and  ${}^5\text{B}$  have so many protons that the repulsive electric force overwhelms the attractive strong force. Finally,  ${}^5\text{H}$  has too many neutrons, and decays via *double* neutron emission



## Shell model

Certain nuclei are especially tightly bound, which means that they have a large binding energy per nucleon, a large  $B/A$ . One of these is  ${}^4\text{He}$ , as well as the other “even-even” nuclei (those with an even number of protons and an even number of neutrons), e.g.,  ${}^{12}\text{C}$ ,  ${}^{16}\text{O}$ , and  ${}^{20}\text{Ne}$ . In an attempt to understand this structure and regularity, quantum mechanics has been used to create a “shell model” of the nucleus, where the nucleons arrange themselves in shells similar to the electron shells in an atom. This model is more complicated than the atomic model because in the atomic model the electrons all orbit in the strong electric field of the nucleus and the inter-electron interaction is weak. In the nuclear shell model, however, there is no central object in a nucleus, however, so each nucleon moves in a “field” due to all the other nucleons combined. This makes the nucleus a “many-body” problem at its most fundamental level.

Electrons in atoms are the most tightly bound in the inert gases, listed in the right-most column of the periodic table. This is because in these atoms the outermost electron shell is filled. Their atomic numbers are

2	10	18	36	54	86	118
He	Ne	Ar	Kr	Xe	Rn	Uuo

where I've listed the element symbol below these special values of the atomic number.

There are also special values of the number of nucleons, and these are called MAGIC NUMBERS. For nuclei, these magic numbers are

2	8	20	28	50	82	126
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<sup>4</sup>1 ys = 1 yoctosecond =  $10^{-24}$  s.

If a nucleus has either  $N$  or  $Z$  equal to one of these “magic numbers,” then that nucleus is especially tightly bound. If both  $N$  and  $Z$  are magic, then that nucleus is called “doubly magic.” For example,  ${}^4\text{He}$  and  ${}^{16}\text{O}$  are both doubly magic. This means that an extra nucleon added to one of these nuclei is especially loosely bound.  ${}^4\text{He}$  takes this to an extreme, since it requires about 20 MeV to remove a proton or neutron,<sup>5</sup> but an additional proton or neutron is not bound at all.<sup>6</sup>

The higher magic numbers are less striking, but they exhibit observable effects, nonetheless. Tin, for example ( $Z = 50$ ), is the element with the largest number of stable isotopes, ten. Also, in  $\alpha$  decay (see Section 3.8), when the emission of an  $\alpha$  particle removes the 125th and 126th neutrons from a nucleus (which should be strongly bound), the resulting energy of the  $\alpha$  particle is much lower than when the 127th and 128th neutrons are removed (which are weakly bound) — see Problem 45. Finally, the heaviest stable nucleus is that of  ${}^{208}\text{Pb}$ , which is doubly magic, and lead ( $Z = 82$ ) is the endpoint of all four naturally radioactive decay series (see Sec. 3.8.1).

## 3.2 Electric Charge

Unlike mass, charge is strictly conserved. This means that the charge of a nucleus is given by

$$Q_{Z,N} = +Ze, \quad (3.9)$$

since each nucleus consists of  $Z$  protons, and each proton has an electric charge of  $+e$ .

## 3.3 Color

All nuclei are color neutral! This is to be expected because their constituents, the protons and neutrons, are color neutral. This means that there is no color interaction between nucleons. As discussed on page 40, there is, however, a *residual* color interaction, called the strong force, which binds nucleons together into nuclei. Also, nuclei interact via this strong force during fusion and fission, for example.

## 3.4 Size

This is one property that is new. Elementary particles are point-like; they have no physical dimension. This is not to say that their influence is point-like: they do interact over long distances via the four fundamental forces, but those forces do not require any structure in the 12 elementary particles. Compound particles, on the other hand, *do* have structure, and this means that they have a certain size and shape. What do I mean by size? It is a definition that depends on the method of measurement. Hence, whenever someone claims that a particular object has a certain size and shape, they must also state how they came

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<sup>5</sup>You can calculate that it takes 20.58 MeV to remove a neutron and 19.81 MeV to remove a proton from an  $\alpha$ -particle.

<sup>6</sup>Recall that there are no stable nuclei with  $A = 5$ .

to that conclusion, i.e., how was it measured?<sup>7</sup> For example, when you hold a magnet in your hand and deduce its size from your sense of touch, you are tracing its outlines by the contact with your hand. However, a second magnet can be affected when it is not touching the first magnet, and therefore you might conclude that the first magnet is *larger* than its physical contact outlines suggests.

Nuclides are approximately spherical with a radius  $R$  given by

$$R = R_0 A^{1/3}, \quad (3.10)$$

where  $R_0$  is a constant. This formula implies that nuclei are “incompressible”: the volume  $V$  of a spherical nucleus with  $A$  nucleons is just

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi R_0^3 A = V_1 A, \quad (3.11)$$

or the volume  $V$  is the volume of one nucleon,  $V_1 = 4\pi R_0^3/3$ , times the number of nucleons,  $A$ . Neutron stars are examples of matter that have been compressed to the nuclear density because they are essentially comprised of neutrons, packed like marbles.

The exact value of  $R_0$  depends on the method by which it is measured. For example,  $R_0 \approx 1.07$  fm when electron scattering is used.<sup>8</sup> That is, if a particular nucleus is bombarded with electrons, the primary interaction is via the electromagnetic force (since electrons do not feel the strong force—both feel the weak force, but that, of course, is weak), and therefore the scattered electron essentially “sees” the positive electric charge distribution during the process of scattering. If nuclear scattering is used (e.g., neutrons), then the two objects interact via the strong nuclear force, and experiments give  $R_0 \approx 1.4$  fm. It makes sense that the electric charge distribution occupies a smaller volume than the nuclear charge distribution due to the fact that all the nucleons carry nuclear charge, i.e., color, but only the protons have electric charge.

Both distributions can be described by an empirical function of the density (either electric charge density or nucleon density) that depends on radial location

$$\rho(r) = \frac{\rho_0}{1 + e^{(r-R)/b}}, \quad (3.12)$$

where  $R$  is given in Eq. (3.10). In the case when  $\rho$  describes the electric charge density,  $b \approx 0.55$  fm, and  $\rho_0 \approx 0.075$  e/fm<sup>3</sup>. The value of the central *mass* density  $\rho_{0,mass}$  is approximately constant for all nuclei. On the other hand, as shown in Fig. 3.1, the value of the central *electric charge* density  $\rho_{0,charge}$  decreases slightly for nuclei with large  $A$ , which is due to the fact that the fraction of protons in the nucleus,  $Z/A$ , decreases as  $A$  increases (see Section 3.9).

## 3.5 Spin

The intrinsic angular momentum of a nucleus is the vector sum of the spins of the individual nucleons (which are spin  $\frac{1}{2}$ ) plus any orbital angular momentum they may have. For nuclei,

<sup>7</sup>That is, how do I know what I know? — epistemology!

<sup>8</sup>1 fm = 1 femtometer =  $10^{-15}$  m.

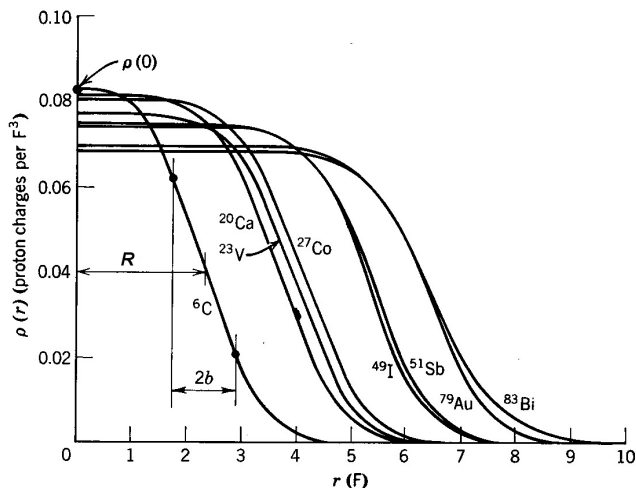


Figure 3.1: Plot of nuclear charge density  $\rho$  versus radial distance  $r$  for several nuclei, from carbon to bismuth. The charge densities  $\rho(r)$  were obtained by fitting experimental scattering data to Eq. (3.12). The radial distance is in femtometers (F), and charge density is in  $e/\text{fm}^3$ . The variables  $\rho(0) = \rho_0$ ,  $R$  and  $b$  are noted for the  ${}^6\text{C}$  nucleus. Note that the atomic number  $Z$  is given in the superscript for each element rather than the atomic mass number  $A$ . Figure 15-6 from Eisberg and Resnick, *Quantum Physics*.

the usual convention is to denote the spin quantum number by  $i$ , rather than  $s$ . It is still intrinsic angular momentum, but simply denoted by a different letter. This means that the magnitude of the nuclear spin is  $I = |\vec{I}| = \sqrt{i(i+1)}\hbar$ , and the  $z$  component is  $I_z = m_i\hbar$ , where  $m_i$  and  $i$  are integers or half integers, and  $|m_i| \leq i$ .

Recall that we determined the proton and neutron spins by adding the spins of the quarks that make up the nucleons. Since the quarks are all spin  $\frac{1}{2}$ , and if we assume that there is no orbital angular momentum, a baryon must have either spin  $\frac{1}{2}$  or spin  $\frac{3}{2}$ . Some baryons, like the  $\Omega^-$ , have spin  $\frac{3}{2}$ . Given this method of adding spins, we have seen from our analysis of the proton-electron model of nuclei (Section 2.3.2) that if the number of constituent particles of spin  $\frac{1}{2}$  is even, then the total spin must be an integer, and if the number of particles is odd, then the total angular momentum must be a half integer. Since both nucleons are spin  $\frac{1}{2}$ , this rule also applies to complex nuclei:

- if  $A$  is even, then  $i = 0, 1, 2, \dots$
- if  $A$  is odd, then  $i = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$

For example, the deuteron (the nucleus of deuterium,  ${}^2\text{H}$ ) has spin  $i = 1$ , and the  ${}^7\text{Li}$  nucleus has spin  $i = \frac{3}{2}$ . These both follow the above rule. (Note that because the orbital angular momentum quantum number  $\ell$  is always an integer, the inclusion of orbital angular momentum will not change the above rule.) If there is no orbital angular momentum in the deuteron, it means that the spins of the proton and neutron are aligned, i.e., they are both spin up or spin down. At first glance, you might think that this violates the

Pauli exclusion principle, but that only applies to *identical* particles, and the neutron and proton are distinguishable.

An interesting fact about nuclei is that pairs of protons like to have their spins aligned in opposite directions, and so do neutrons—this puts them in a lower energy state. This is similar to the behavior of electrons in atoms, and the fact that each subshell can contain an even number of electrons, but each pair must have their spins anti-parallel. The observational consequence of this is that all even-even nuclei have zero spin! The nuclei of  ${}^4\text{He}$ ,  ${}^{12}\text{C}$ ,  ${}^{16}\text{O}$ , etc., have  $i = 0$ . These nuclei also have large binding energies—they are very stable and tightly bound, which is another indication that the nucleons *like* to be in this configuration. Conversely, nuclei do not like to consist of an odd number of protons and neutrons. In fact, there are only four stable nuclei that are “odd-odd,” with an odd number of protons and an odd number of neutrons:  ${}^2\text{H}$ ,  ${}^6\text{Li}$ ,  ${}^{10}\text{B}$ , and  ${}^{14}\text{N}$ .

## 3.6 Magnetic Moment

**Proton** What is the magnetic dipole moment of the proton, the simplest atomic nucleus? To answer this question, a knowledge of its internal structure is needed. On the other hand, a measurement of  $g_p$  can reveal information about its internal structure. As we have seen in Sec. 2.4, the classical result for a uniformly charged sphere would lead us to expect that

$$\vec{\mu}_p = \frac{+e}{2m_p} \vec{S}, \quad (3.13)$$

where the proton mass is used instead of the electron mass, as is appropriate. But, if the proton is a point particle, then  $g \approx 2$ , and Dirac’s theory would predict twice the value in Eq. (3.13). Both predictions, however, turn out to be wrong! The  $g$ -factor for the proton, first measured in 1933 by Otto Stern [Nobel Prize, Physics, 1943], Robert Frisch (the nephew of Lise Meitner, who worked out the theory of nuclear fission with her in 1938 after she had fled Nazi Germany), and Immanuel Estermann,<sup>9</sup> is  $g_p = 5.585\,694\,702(56)$ , and it is defined by



$$\vec{\mu}_p = g_p \left( \frac{+e}{2m_p} \right) \vec{S}, \quad (3.14)$$

Considering just the  $z$  component, and noting that  $S_z = m_s \hbar$ , allows us to express  $\mu_z$  of the proton in terms of the “nuclear magneton,”  $\mu_N$ ,

$$\mu_z = g_p \left( \frac{e\hbar}{2m_p} \right) m_s = g_p m_s \mu_N, \quad (3.15)$$

<sup>9</sup>In 1933, using the molecular beam method—the same as the Stern-Gerlach experiment (see App. C)—Stern measured  $g = 5.0 \pm 10\%$ . Six years later the magnetic resonance method, pioneered by Rabi and Ramsey, was much more accurate:  $5.570 \pm 0.04$ . In 1933, Pauli and other theorists advised Stern to not make this measurement because, he said, “we know the moment of the proton, because we know the difference in mass between the proton and the electron, and we know the magnetic moment of the electron.” Of course, they were wrong!



where  $\mu_N \equiv e\hbar/2m_p = 5.050\,783\,43(43) \times 10^{-27}$  J/T.<sup>10</sup> This result tells us two things. First, the proton is *not* a point particle, which we already knew because it is comprised of three quarks, which *are* point particles. Second, it is not a uniform sphere—the charged quarks have spin and orbital angular momentum.

Why does the proton’s magnetic moment have this value? It must come from the addition of the magnetic moments due to the orbital and spin angular momenta of the three quarks. But the actual quantum state that the quarks are in is not well known. In fact, as we saw in Sec. 2.1, not even the masses of the quarks are well known. To understand the proton’s  $g$ -factor requires a detailed study of QCD, the quantum field theory of the strong force, and how it interacts with the electromagnetic force associated with the quarks’ spins.

**Neutron** What about the neutron? The classical prediction is that its magnetic dipole moment is *zero* since it is an electrically neutral particle. However, just like the proton, we know that it is not a point particle, but consists of charged quarks, which not only move around but also have their own spins and magnetic dipole moments. The measured value for the neutron’s  $g$ -factor is  $g_n = -3.826\,085\,46(90)$ , which implies that neutrons have some internal structure, as expected. The fact that it is negative means that  $\vec{\mu}$  points anti-parallel to  $\vec{S}$ .

Even though the neutron’s electric charge is  $q = 0$ , its  $g$ -factor is defined in the same way as the proton, Eqs. (3.14) and (3.15). In fact, since both the proton and neutron are spin  $\frac{1}{2}$  particles, and  $m_s = \pm\frac{1}{2}$ , Eq. (3.15) shows that measured values of  $\mu_z$  are multiples of half a nuclear magneton,

$$\mu_z = \pm\frac{g}{2}\mu_N. \quad (3.16)$$

For this reason it is common to quote  $g/2$  rather than  $g$  itself

$$\frac{g_p}{2} \approx 2.793 \quad \text{and} \quad \frac{g_n}{2} \approx -1.913. \quad (3.17)$$

**Deuteron** With this knowledge, can we predict the magnetic moment of the deuteron? Yes, if we remember that the spin of the deuteron consists of the spins of its constituents plus any orbital angular momentum that they may have. Since the deuteron spin is measured to be  $i = 1$ , the simplest explanation is that the neutron spin is aligned with the proton spin and there is no orbital angular momentum (i.e., they are in an  $\ell = 0$  state, which for those of you who remember your atomic orbital theory is another way of saying that they are in an “ $S$  state”<sup>11</sup>). Therefore, the deuteron’s  $g$ -factor *should* be (keeping only six decimal places)

$$\begin{aligned} g_D(\text{predicted}) &= \frac{g_p}{2} + \frac{g_n}{2} \\ &= 0.879\,805. \end{aligned} \quad (3.18)$$

<sup>10</sup>Notice that the proton magnetic moment (and nuclear magnetic moments in general) are smaller than electron magnetic moments by a factor equal to the mass ratio,  $m_e/m_p$ . For this reason, it is much more difficult to measure magnetic moments of nuclei than electrons.

<sup>11</sup>The subshells for the electronic configuration of an atom are denoted by the letters  $s, p, d, f, \dots$ , which correspond to the angular momentum values  $\ell = 0, 1, 2, 3, \dots$ , respectively. This notation is discussed in more detail in Section 4.3

However, the *measured* value is<sup>12</sup>

$$g_D(\text{observed}) = 0.857\,438. \quad (3.19)$$

The two values differ by 2.6%, which is much larger than the experimental uncertainty, so our assumption that there is zero orbital angular momentum must be wrong. It turns out that the deuteron is not in a pure  $\ell = 0$  state, but is, in fact, in a “superposition state,” with 96% zero angular momentum, and 4% in a “*D*” state, which is characterized by  $\ell = 2$ . (See Problem 50.) Superposition means linear combination, and the deuteron’s orbital angular momentum is a linear combination of  $\ell = 0$  and  $\ell = 2$ . If we calculate the deuteron’s magnetic moment including this small amount of orbital angular momentum, then the electric current due to the proton’s orbital motion gives a contribution to the total magnetic moment, and the agreement with experiment is almost exact.

As is common in science, this new information leads us to two new questions: Why is the deuteron in this superposition state? and, How do we know it is? The answer to the first question is to be found in advanced quantum mechanics, and will not be answered here, but to the second question I can give a partial answer. In 1939, Kellogg, Rabi, Ramsey, and Zacharias were able to determine that the deuteron had a non-zero *electric* quadrupole moment. What does this mean? It means that the shape of the deuteron is not spherical, but more like a football. Specifically, it is a “prolate spheroid,” which is an ellipsoid of revolution about an ellipse’s major axis.<sup>13</sup> If you recall the orbital shapes of the electron clouds in atoms, the  $\ell = 0$  orbitals are spherically symmetric, so that any asymmetry (i.e., anything other than perfect spherical symmetry) implies nonzero orbital angular momentum.

### 3.7 Radioactivity

*To the chemists of the 19th century the atom and the element represented each in its sphere the uttermost limit of chemical subdivision or disintegration, and at the same time the point beyond which it was impossible for experimental investigation to proceed. If it were queried what there was beyond, nothing but more or less vague and fruitless speculations were forthcoming. This line of demarcation, for so long regarded as insurmountable, has now been swept away, at all events in principle. Nowadays the inner structure of atoms and the laws regulating that structure belong to the problems that can be made the subject of discussion in a thoroughly practical and at the same time fully scientific manner, thanks to the exactness of the measurements which have been taken. The results already arrived at are not only of the utmost importance in themselves, but derive perhaps a still greater significance from the numerous possibilities, wholly unsuspected ten or twelve years ago, which have been thrown open for the continuance of the work of investigation in this department of science. —*  
Presentation of 1908 Nobel Prize in Chemistry to Ernest Rutherford

<sup>12</sup>First measured by Estermann and Stern in 1933.

<sup>13</sup>For reference, the Earth’s shape is an oblate spheroid, which again is a figure of revolution, but this time the ellipse is revolved around its minor axis, and resembles a pancake.

## Historical Background

On March 1, 1896, Antoine Henri Becquerel discovered radioactivity. His motivation was to look for X-rays (recently discovered in November 1895 by Roentgen) from phosphorescent materials, and he was familiar with the phosphorescent properties of uranic salts,<sup>14</sup> which, of course, contain uranium. He wrapped a photographic plate (a piece of glass covered with a photographic emulsion) in black paper, and placed on the paper a piece of a phosphorescent substance. He exposed the combination to the sun for several hours, in the expectation that the sunlight would cause the uranium to phosphoresce, and that phosphorescent light from the uranium would penetrate the black paper and leave an image on the emulsion. It worked as expected, but then there came a week of cloudy weather and the sun did not shine. Becquerel put his plates and uranium in the cupboard for a week (without being exposed to sunlight), and for some reason he decided to develop those plates, even though he expected nothing. However, his intuition was correct, and he discovered that the plates showed an image, just as if it had been in the sun!

This was the first step in the discovery and understanding of radioactivity, and the “rays” that must have been emanating from the uranium were called “Becquerel rays.”

One week later, on March 9, Becquerel discovered that the rays could discharge an electroscope, which meant that the rays were charged. At that time there were two types of rays known, cathode rays (which a year later would be shown by Thomson to be electrons) and light rays (which had been shown by Maxwell and Hertz to be electromagnetic waves). Of course, the “X-rays” of Roentgen would turn out to be high-frequency electromagnetic waves, and the Becquerel rays were nothing but electrons, but that was not clear for quite a while. In fact, the uranium sample emitted both electrons and  $\alpha$ -particles, but the  $\alpha$  particles were easily stopped by the paper and so did not contribute to the darkening of the emulsion.

Becquerel’s family was quite prodigious. Along with his grandfather, Antoine César, his father, Alexandre Edmond, and his son, Jean, the four of them continuously held the chair of physics at the Museum of Natural History in Paris from 1838-1948, a span of 110 years! The four of them studied many aspects of physics, including thermoelectric phenomena, luminescence, infrared spectroscopy, magnetic polarization by crystals, and magneto-optics. In fact, after his discoveries, Antoine Henri said, “These discoveries are only the lineal descendants of those of my father and grandfather on phosphorescence, and without them my own discoveries would have been impossible.”

The second step in the understanding of radioactivity came in 1898 when Marie and Pierre Curie found that the element thorium ( $Z = 90$ ) was also radioactive. In addition, they discovered two new elements due to their radioactivity, which they named polonium and radium. These latter two they found by chemically isolating them from their sample of pitchblende. Pitchblende is a black mineral, mainly  $\text{UO}_2$ , but it also has some impurities, and these are what the Curies found. The Curies won the 1903 Nobel Prize in Physics, jointly with Becquerel, for their investigation into radioactivity. In addition, Marie won the 1911 Nobel Prize in Chemistry for the discovery of polonium and radium. At this point, even though radioactivity was not at all understood yet, two questions had become common: Where did the energy associated with the activity come from? and: Were *all*

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<sup>14</sup>Specifically, Becquerel used uranyl disulfate,  $\text{K}_2\text{UO}_2(\text{SO}_4)_2 \cdot 2\text{H}_2\text{O}$ .

elements radioactive (but perhaps with very long lifetimes)?

The third event in our story occurred in 1899 when Rutherford deduced that there were two different types of Becquerel rays:  $\alpha$  rays and  $\beta$  rays. They were distinguished by their ability to penetrate matter:  $\alpha$  rays were easily absorbed in a few centimeters of air (Becquerel's black paper absorbed them);  $\beta$  rays were more penetrating—it took several cm of air before they were absorbed. Later it was determined that  $\alpha$  rays were actually the nuclei of  ${}^4\text{He}$ , and  $\beta$  rays were electrons. In 1900, Paul Villard in Paris observed a third type of ray emitted by radium that was even more penetrating than  $\beta$  rays (but it was not charged), and he called them  $\gamma$  rays. These, of course, were photons, but that was not determined until 1914.

The final piece of the puzzle, the fourth step, was put in place in 1902 when Rutherford and Frederick Soddy [Nobel Prize, Physics, 1921] developed their “transformation theory.” This theory was an explanation of what was occurring during radioactive decay: In modern terminology, a “parent” nucleus was transformed into a “daughter” nucleus when an  $\alpha$  or  $\beta$  ray was emitted. Soddy had originally suggested the term “transmutation theory,” but Rutherford objected, believing that people would think they were proposing medieval alchemy. In fact, though, that was *exactly* what they were doing: radioactivity was changing one element into another! Another part of the transformation theory was the observation that the process of transformation decayed exponentially with time. They discovered this while investigating a gas called “thorium emanation,” which we now know was an isotope of radon,  ${}^{220}\text{Rn}$ . Most of the daughter elements were solids at room temperature, so that they remained locked in the original rock. Radon, however, is a gas, and so when it is created as a part of a series of radioactive decays it can be easily isolated. Rutherford and Soddy found that no matter when they started observing, the activity of  ${}^{220}\text{Rn}$  was reduced by half in one minute, and this allowed them to describe radioactivity mathematically as an exponential decay.



### 3.8 Radioactive decay

Regardless of the type of decay ( $\alpha$ ,  $\beta$ , or  $\gamma$ ) that a nucleus undergoes, it is a random process. That is, given a single nucleus, I can't tell you *when* that nucleus will decay; all I can tell you is the *probability* that it will decay within a certain time. Given a large number of identical nuclei, I can tell you what fraction of those nuclei will have decayed after a certain time has passed. The reason for this probabilistic description is that the underlying physical process that is occurring is quantum mechanical in nature. Our current understanding is that the predictions of quantum mechanics are strictly probabilistic. That is, while the Schrodinger equation (for example) is a deterministic differential equation that predicts the future state of a particle given the current state, that future state is usually a superposition state, i.e., a linear combination of possible states. When we make a measurement of a certain property of that particle, we measure only one value, not a superposition of values, and quantum mechanics is, in part, a prescription for turning the deterministic solution of the Schrodinger equation into a set of probabilities for measuring

particular values of a given property. We will discuss this in more detail in Chapter 7.

Experimentally, it is found that the number of unstable radioactive nuclei  $N(t)$  in a sample decays exponentially with time

$$N(t) = N_0 e^{-\lambda t}, \quad (3.20)$$

where  $N_0$  is the number of nuclei at time  $t = 0$  and  $\lambda$  is called the decay constant. You can't, of course, measure the number of nuclei in a sample (of rock, say), but you *can* measure the activity  $\mathcal{A}$  as a function of time. With an electroscope, for example, you can measure how quickly the electroscope is charged, which means you are really measuring the number of charged particles per second that your sample is emitting. The number of decays per unit time is just the time derivative of  $N$ , which is called the "activity,"<sup>15</sup>

$$\mathcal{A}(t) \equiv -\frac{dN}{dt} = (\lambda N_0) e^{-\lambda t}, \quad (3.21)$$

where  $\lambda N_0 \equiv \mathcal{A}_0$  is the initial activity, and the minus sign is there to denote the fact that while  $dN/dt$  is negative,  $\mathcal{A}$  should be positive. Thus, the activity also decreases exponentially with time.

The decay constant  $\lambda$  is related to the more familiar "half life,"  $\tau$

$$\tau = \frac{\ln 2}{\lambda}. \quad (3.22)$$

How is this relation obtained? Simply from the definition of half life, which is that after a time interval equal to one half life, the activity of a given sample decreases by a factor of 2. Setting  $t = \tau$  in Eq. (3.21) and requiring that  $\mathcal{A}(\tau) = \mathcal{A}_0/2$  results in Eq. (3.22). Another, equivalent, definition is that after a time interval equal to  $\tau$  only one half of the original nuclei remain (see Figure 3.2).

### Compound Interest

The mathematics of radioactive decay are identical with the mathematics of compound interest. Consider \$1 invested for one year at an interest rate of 20%. After one year, the bank returns your \$1 and in addition gives you \$0.20 interest so that you have \$1.20 total. If the bank compounds the interest semiannually, then after six months your account is credited with \$0.10, half the annual rate of interest, and during the second six months not only does your principal earn interest, but the *interest* earns interest. At the end of the year you receive  $\$(1.10)(1.10) = \$1.21$ , which is an extra cent more than if the bank compounds only annually. In general, if the annual interest rate is  $p$ , where  $0 < p < 1$ , and the number of times per year that your investment is compounded is  $N$ , then after one year you receive

$$\left(1 + \frac{p}{N}\right)^N$$

<sup>15</sup>What are the units of activity? Well, the SI unit of radioactivity is one decay per second, also known as 1 Bq (1 bequerel) in honor of Antoine. An older unit is 1 Ci (1 curie), which is equal to  $3.7 \times 10^{10}$  Bq. Why this strange number of decays? Because one gram of pure radium (which is mostly  $^{226}\text{Ra}$ ) has this activity. You can show that 1 g contains approximately  $2.26 \times 10^{21}$  atoms, and with a half life of  $5 \times 10^{10}$  s,  $^{226}\text{Ra}$  has an activity of 1 Ci. See Problem 59.

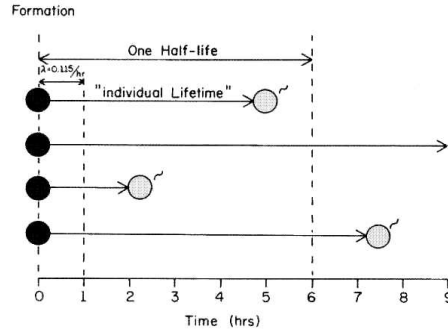


Figure 3.2: Four individual nuclei are “born” at  $t = 0$ . Since their half life is  $\tau = 6$  hours, the decay constant is  $\lambda = \ln 2/\tau \approx 0.116/\text{hr}$ , and after 6 hours half of them have decayed.

times your original principal. This looks rather fishy, because it implies that we can obtain more interest at the end of the year simply by requiring that compounding is performed more often. This is true, but what happens if the bank compounds continuously? Does the final amount approach infinity? No, because of the fact that

$$\lim_{N \rightarrow \infty} \left(1 + \frac{p}{N}\right)^N = e^p. \quad (3.23)$$

In the case of an interest rate of 20% this results in the actual interest paid of \$0.22 per dollar.<sup>16</sup> Of course, for radioactive decay, the percentage rate  $p$  is negative and is related to the decay constant  $\lambda$ .

### Average lifetime

As can be seen from Figure 3.2, each individual nucleus survives for a time equal to its “individual lifetime,” and these individual lifetimes are randomly distributed, and vary considerably within a sample. We have already encountered one useful type of average — the half life  $\tau$ . However, another useful quantity is the “average lifetime” of a nucleus. That is, consider a nucleus that is born at  $t = 0$ . What is its life expectancy? In other words, how long, on average, do we expect it to live before it decays? To calculate this, we need to discuss the mathematics of probability, and the simplest system to consider is a six-sided die.

If you roll a die many times, what is the average  $\langle n \rangle$  of all the numbers that are rolled? If the die is not loaded, you simply add all the numbers you roll, and divide by the number of rolls

$$\langle n \rangle = \frac{\sum_{i=1}^N n_i}{N}, \quad (3.24)$$

where  $n_i$  is the value of the  $i$ th roll, there are  $N$  total rolls, and the sum runs from  $i = 1$  to  $i = N$ . This simple formula, however, hides many of the subtle details of probability theory. First, the denominator should be written as  $\sum_{i=1}^N w_i$ , where  $w_i$  is the statistical

<sup>16</sup>In the parlance of banks, 20% is called the annual percentage rate (APR), and 22% is called the annual percentage yield (APY).

weight (or probability) of the  $i$ th roll. In the case of an honest die,  $w_i = \frac{1}{6}$  for all  $i$ . so you are really summing a lot of “ $\frac{1}{6}$ ”s. What if the die *is* loaded, and each side has a different probability of coming up? In that case, each term in the sum of the numerator must be multiplied by its probability  $w_i$

$$\langle n \rangle = \frac{\sum_{i=1}^N n_i w_i}{\sum_{i=1}^N w_i}. \quad (3.25)$$

The denominator is sometimes called the “normalization.”<sup>17</sup>

Radioactive decay can be treated in exactly the same manner as a loaded die. The probability that a particular nucleus will decay changes with time, and this change must be taken into account. For a radioactive nucleus that is born at  $t = 0$ , the probability that it will decay between time  $t$  and time  $t + dt$  is

$$\begin{aligned} \text{probability} &= \mathcal{P}(t)dt \\ &\propto e^{-\lambda t} dt, \end{aligned} \quad (3.26)$$

where  $\mathcal{P}(t)$  is called the “probability density.” Note that the probability for a nucleus to decay *exactly at* time  $t$  is zero (i.e., take the limit as  $dt \rightarrow 0$ ), because if you are able to measure accurately, you will never obtain a particular time exactly. Now, since time is continuous (whereas dice rolls are discrete), the sums in Eq. (3.24) become integrals, and the average lifetime  $\langle t \rangle$  of one radioactive nucleus is

$$\langle t \rangle = \frac{\int_0^\infty t e^{-\lambda t} dt}{\int_0^\infty e^{-\lambda t} dt}. \quad (3.27)$$

The integrand in the numerator consists of two factors, the quantity that we are averaging,  $t$ , and the weighting factor,  $e^{-\lambda t}$ . The integrand in the denominator only includes the weighting factor because it is there to normalize the answer. Also, notice that we only need the functional form of the probability density  $\mathcal{P}(t)$  up to an unknown multiplicative constant. This is because we will always divide by the normalization, and any constant factor appears in both the numerator and denominator.

You can show (see Problem 52) that when evaluated, the integrals in Eq. (3.27) result in

$$\langle t \rangle = \frac{1}{\lambda} = \frac{\tau}{\ln 2}. \quad (3.28)$$

The half life  $\tau$  is the quantity that can be experimentally measured from a sample of many radioactive nuclei, but the average lifetime  $\langle t \rangle$  is a quantity that gives theoretical (statistical) information regarding each individual nucleus.

### Two-particle decay

The simplest example of radioactivity is “two-particle decay,” where an unstable nucleus (the “parent”) decays into a stable nucleus (the “daughter”), and we wish to keep track

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<sup>17</sup>We could also set  $w_i = 1/6$  for all  $i$  if the die was not loaded. In that case the denominator would be unity, and the weights  $w_i$  would be properly normalized.

of how many of each type there are, as a function of time. For example, polonium-211 decays into a stable isotope of lead via  $\alpha$ -decay



with a half life of  $\tau = 516$  ms. In this case,  ${}^{211}\text{Po}$  is the parent nucleus and  ${}^{207}\text{Pb}$  is the daughter.

Let's go through the mathematics of this case, where  $N_1(t)$  denotes the number of parent nuclei as a function of time, and the parent nuclei have a decay constant  $\lambda_1$ . These parents decay into daughter nuclei, whose number as a function of time is  $N_2(t)$ , and they are stable.<sup>18</sup> The differential equations that govern the number of each type of nucleus are

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \quad (3.30)$$

$$\frac{dN_2}{dt} = +\lambda_1 N_1. \quad (3.31)$$

Equation (3.30) is simply a reproduction of Eq. (3.21), describing how the number of parent nuclei changes with time. Equation (3.31) describes how the number of daughter nuclei changes, and the fact that the right-hand-side is positive shows that they increase with time, but their increase depends only on the decrease of parent nuclei. Note that the sum of the two equations is zero

$$\frac{d}{dt} (N_1 + N_2) = 0, \quad (3.32)$$

which shows that the total number of nuclei is constant—they are simply changing from one type to another.

The solution to Eq. (3.30) can be obtained by straightforward separation of variables

$$N_1(t) = N_{10} e^{-\lambda_1 t}, \quad (3.33)$$

where  $N_{10}$  is the number of parents at  $t = 0$ . Once  $N_1(t)$  is known, then Eq. (3.31) can be solved by straightforward integration

$$N_2(t) = N_{20} + N_{10} (1 - e^{-\lambda_1 t}), \quad (3.34)$$

where  $N_{20}$  is the number of daughters at  $t = 0$ . Plots of the two solutions are shown in Fig. 3.3.

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<sup>18</sup>In general, of course, the daughters are also unstable, and decay into “granddaughters,” which may also be unstable. As Abraham Pais puts it in *Inward Bound* (page 113)

*Radioactive bodies contain unstable atoms of which a fixed fraction decay per unit time. The rest of the decayed atom is a new radio-element which decays again, and so forth, 'till finally a stable element is reached.*



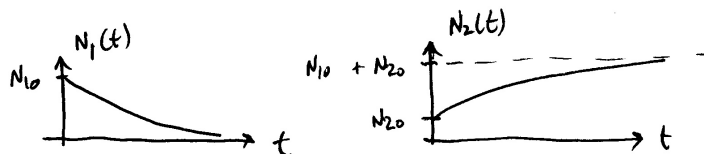


Figure 3.3: Time histories of parent and daughter nuclei.

### Three-particle decay

When the daughter nuclei are also radioactive — with a different half-life — the problem becomes describable by *three* coupled differential equations. If the daughter nuclei have a decay constant of  $\lambda_2$ , and they decay into stable “granddaughters,” then the system of equations is

$$\frac{dN_1}{dt} = -\lambda_1 N_1, \quad (3.35)$$

$$\frac{dN_2}{dt} = +\lambda_1 N_1 - \lambda_2 N_2, \quad (3.36)$$

$$\frac{dN_3}{dt} = +\lambda_2 N_2, \quad (3.37)$$

where  $N_3(t)$  denotes the number of granddaughters as a function of time. The solution procedure is the same as with two particles. First, Eq. (3.35) is solved for  $N_1(t)$  via separation of variables. Then, Eq. (3.36) can be solved for  $N_2(t)$  using the integrating factor technique (since it's a first-order equation with constant coefficients). Finally, when  $N_2(t)$  is known, Eq. (3.37) can be directly integrated to obtain  $N_3(t)$ . See Problem 55.

#### 3.8.1 Natural and artificial radioactivity

When radioactivity was initially being investigated, and it was realized that the “rays” carried enormous amounts of energy, the answers to two questions were being sought by most scientists. First, where did the energy come from? Initially, the energy was thought to be contained in the atom, but in 1903 Pierre Curie and Albert Laborde showed that 1 g of radium could heat 1.3 g of water from melting to boiling in 1 hour.<sup>19</sup> This was quite a bit of energy, and it caused some to consider abandoning the principle of the conservation of energy. Second, were all elements radioactive? It was possible that elements only appeared to be stable, but in reality had very long half lives.

#### Natural radioactivity

If we were to wait a long enough time, then all radioactive elements would decay, and only stable elements would be left. The age of the Earth is finite, however, and any radioactive

<sup>19</sup>P. Curie and A. Laborde, *C. R. Acad. Sci. Paris* **136** 673 (1903). An excellent discussion of the debate involving the conservation of energy can be found in Pais, “Introducing Atoms and Their Nuclei,” Chapter 2 in Volume I of *Twentieth Century Physics*, New York, 1995. (ERAU QC 7 .T84 1995) See also Problem 59.

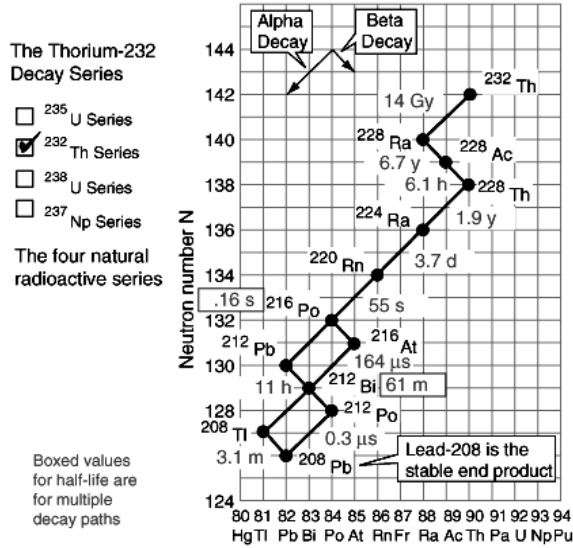


Figure 3.4: A nuclide chart for the  $4n$  decay series, which starts with  $^{232}\text{Th}$  and ends with the stable element  $^{208}\text{Pb}$ . In between it creates several other nuclei that were observed by Rutherford and Soddy, for example,  $^{224}\text{Ra}$  and  $^{220}\text{Rn}$ . Note that some nuclei, for example  $^{216}\text{Po}$  and  $^{212}\text{Bi}$ , can decay in two different ways. From HyperPhysics, <http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/radser.html>

elements that were present at the time of the Earth’s formation must have a sufficiently long half life in order to still be around in sufficient quantities to be observed. There are three nuclides that have half lives that are comparable to the 4.5 Gy age of the Earth. Those three are listed in the following table (along with  $^{237}\text{Np}$ ).

element	$\tau$	series	stable end
$^{232}\text{Th}$	14 Gy	$4n$	$^{208}\text{Pb}$
$^{237}\text{Np}$	2.3 My	$4n + 1$	$^{209}\text{Bi}$ ( $\tau = 19 \text{ Ey}$ )
$^{238}\text{U}$	4.5 Gy	$4n + 2$	$^{206}\text{Pb}$
$^{235}\text{U}$	0.71 Gy	$4n + 3$	$^{207}\text{Pb}$

Each of these elements is at the start of a “radioactive decay series,” in which successive  $\alpha$  and  $\beta$  decays occur until a stable element is reached. Since each  $\alpha$  decay changes  $A$  by 4 units, and  $\beta$  decays do not change  $A$ , this means that each step in a given series will consist of isotopes of atomic mass number  $A$  that differ by 4. Hence, all the elements in the  $^{232}\text{Th}$  series, for example, will have mass numbers that are multiples of 4, or given by  $4n$ . Those of the  $^{237}\text{Np}$  series will have mass numbers given by  $4n + 1$ , etc.

The isotopes in the  $^{237}\text{Np}$  series are not naturally occurring on Earth because of the short half life of  $^{237}\text{Np}$  compared to the age of the Earth. All other elements *are* observed, with varying abundances. The reason is because it is thought that a supernova (or supernovas) provided the material that eventually condensed to form the solar system. The physics of supernovas is fairly well understood, including the heavy elements that are produced in nuclear reactions during the violent explosion. Calculations show that

approximately equal numbers of  $^{238}\text{U}$  and  $^{235}\text{U}$  are produced. However, the half life of  $^{235}\text{U}$  is much shorter than that of  $^{238}\text{U}$ , so that today in the Earth, there is significantly more  $^{238}\text{U}$  than  $^{235}\text{U}$ .

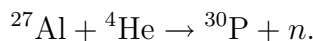
There are no isotopes of any element above lead ( $Z = 82$ ) in the periodic table that are stable. This is why lead is the common end product of each series. Bismuth ( $Z = 83$ ) has one “quasi-stable” isotope,  $^{209}\text{Bi}$ , whose half life is  $19 \times 10^{18}$  years. It was originally thought that  $^{209}\text{Bi}$  was stable, since no radiation had been detected. However, the mass excess of  $^{209}\text{Bi}$  predicts that it should  $\alpha$ -decay into an isotope of thallium,  $^{205}\text{Tl}$ , which it does.

Rutherford and Soddy, for example, started with thorium ( $^{232}\text{Th}$ ), which decays after several steps into thorium X ( $^{224}\text{Ra}$ ), and this then  $\alpha$  decays into  $^{220}\text{Rn}$ , “thorium emanation.” (See Fig. 3.4.) This thorium emanation, a gas,<sup>20</sup> is what led them to their discovery of the exponential decay law. They observed that the activity of this gas decreased rapidly, with a half life of about 1 minute (today it is measured at  $\tau = 55.6$  s). Determining this sequence of events was not simple. At first, they thought that thorium itself transformed into the emanation. However, they soon discovered a previously unknown component of thorium compounds, which they called thorium X and which could be chemically separated from thorium. After separation, they found that it was the *thorium X* that produced the emanation. This led them to believe that thorium itself was inactive. A second discovery showed that the separated thorium continued to produce thorium X, and the activity of the separated thorium X decreased with time. A glance at Fig. 3.4 reveals that the situation is more complex than this, and it’s a wonder that Rutherford and Soddy were able to deduce what they did.

There are two other long-lived radioactive isotopes that act as “clocks” and allow us to determine the ages of rocks. These are rubidium ( $^{87}\text{Rb}$ ) and potassium ( $^{40}\text{K}$ ). Finally,  $^{14}\text{C}$  is continually produced in the atmosphere from the bombardment of cosmic rays, and this is the basis for “carbon dating,” which can determine the ages of objects that have been alive in the past, such as trees. We will discuss these techniques later.

### Artificial radioactivity

In 1934, Irène Joliot-Curie and Frédéric Joliot produced the first “artificial” radioactive substance, phosphorus-30. They bombarded aluminum<sup>21</sup> with  $\alpha$ -particles from the decay of polonium

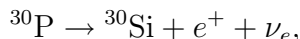



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<sup>20</sup>Radioactive uranium and thorium is spread throughout the Earth’s crust in the form of solid rock. During the subsequent decays, most of the daughter product elements also chemically bind themselves in solid form, with two exceptions. The first is, of course, the helium that is the byproduct of every  $\alpha$  decay. At standard temperature and pressure, helium is a gas, and it trapped in the rock in pockets. All of the helium that we have was obtained from gas deposits, usually along with combustible ‘natural gas.’ The helium in your party balloons, therefore, is really the by-product of radioactivity. The second is radon. It is also a gas at STP, and this gas is also trapped in the rock. It escapes from the ground into the atmosphere, as does helium in small amounts, but radon is radioactive, and is the cause of about half of the radiation exposure received by humans. When it builds up to high concentrations in the closed basements of houses, it can be a serious health hazard, requiring ‘mitigation’ for safety.

<sup>21</sup>Their sample was 100% of the isotope  $^{27}\text{Al}$  because it is the only stable isotope of aluminum.

$^{31}\text{P}$  is the only stable isotope of phosphorus, so the  $^{30}\text{P}$  nucleus produced in the bombardment undergoes  $\beta^+$  decay into the stable isotope  $^{30}\text{Si}$



with a half life of about 2.5 minutes. Thus, they were able to “activate” normal matter, i.e., take stable aluminum and create radioactive phosphorus, and for this they received the Nobel Prize in Chemistry for 1935. They had found, in effect, the “philosopher’s stone,” that age old quest to turn one element into another. Although they didn’t create gold, their work had profound implications for the human race.

Subsequently, Enrico Fermi and his laboratory in Rome bombarded stable elements with neutrons, and were able to create many new radioactive isotopes. For this he won the Nobel Prize in Physics for 1938. It turns out that many particles will work as a tool to transmutate a nucleus: protons, deuterons,  $\alpha$ -particles, neutrons; but neutrons, due to their neutral electric charge, tend to have the easiest time penetrating the nucleus.

### 3.8.2 $\gamma$ decay

Nuclei have quantum mechanical excited states just like atoms do. From your studies of chemistry, you know that an electron in a hydrogen atom, for example, can exist in different discrete states, each of which has a different energy. Therefore, when the electron makes a transition from a higher energy state to a lower energy state, it must, to conserve energy, emit a photon with the proper energy. This is why atoms emit light. Nuclei also have discrete energy levels, i.e., shells and subshells (see page 58), that can be calculated using quantum mechanics, and when they make transitions between states, they too must emit or absorb photons of the proper energy.

For electrons in atoms, the typical difference between energy levels is a few electron volts, say 10 eV. For nuclei, however, the transition energy is much larger, thousands or millions of electron volts. What type of photons do nuclei therefore emit? Since the energy of a photon is proportional to its frequency,  $E = h\nu$ , the larger the energy the larger the frequency (and the smaller the wavelength). If a nucleus undergoes a transition that gives  $\Delta E = 10^6$  eV, for example, the wavelength of the light (photon) emitted will be

$$\lambda = \frac{hc}{\Delta E} \approx 10^{-12} \text{ m},$$

which is a  $\gamma$ -ray. In our labeling of the electromagnetic spectrum, it is usually customary to call light with a wavelength shorter than 100 pm a “gamma ray.”

What about light from electronic transitions in atoms? If  $\Delta E = 10$  eV, then  $\lambda \approx 10^{-7} \text{ m} = 100 \text{ nm}$  which is in the ultraviolet region of the spectrum. This, in fact, is one way to determine how a photon was created: short-wavelength photons tend to come from nuclear transitions, while long-wavelength photons come from atomic (electronic) transitions, and extremely long wavelength photons (in the microwave or radio region of the spectrum) come from molecular transitions between different vibrational or rotational energy levels.

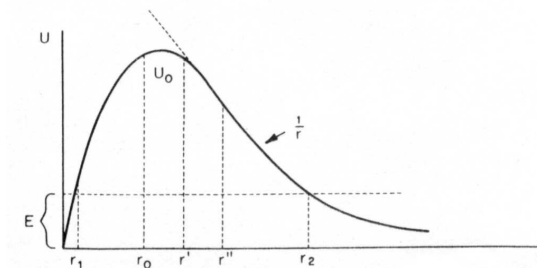


Figure 3.5: An approximation to the potential energy function  $U(r)$  between an  $\alpha$ -particle and its nucleus. When the separation distance between the two is small, the potential energy is confining - this is similar to the color force potential energy in Eq. (2.15). However, when the separation is large, the potential energy is repulsive (and is proportional to  $1/r$ ) because of the Coulomb force. The dashed line labeled “ $E$ ” denotes the typical kinetic energy of an  $\alpha$ -particle. From G. Gamow, *Quantum Theory of the Atomic Nucleus*, 1928.

### 3.8.3 $\alpha$ decay

The process of  $\alpha$  decay consists of a parent nucleus breaking apart into two pieces, one of which is an  $\alpha$ -particle, the other of which is a smaller, daughter nucleus. The reaction can be written



What causes this breakup? You can envision the  $\alpha$ -particle as a separate, tightly bound entity existing within the parent nucleus, and as it bounces around inside the nucleus, it attempts to escape the barrier of potential energy that exists due to the attractive strong force. Of course, if it *does* escape the nucleus, the Coulomb repulsion between it and the daughter nucleus will drive these two particles apart. It turns out that the height of the potential energy barrier is larger than the kinetic energy of the  $\alpha$ -particle when it is inside the nucleus, so that from a classical point of view it will never escape. (See Fig. 3.5.) Quantum mechanically, however, it can “tunnel” through the barrier with some nonzero probability. We will study the details of that “quantum tunneling” process in Chapter 7, but at this point we can determine if this decay is possible or not simply by calculating  $Q$  for the reaction<sup>22</sup>

$$Q = (M_{Z,A} - M_{Z-2,A-4} - M_{2,4}) c^2, \quad (3.39)$$

where  $M_{Z,A}$  is the mass of a nuclide with  $Z$  protons and  $A - Z$  neutrons, and  $M_{2,4}$  is the mass of the  $\alpha$ -particle. Even though  $Q$  should be calculated with nuclear masses, we can instead use the atomic masses, since the number of electrons cancels out.

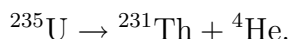
How can  $Q$  be measured? If the masses of all the particles involved are known, then in principle  $Q$  can be calculated. In practice, however, only some of the masses are known, and it is necessary to measure the kinetic energy of the ejected  $\alpha$ -particle to determine  $Q$ . However, the  $\alpha$ -particle does not take away the entire reaction energy because not

<sup>22</sup>Of course, the probability of it occurring, i.e., the half-life, is more difficult to determine. George Gamow developed a model of  $\alpha$  decay in 1928.

only must energy be conserved, but momentum as well. If the parent nucleus is initially at rest (which it is in some reference frame), then the daughter nucleus and the emitted  $\alpha$ -particle must have equal momentum (in opposite directions) and they must share the energy released. Therefore, the kinetic energy of the  $\alpha$  particle must be less than  $Q$ . You can show, using straightforward nonrelativistic mechanics, that

$$K_\alpha \approx \frac{A-4}{A}Q, \quad (3.40)$$

where  $A$  is the mass number of the parent nucleus. For example,  $^{235}\text{U}$  decays into  $^{231}\text{Th}$  with a half-life of  $7.04 \times 10^8$  years



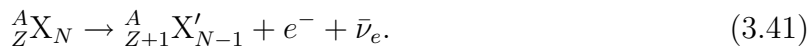
From a knowledge of the atomic masses, the value of  $Q$  can be calculated to be 4.68 MeV. The kinetic energy of the emitted  $\alpha$  particle is therefore  $K_\alpha = 4.60$  MeV. In fact, this is one method that is used to obtain nuclear masses: measure the kinetic energy of the emitted  $\alpha$  particle, calculate  $Q$ , and therefore obtain the mass difference between the parent and daughter nuclei.

### 3.8.4 $\beta^-$ decay, $\beta^+$ decay, and electron capture

*... the uranium radiation is complex, and there are at present at least two distinct types of radiation—one that is very readily absorbed, which will be termed for convenience the  $\alpha$ -radiation, and the other of a more penetrating character, which will be termed the  $\beta$ -radiation. — Ernest Rutherford, 1899*

#### $\beta^-$ decay

The fundamental  $\beta^-$  decay, which we have seen on page 22, is the decay of a neutron into a proton  $n \rightarrow p + e^- + \bar{\nu}_e$ , where a free neutron has a half life of about 10 minutes. In general, neutrons inside nuclei are stable, but in a nucleus that is unstable to  $\beta^-$  decay, one of the neutrons transforms into a proton, giving the following general reaction



As before,  $Q$  must be positive for the isotope  ${}^A_Z\text{X}_N$  to be unstable to  $\beta^-$  decay. For example, the reaction



has a half life of 5.75 years (can you calculate  $Q$ ?). Unlike in  $\alpha$  decay, the electron is not emitted with a definite energy. This is because there are *three* particles after the decay, and this means that there are many ways to distribute the energy and momentum.

Equations (3.38) and (3.41) represent the heart of Rutherford and Soddy's "transformation theory." If an element emits  $\alpha$  radiation, it is transformed into a different element that is two spaces to the left on the periodic table ( $Z$  decreases by 2). On the other hand, if it emits  $\beta^-$  radiation, it is transformed into a different element one space to the right. This transformation was part of the proof they needed to conclude that radioactivity was an internal atomic process (recall that in 1902 Rutherford hadn't yet discovered the nucleus).

**$\beta^+$  decay**

Nuclei can also decay via  $\beta^+$  (or positron) emission. In this case, a proton in the nucleus transforms into a neutron. How does this occur? One of the rules of nuclear reaction theory is that if you have one valid reaction (it conserves all the proper quantum numbers), then you can form another possible reaction by moving particles from one side of the reaction to the other, as long as you change that particle to its antiparticle. In addition, the reaction arrow can be in either direction, as long as there is enough energy. Therefore, we can manipulate Eq. (2.3) to give

$$p \rightarrow n + e^+ + \nu_e. \quad (3.43)$$

Of course,  $Q < 0$  for this reaction, which means that the proton is stable.<sup>23</sup> It can occur, however, as part of a nucleus, as long as the binding energies of the respective nuclei result in  $Q > 0$ . The general reaction can be written

$${}^A_Z X_N \rightarrow {}^A_{Z-1} X'_{N+1} + e^+ + \nu_e. \quad (3.44)$$

For example, the reaction

$${}^{40}\text{K} \rightarrow {}^{40}\text{Ar} + e^+ + \nu_e \quad (3.45)$$

has a half life of 1.251 Gy (can you calculate  $Q$ ?). In this case, the element moves one space to the left in the periodic table.

**Electron capture**

“Electron capture,” or “inverse beta decay,” is a reaction that can be obtained by moving the electron to the left side in Eq. (3.43)

$$p + e^- \rightarrow n + \nu_e. \quad (3.46)$$

As you can show,  $Q < 0$  here as well, so this will not happen to a stationary proton and electron, but this reaction *can* occur as long as the electron and proton collide with enough kinetic energy. The general nuclear reaction for electron capture is

$${}^A_Z X_N + e^- \rightarrow {}^A_{Z-1} X'_{N+1} + \nu_e. \quad (3.47)$$

Where does the electron come from? If the nucleus is in a neutral atom with orbiting electrons, usually one of the innermost orbital electrons (in the  $K$  shell) is “appropriated” by the nucleus for this task, although any electron will do. Any passing electron, however, must penetrate the nucleus (remember that the weak force is very short range) for the reaction to occur. Quantum mechanics shows that a  $K$  shell electron has a probability distribution that significantly overlaps the nucleus, which means that it is already in a good location for the weak force to do its job. In practice, although some nuclei are unstable to *both* positron emission and electron capture, one or the other is usually much more probable.

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<sup>23</sup>There is a possibility that a proton might decay into a lighter hadron, such as a pion, but the rule that the number of baryons is conserved seems to be correct. Observationally, the proton’s half life is at least  $6.6 \times 10^{33}$  years.

### Reaction energy

What are the reaction energies for the three different types of beta decay? As is the case in  $\alpha$  decay, you must use *nuclear* masses, because radioactivity is a purely nuclear process and doesn't involve the orbiting electrons. This means, in principle, if you are using a table of *atomic* masses, such as NUBASE, you need to subtract the proper number of electron masses from each term in the calculation. However, also like  $\alpha$  decay, we can judiciously add and subtract electron masses, which will therefore turn the calculation into one that uses atomic masses. For example, calculating  $Q$  for Eq. (3.41) using nuclear masses gives

$$Q = (M_{Z,A} - M_{Z+1,A} - m_e) c^2, \quad (3.48)$$

where the capital  $M$  stands for the nuclear mass. I will use a lower-case  $m$  to denote the atomic mass, so that

$$m_{Z,A} = M_{Z,A} + Zm_e, \quad (3.49)$$

(compare to Eq. (4.2)). Now, if I both add and subtract  $Zm_e$  to the right-hand-side of Eq. (3.48), I get

$$Q = [(M_{Z,A} + Zm_e) - (M_{Z+1,A} + \{Z + 1\}m_e) - m_e + m_e] c^2, \quad (3.50)$$

which reduces to

$$Q(\beta^-) = (m_{Z,A} - m_{Z+1,A}) c^2. \quad (3.51)$$

Hence, if you use atomic masses, the reaction energy is simply the mass difference between the parent and daughter atoms. This result means that in order to determine if an isotope is unstable to  $\beta^-$  decay, all that needs to be done is to compare its mass to the isobar with one more proton. With a table of atomic masses, this can be done “by inspection.”

How about  $\beta^+$  decay? Unfortunately, in this case, the transformation means that if the parent nucleus has  $Z$  protons, the daughter has  $Z - 1$  protons, rather than  $Z + 1$  protons. As before, the reaction energy using nuclear masses is

$$Q = (M_{Z,A} - M_{Z-1,A} - m_e) c^2. \quad (3.52)$$

(Note that the positron mass is identical to the electron mass.) If we use the same trick of adding and subtracting electron masses, then the reaction energy can be expressed in terms of atomic masses

$$Q = (m_{Z,A} - m_{Z-1,A} - 2m_e) c^2. \quad (3.53)$$

Only the masses of  $(Z - 1)$  electrons combined with  $M_{Z-1,A}$  to give the atomic mass, leaving two  $m_e$  unattached. Does this mean that we can't determine  $\beta^+$  instability “by inspection” as we could with  $\beta^-$  instability? No, because whenever positron emission occurs, electron capture can also occur, and the reaction energy for electron capture is always greater than the reaction energy for positron emission by exactly  $2m_e c^2$ ! This means that if we use atomic masses, the  $Q$  value for inverse beta decay is analogous to Eq. (3.51), e.g., just the atomic mass difference

$$Q(\text{e.c.}) = (m_{Z,A} - m_{Z-1,A}) c^2. \quad (3.54)$$



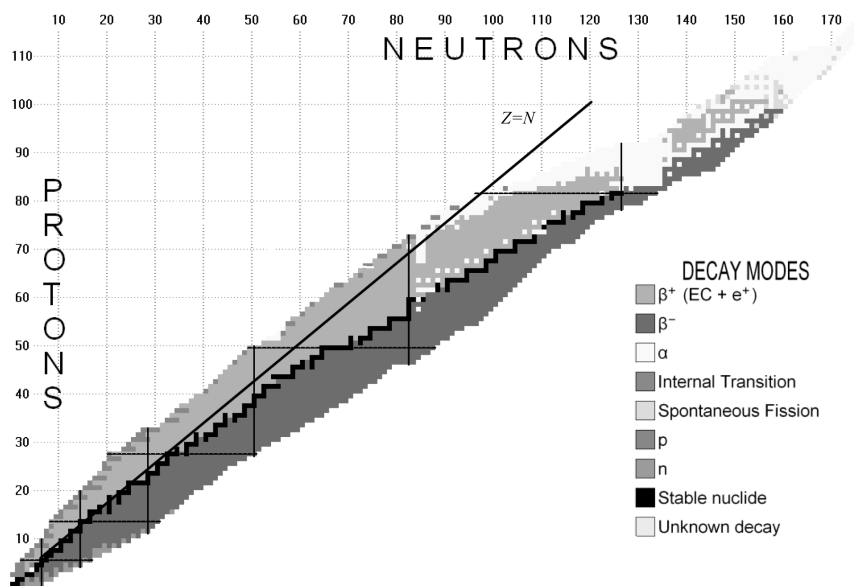


Figure 3.6: A nuclide chart depicting all stable nuclei in black,  $\beta^-$  emitters in dark grey, and  $\beta^+$  emitters in light grey. Note the “line of stability,” or the bottom of the “valley of stability” running approximately along the line where  $Z = N$ , marked by the thin straight line. Figure 6 from Audi et al., NUBASE.

These two results, Eqs. (3.51) and (3.54) imply that if you have two neighboring isobars,  ${}^A_Z X$  and  ${}^A_{Z\pm 1} X$ , the one with the greater atomic mass is unstable to  $\beta$ -decay into the other. If you look through the NUBASE table of atomic masses (which are grouped by  $A$ ), you will see that there are *never* two neighboring isobars that are both stable. As Hans Bethe says,

“There are, however, many (about fifty) isobar pairs in nature of the type  ${}^A_Z X$  and  ${}^A_{Z-2} X$ , with both  $Z$  and  $A$  even. The intervening nucleus,  ${}^A_{Z-1} X$ , of odd charge, decays to one or the other of its neighbors, or sometimes to both.”<sup>24</sup>

This analysis of  $\beta$  decay comes to the conclusion that for any set of isobars — nuclei with the same value of  $A$  — there is only one, or at most two, that are stable. For example,  ${}^{129}\text{Xe}$  is the only stable isobar with  $A = 129$ , but both  ${}^{130}\text{Xe}$  and  ${}^{130}\text{Ba}$  are stable with  ${}^{130}\text{Cs}$  decaying into both. Since  ${}^{208}\text{Pb}$  is the heaviest stable nucleus, the above considerations mean that you could guess that there are between 200 and 300 stable nuclei in nature. In fact, there are exactly 255.

### 3.9 The Valley of Stability

This conclusion, that all isobars are unstable to either  $\beta^-$  or  $\beta^+$  decay, except for one or two, explains the existence of the so-called “valley of stability” in a nuclide chart, as shown

<sup>24</sup>Bethe, *Elementary Nuclear Theory*, page 11.

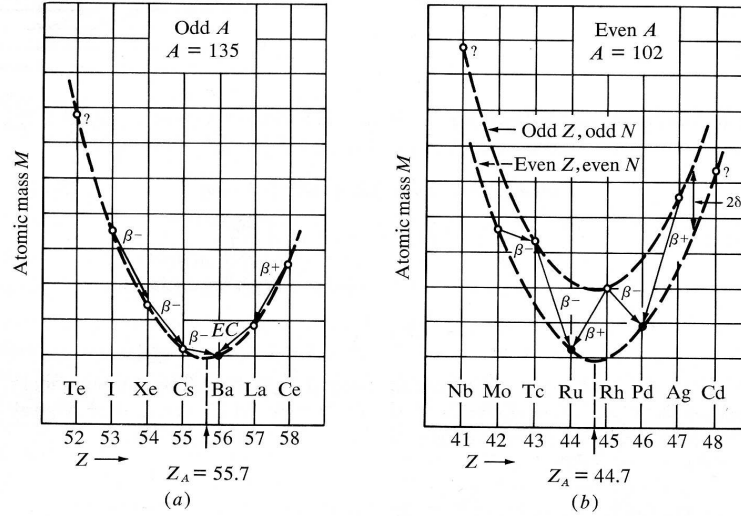


Figure 3.7: Two “mass parabolas” for isobars, one odd ( $A = 135$ ) and one even ( $A = 102$ ). Filled circles represent stable nuclides and open circles represent radioactive nuclides. The dashed lines represent the theoretical prediction of the liquid drop model. Note that there is only one stable nuclide for odd  $A$ , yet two for even  $A$ , as discussed in the text. The vertical axis on both plots is approximately 1 MeV per division. Figure 2-15 from Meyerhoff, *Elements of Nuclear Physics*.

in Fig. 3.6. The stable nuclei form a line in a plot of  $Z$  versus  $N$ , and any nucleus not on this line is unstable to  $\beta$ -decay. The line is also the minimum mass line (atomic masses), which means that the atomic masses of the isotopes become larger as you move away from the line. If you were to produce a three-dimensional plot that showed the atomic mass on the third axis (as a function of  $N$  and  $Z$ ), the black line would be the bottom of a valley, which is where the name “valley of stability” originates. Easier to draw, of course, is a two-dimensional plot, selecting out one value of  $A$  at a time. Two of these plots are shown in Fig. 3.7 for  $A = 135$  and  $A = 102$ . For odd values of  $A$ , the nucleus is “odd-even,” with either  $Z$  odd and  $N$  even, or vice-versa. Each of these configurations is equally undesirable, so that there is only one curve denoting mass versus  $Z$ . For even values of  $A$ , on the other hand, the nuclei must be either odd-odd or even-even. As we have seen from a consideration of spin, the even-even configuration is much more desirable — i.e., a lower potential energy — while the odd-odd configuration is so undesirable that there are only four such stable nuclei. In the  $A = 102$  case, the masses of the odd-odd nuclei are significantly larger (i.e., a larger potential energy) so that they will always decay to a lower energy state (lower mass), which means  $\beta$  decay into an even-even nucleus with the same  $A$ . As you can see, ruthenium and palladium are the only stable nuclei with  $A = 102$ , and rhenium-102 decays into both of them.

## Problems

**38.** From the values of the atomic mass excess,  $\Delta c^2$ , listed in NUBASE, calculate the atomic masses of the four stable isotopes of hydrogen and helium. Compare your answers with the nuclear masses listed by CODATA.

**39.** Calculate and plot the values of  $B/A$  for the 20 stable nuclei with the lowest  $A$ . That is, for all the stable nuclei of hydrogen (H) through neon (Ne), plot  $B/A$  versus  $A$ . NUBASE lists the stable isotopes as “STABLE.” If you use these values, you’ll need to derive an equation relating  $B/A$  to the mass excess  $\Delta$ .

**40.** Look up the Sun’s luminosity (defined as the total power radiated), and calculate the number of net reactions ( $4^1\text{H} \rightarrow ^4\text{He} + 6\gamma + 2\nu_e$ ) that occur in the Sun’s core per second.

**41.** From the answer to the previous question, calculate the flux (number per second per square meter) of neutrinos at the Earth due to reactions in the Sun. HINT: You’ll have to take into account the geometrical spreading of the neutrinos emitted by a point source. NOTE: these neutrinos are continually passing through us and the Earth since they interact so weakly.

**42.** “Mirror nuclei” are pairs of nuclei that have the same  $A$  (isobars) but switch their proton and neutron number  $N \leftrightarrow Z$ . Consider two nuclei with identical  $A$ , where the first has  $Z$  protons, and the second has  $Z + 1$  protons. Assuming that the protons (and hence the positive charge) are uniformly distributed over a sphere of radius  $R$ , show that the *extra* Coulomb repulsive energy of the second nucleus (compared to the first nucleus) is given by

$$\frac{6}{5} \frac{(Z + \frac{1}{2})e^2}{4\pi\epsilon_0 R},$$

where the formula for the potential energy of a uniformly charged sphere is given in Problem 8.

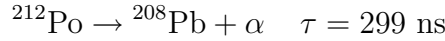
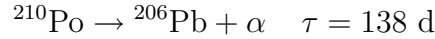
**43.** The energy in Problem 42 is not the only difference in the masses of the two nuclei. We can solve the binding energy equation (3.2) for  $M_{Z,N}$

$$\begin{aligned} M_{Z,N} &= (Zm_p + Nm_n) - B/c^2 \\ &= (Zm_p + Nm_n) - (B_{Coulomb} + B_{strong})/c^2, \end{aligned}$$

where I’ve split up the binding energy  $B$  into a Coulomb component (due to electrostatic forces) and a strong component (due to the strong force). QCD proposes (and experiments confirm) that the strong force affects protons and neutrons equally, which means that as long as  $A$  is constant (true for our mirror nuclei pair) then the  $B_{strong}$  term is the same for both nuclei. But the Coulomb energy is different, as you have calculated. (a) Obtain an expression for the difference in mass between two mirror nuclei, i.e.,  $M_{Z+1,N-1} - M_{Z,N}$ . (b) Evaluate this predicted mass difference for the pair  $^{11}\text{B}$  and  $^{11}\text{C}$ , and compare it with the measured mass difference, which you can find in NUBASE. To evaluate the mass difference, you’ll need to assume a value for nuclear radius, Eq. (3.10), using the appropriate value of  $R_0 = 1.07$  fm, since the radius  $R$  appears in the expression for the Coulomb energy.

**44.** Problem 43 can be turned around, and, taking the experimentally measured mass difference of two mirror nuclei to be correct, use that value to determine the value of  $R_0$ . Do this for the mirror pair  $^{11}\text{B}$  and  $^{11}\text{C}$  in Problem 43 as well as the mirror pair  $^{15}\text{N}$  and  $^{15}\text{O}$ .

**45.** Polonium has two isotopes that are unstable to  $\alpha$  decay,  $^{210}\text{Po}$  and  $^{212}\text{Po}$ . The reactions, along with their half lives are



Calculate the reaction energy  $Q$  for both reactions, and explain the different results as well as the different half lives in terms of nuclear magic numbers.

**46.** List the elements that have  $Z$  equal to a nuclear magic number. How many stable isotopes of each are there? NOTE: the average value is 3.2 isotopes per element.

**47.** If the nuclear radius is given by  $R = R_0 A^{1/3}$  with  $R_0 = 1.4 \text{ fm}$ , what is the density of nuclear matter in  $\text{kg}/\text{m}^3$ ? How does this compare with the density of water?

**48.** (a) Show that when  $r = R + b$ , the nuclear density in Eq. (3.12) is approximately one fourth of its central value,  $\rho(R + b) \approx \rho_0/4$ . (b) Show that when  $r = R - b$ , the nuclear density is approximately three fourths of its central value,  $\rho(R - b) \approx 3\rho_0/4$ . (c) Sketch (by hand) the nuclear density  $\rho$  as a function of  $r$  as given by Eq. (3.12). Be sure to label the axes and any special locations on the curve.

**49.** Derive the first part of Eq. (3.18) by adding the magnetic moment  $z$  components of the proton and neutron to obtain the  $z$  component of the deuteron. Use the definition of  $g$  found in Eq. (3.15).

**50.** The quadrupole moment  $q$  of a charge distribution  $\rho(x, y, z)$  is defined as

$$q \equiv \int \rho(3z^2 - r^2)dV. \quad (3.55)$$

Kellog et al. made the following measurement for the deuteron

$$\frac{\langle z^2 \rangle}{\langle r^2 \rangle} = \frac{\int \rho z^2 dV}{\int \rho r^2 dV} \approx \frac{1}{3} \quad (1.14). \quad (3.56)$$

This measurement is consistent with the  $D$  and  $S$  state superposition discussed in Section 3.6. For this problem, show that if  $\rho$  is spherically symmetric, that is  $\rho(r)$  is only a function of  $r$ , then the Kellog measurement should be exactly  $\frac{1}{3}$ .

**51.** (a) Derive Eq. (3.22) from Eq. (3.21) using the definition of half life. (b) Show that the exponential decay rate of  $e^{-\lambda t}$  is equal to  $\left(\frac{1}{2}\right)^{t/\tau}$ .

**52.** Show that the average lifetime  $\langle t \rangle$  of a given radioactive nucleus is *not* the half life  $\tau$ , but is  $\langle t \rangle = \tau/\ln 2$ . That is, evaluate the integrals in Eq. (3.27) [HINT: Show that  $\int_0^\infty t \exp(-\lambda t) dt = \lambda^{-2}$ .]

**53.** Muons have a decay half life of  $\tau = 1.523 \mu\text{s}$ . What is their average lifetime?

**54.** The number  $e$  can be defined by the property

$$e^\varepsilon \approx 1 + \varepsilon \quad \text{for } \varepsilon \ll 1.$$

For any small value of  $\varepsilon$ , any number  $r$  raised to the power  $\varepsilon$  will differ from 1 by an amount proportional to  $\varepsilon$ . However, only for  $r = e$  will the proportionality constant be 1. (a) Using the definition equation above, show that

$$\varepsilon \log_{10} e \approx \log_{10}(1 + \varepsilon).$$

(b) Use this expression to calculate  $e$  by using small values for  $\varepsilon$ , i.e., let  $\varepsilon = 0.1, 0.01, 0.001, \dots$ . Make a table showing that your value for  $e$  gets more and more precise as  $\varepsilon$  gets smaller and smaller. (c) Show that this way of calculating  $e$  is equivalent to

$$e = \lim_{N \rightarrow \infty} \left(1 + \frac{1}{N}\right)^N.$$

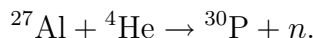
**55.** Consider a three-isotope radioactive decay chain. There are  $N_1$  atoms of isotope 1,  $N_2$  atoms of isotope 2 and  $N_3$  atoms of isotope 3; isotope 1 has a decay constant  $\lambda_1$ , isotope 2 has a decay constant  $\lambda_2$ . Isotope 1 decays into isotope 2, which then decays into isotope 3, which is stable.

(a) Write down the three differential equations that govern  $N_i(t)$ , and then solve them, assuming that the initial values for  $N_i$  are arbitrary.

(b) Investigate two interesting limits. First, what happens if  $\lambda_1 \gg \lambda_2$ . Second, what happens if  $\lambda_1 \ll \lambda_2$ . For each case, assume  $N_2$  and  $N_3$  are initially zero, and sketch  $N_i(t)$  for all three species.

**56.** Some nuclei can decay in two different ways. For example,  $^{212}\text{Bi}$  can both  $\alpha$  decay into  $^{208}\text{Tl}$  (36% of the time) as well as  $\beta$  decay into  $^{212}\text{Po}$  (64% of the time). (a) If the decay constants for these processes are  $\lambda_2$  and  $\lambda_3$ , respectively, obtain a formula for the total half-life  $\tau$  for the decay of  $^{212}\text{Bi}$  (in terms of  $\lambda_2$  and  $\lambda_3$ )? (b) If the half-life of  $^{212}\text{Bi}$  is 60.55 minutes, calculate values for  $\lambda_2$  and  $\lambda_3$ .

**57.** (a) Calculate the  $Q$  value for the reaction that produced the first artificial radioisotope



Since the aluminum was stationary in the target, and since  $Q < 0$ , the  $\alpha$ -particle needed to impact the aluminum with a kinetic energy at least as large as  $-Q$  in order to make the reaction possible. (b) Where did the Joliot-Curies get an  $\alpha$ -particle with this much energy? They obtained it from a sample of polonium undergoing  $\alpha$  decay. The question is, which isotope? Well, there are five possibilities. In the U-238 and Th-232 decay series there occur five polonium isotopes that decay via  $\alpha$  emission, and their mass numbers are 210, 212, 214, 216, and 218. For each of these isotopes, calculate the kinetic energy of the emitted  $\alpha$  particle, see Eq. (3.40), and determine if it is large enough to produce “radio-phosphorus” from the above reaction.

**58.** In the earth today, natural uranium is composed of 0.72%  $^{235}\text{U}$  and 99.27%  $^{238}\text{U}$ . (There is a trace of  $^{234}\text{U}$ , approximately 0.01%, which has a short half life of 245.5 ky for  $\alpha$ -decay, so it should be nonexistent, but it is populated via the radioactive decay series starting with  $^{238}\text{U}$ .) Where did these two isotopes of uranium come from, and why do they exist in this proportion? Uranium is produced in supernovae through a rapid neutron capture process (the so-called  $r$ -process), and astrophysicists predict that the production ratio in supernovae of  $^{235}\text{U}$  to  $^{238}\text{U}$  is 1.65. Given that the half life for  $^{235}\text{U}$  is 704 My, and that for  $^{238}\text{U}$  is 4.468 Gy, your task is to extrapolate back in time to determine when the supernova occurred that provided the material for the cloud of gas and dust that eventually formed the solar system. (Unfortunately, there is other evidence, mostly from meteorites, that suggests that there was not just one supernova, but several over a period of time, so that the current proportion of uranium isotopes is less exact.)

**59.** In 1903 Pierre Curie and Albert Laborde found that 1 g of radium can heat 1.3 g of water from its melting point to its boiling point in 1 hour! (a) How much energy is this (in Joules)? Use the known specific heat of liquid water. (b) What percentage of the mass of the radium is converted into energy in 1 hour? Simply use  $E_0 = mc^2$  — no radioactivity yet.

Now we want to explain this experiment using radioactivity theory. The radium of the Curies was actually  $^{226}\text{Ra}$ , which is part of the  $^{238}\text{U}$  natural decay series. It has a half life of 1.6 ky, and it undergoes  $\alpha$ -decay into  $^{222}\text{Rn}$ . (c) What is the activity of 1 g of  $^{226}\text{Ra}$ ? Use the known half-life. (d) In 1 hour, how many decays occur? (e) Given the energy observed by Curie and Laborde, what must be the energy released in each decay? (f) Calculate the  $Q$  value of the  $\alpha$ -decay:  $^{226}\text{Ra} \rightarrow ^{222}\text{Rn} + \alpha$ . How does this compare with the value calculated in part (e)?

[The reason why your answers don't match is that  $^{222}\text{Rn}$  undergoes  $\alpha$ -decay into  $^{218}\text{Po}$  with half life of 3.8 days. (The Curies called  $^{222}\text{Rn}$  "radium emanation.") The situation, therefore, is more complicated than a parent decaying to a daughter because the daughter is itself radioactive. In this case, when the parent is much longer-lived than the daughter, both parent and daughter decay at the rate of the parent, so you need to include the energy of the  $^{222}\text{Rn}$  decay in your calculation in parts (e) and (f).]

**60.** Classical decay kinematics. This problem allows you to work through the formalism of particle decay kinematics using nonrelativistic kinematics that you are familiar with. Consider a ball of mass  $M$  moving to the right with speed  $V$ . Inside the ball is a firecracker, and at time  $t = 0$  it explodes, breaking the ball into two equal pieces, each of mass  $M/2$ . Assume that the two pieces head off symmetrically in opposite directions, each at an angle  $\theta$  with respect to the velocity direction of the original ball. If the firecracker released energy  $Q$  into the system, then the kinetic energy of the two pieces is larger than the initial kinetic energy. (a) Apply the conservation of momentum (vector) and energy (scalar) to obtain an expression for the angle  $\theta$  (or simply  $\cos \theta$ ) as a function of  $Q$ . (b) What is the result when  $Q = 0$ ? Does it make sense? (c) Use a Taylor expansion of your result to obtain an approximate expression for  $\theta$  as a function of  $Q$  in the limit when  $Q \ll MV^2/2$ . Do the same in the limit when  $Q \gg MV^2$ . What information do these limits give you? (d) Evaluate numerically the angle  $\theta$  in the case where a  $^8\text{Be}$  nucleus is traveling with a kinetic energy of 100 keV when it breaks up into two  $^4\text{He}$  nuclei. See Problem 13.

**61.** Derive Eq. (3.40). Assume that the parent nucleus has mass  $\mathcal{M}$ , is at rest, and decays into two particles of masses  $M$  (the daughter nucleus) and  $m$  (the  $\alpha$  particle), which are moving in opposite directions (with speeds  $V$  and  $v$ , respectively) with a total kinetic energy  $Q$ . First, show that in the non-relativistic limit ( $K_\alpha = \frac{1}{2}mv^2$  and  $K_M = \frac{1}{2}MV^2$ ) the kinetic energy of the  $\alpha$  particle is exactly

$$K_\alpha = Q \frac{M}{M + m}.$$

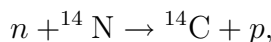
Second, make reasonable approximations to the masses to obtain Eq. (3.40).

**62.** Can you determine a series of radioactive decays ( $\alpha$  or  $\beta$  or a combination of both) that will transform an isotope of lead into an isotope of gold? Use the NUBASE table to determine the decay processes of each isotope.

**63.** Estimate the number of positrons emitted by a banana each second due to the decay of radioactive potassium that it contains.

**64.** Plot  $\Delta$  versus  $Z$  for  $A = 184$  and  $A = 185$  on two separate graphs. Do this for elements 72 through 79 (i.e., Hf through Au). Label the stable isotopes in your plot. Do you obtain something similar to Figure 3.7?

**65.** There are approximately 7 kg of radioactive  $^{14}\text{C}$  produced each year in the atmosphere produced via the following reaction



where the neutrons come from cosmic ray bombardment reactions. How many atoms is that?

**66.** There are two stable isotopes of carbon:  $^{12}\text{C}$  and  $^{13}\text{C}$ . A third isotope,  $^{14}\text{C}$ , is unstable to  $\beta$  decay in the following reaction:  $^{14}\text{C} \rightarrow {}^{14}\text{N} + e^- + \bar{\nu}_e$ , with a half life of 5700 years, but it exists in the atmosphere due to the fact that it is continually created from cosmic ray bombardment. It turns out that the two processes of creation and decay are in equilibrium, because the steady-state concentration of  $^{14}\text{C}$  in the environment is observed to be constant. This fact can be used to “carbon date” organic materials; however, we need first to determine the steady-state concentration of  $^{14}\text{C}$  in the environment. For this problem, we will assume that there are no  $^{13}\text{C}$  atoms, only  $^{12}\text{C}$  and  $^{14}\text{C}$ , because the percentage of  $^{13}\text{C}$  atoms is very small. Therefore, “pure” carbon consists of mostly  $^{12}\text{C}$  with a small fraction of  $^{14}\text{C}$  atoms. This fraction is what we wish to determine.

(a) One gram of pure carbon emits 15.3 electrons per minute (that is, its activity is  $15.3 \text{ min}^{-1}$ ). From this piece of information, calculate how many  $^{14}\text{C}$  atoms there are in one gram of pure carbon. (b) Calculate how many total carbon atoms (of either isotope) there are in one gram. (c) Therefore, what is the steady-state fraction of  $^{14}\text{C}$  atoms in the environment?

## Solutions

**38.** For example, the mass excess of  $^4\text{He}$  is  $\Delta c^2 = 2424.9156 \text{ keV}$ , which means that the helium *atom* has a mass, from Eq. (3.6)

$$M_{4,2} = \Delta + 4 \text{ u} = 2.4249156 \text{ MeV}/c^2 + 4 \times 931.494043 \text{ MeV}/c^2 = 3728.401 \text{ MeV}/c^2.$$

This is the *atomic* mass, so to compare with CODATA, I need to subtract the masses of the two electrons, i.e., subtract  $2 \times 0.510998 \text{ MeV}/c^2$ , which gives me an  $\alpha$ -particle mass of  $3727.379092 \text{ MeV}/c^2$ . This matches the CODATA value. This calculation also shows that the binding energy of the two electrons must be smaller than the precision of these values. In fact, it’s on the order of 10 eV, which is barely resolvable in the Nubase and CODATA data.

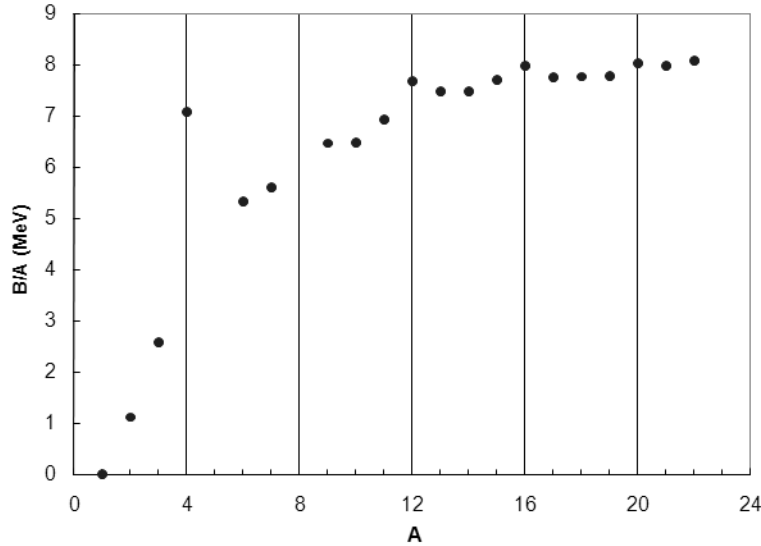
**39.** A sample calculation uses Eq. (3.2), which says

$$B = (Zm_p + Nm_n - \{A(1 \text{ u}) + \Delta - Zm_e\}) c^2,$$

where I've assumed that  $\Delta$  is the atomic mass excess, exactly what is listed in NUBASE. Or, since we want the quantity  $B/A$ ,

$$\frac{B}{A} = \left( \frac{Z(m_p + m_e) + Nm_n}{A} - (1 \text{ u}) - \frac{\Delta}{A} \right) c^2.$$

For  $^{16}\text{O}$ ,  $Z = 8$ ,  $N = 8$ ,  $A = 16$ , and  $\Delta c^2 = -4737.0014 \text{ keV}$ . Plugging these values in, I obtain  $B/A = 7.976208 \text{ MeV}$ . See the figure for a plot of all the results. Notice that there is a general increasing trend:  $B/A$  increases with increasing  $A$  which means that *fusion* reactions will be exothermic! We'll see that for higher values of  $A$ , beyond iron, there is a decreasing trend, which means that fission reactions are exothermic.



**40.** As calculated in the text, there is  $26.21 \text{ MeV} = 4.199 \times 10^{-12} \text{ J}$  released per fusion reaction. The solar luminosity is  $L_{\odot} = 3.846 \times 10^{26} \text{ W}$ , which implies  $9.159 \times 10^{37}$  reactions per second. Or, since  $10^{12}$  is one trillion, this is about 100 trillion trillion trillion reactions per second.

**41.** For each fusion reaction two neutrinos are emitted ( $1.832 \times 10^{38} \nu/\text{s}$ ), but they are emitted into  $4\pi$  steradians, which means that at the distance of the Earth, the area that they have spread out into is  $4\pi R_{ES}^2$ , where  $R_{ES} = 149.9$  million km. For the flux of neutrinos at the Earth, I get

$$\text{flux} = \frac{1.832 \times 10^{38} \text{ s}^{-1}}{4\pi R_{ES}^2} = 6.487 \times 10^{14} \frac{\text{neutrinos}}{\text{s m}^2},$$

which, for a thumbnail of area  $1.5 \text{ cm}^2$ , implies that about 100 billion neutrinos pass through my thumbnail each second.

**42.** The Coulomb potential energy, from Problem 8, is

$$U_i = \frac{3}{5} \frac{Q_i^2}{4\pi\epsilon_0 R},$$

where  $Q_1 = Ze$  and  $Q_2 = (Z+1)e$ . The difference in these energies is

$$\Delta U = U_2 - U_1 = \frac{3}{5} \frac{e^2}{4\pi\epsilon_0 R} [(Z+1)^2 - Z^2] = \frac{3(2Z+1)e^2}{5 \cdot 4\pi\epsilon_0 R}.$$



43. (a) Using the formula in the problem statement, I obtain a mass difference of

$$M_{Z+1,N-1} - M_{Z,N} = m_p - m_n - \Delta B/c^2.$$

First, the result you obtained in Problem 42 is actually  $-\Delta B$ , because the Coulomb energy is *repulsive*, and so counts as “negative” binding energy. Second, this is the *nuclear* mass difference, but of course when we compare with the data tables they list *atomic* masses. There is a one electron difference between our nuclei/atoms, so assuming we have atomic masses, I get

$$M_{Z+1,N-1}(\text{atomic}) - M_{Z,N}(\text{atomic}) = m_p - m_n + m_e + \frac{3(2Z+1)e^2}{5 \cdot 4\pi\epsilon_0 R} \frac{1}{c^2}.$$

Since the mass difference between a proton and neutron is -1.293 MeV, and taking  $R_0 = 1.07$  fm and  $A = 11$ , I get

$$M_{Z+1,N-1}(\text{atomic}) - M_{Z,N}(\text{atomic}) = (-1.293 + 0.511 + 3.994) \text{ MeV}/c^2 = 3.211 \text{ MeV}/c^2,$$

and the tabulated mass difference is  $0.002129 \text{ u} = 1.983 \text{ MeV}/c^2$ . This is not too close, but we did make an assumption about  $R_0$ . One technique that is used is to *solve* for  $R_0$ , and this is an independent determination of the nuclear size, separate from that obtained by scattering electrons.

44. What is the radius prediction for  $^{11}\text{B}$  and  $^{11}\text{C}$ ?

45. From NUBASE, the mass excesses (in keV) are

$^{206}\text{Pb}$	-23 785.4
$^{208}\text{Pb}$	-21 748.5
$^{210}\text{Po}$	-15 953.1
$^{212}\text{Po}$	-10 369.4
$^4\text{He}$	2424.9

This gives  $Q = 5.407$  MeV for the decay of  $^{210}\text{Po}$ , and  $Q = 8.954$  MeV for the decay of  $^{212}\text{Po}$ . The long half life of  $^{210}\text{Po}$  and the low energy of the emitted  $\alpha$  means that the  $\alpha$  particle doesn't really have that much desire to leave the cozy nucleus — its neutrons make the 125th and 126th neutrons in  $^{210}\text{Po}$ , and so it's fairly tightly bound (recall that 126 is a magic number). On the other hand, the  $\alpha$  particle can't wait to get out of  $^{212}\text{Po}$  — it leaves in 299 ns and has almost twice as much energy as the  $\alpha$  particle from  $^{210}\text{Po}$ .

46.

$Z$	2	8	20	28	50	82	126
X	He	O	Ca	Ni	Sn	Pb	-
#	2	3	6	5	10	4	-

Tin (Sn) has the most number of stable isotopes (10) of any element, and lead (Pb) is the last element with any stable isotopes. In fact, lead-208 is “doubly magic.” The average number of stable isotopes per element is 3.2, so most of these beat that average. For calcium, I counted  $^{48}\text{Ca}$  because it has a half life of  $53 \text{ Ey} = 53 \times 10^{18}$  years, which, for all intents and purposes, is stable.

**47.** The density is the mass per-unit-volume, so if we take the mass of one nucleon to be the average of the proton and neutron mass,  $m_1 = 1.674 \times 10^{-27}$  kg, and the volume of a nucleus to be  $V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi R_0^3 A$ , I get

$$\rho = \frac{m_1}{\frac{4}{3}\pi R_0^3} = 1.46 \times 10^{17} \text{ kg/m}^3.$$

This is about  $10^{14}$  times denser than water.

**48.** (a) When  $r = R + b$ , then  $\rho/\rho_0 = (1 + e)^{-1} = 3.71^{-1} \approx 4^{-1}$ . (b) When  $r = R - b$ , then  $\rho/\rho_0 = (1 + e^{-1})^{-1} = 1.37^{-1} \approx (4/3)^{-1}$ .

**49.** Even though the spin vectors add, since we do not know their directions, we can't add them. However, the  $z$  components also add, and we do know their values. So the  $z$  component of a deuteron's magnetic moment is

$$\mu_{z,d} = \mu_{z,p} + \mu_{z,n},$$

where I've assumed there is no magnetic moment due to orbital angular momentum. In units of the nuclear magneton  $\mu_N$  this becomes

$$m_{s,d}g_d = m_{s,p}g_p + m_{s,n}g_n.$$

Since the proton and neutron are spin  $\frac{1}{2}$  but the deuteron is spin 1, the values of for  $m_s$  give Eq. (3.18)

$$g_D = \frac{g_p}{2} + \frac{g_n}{2}.$$

**51.** (a) Evaluating Eq. (3.21) at  $t = \tau$  gives

$$\mathcal{A}(\tau) = \frac{\mathcal{A}_0}{2} = \mathcal{A}_0 e^{-\lambda\tau},$$

and taking the natural log of both sides results in Eq. (3.22)

$$\tau = \frac{\ln 2}{\lambda}.$$

(b) Applying the well-known laws of exponents, as well as the relation between  $\tau$  and  $\lambda$ , gives

$$e^{-\lambda t} = (e^{-\lambda})^t = (e^{-\ln 2/\tau})^t = (e^{-\ln 2})^{t/\tau} = \left(\frac{1}{2}\right)^{t/\tau}.$$

**52.** The average lifetime is defined as

$$\langle t \rangle = \frac{\int_0^\infty t e^{-\lambda t} dt}{\int_0^\infty e^{-\lambda t} dt} = \frac{1/\lambda^2}{1/\lambda} = \frac{1}{\lambda}.$$

The normalizing denominator is straightforward to evaluate:  $\int_0^\infty \exp(-\lambda t) dt = \lambda^{-1}$ . The numerator can be evaluated by a method called "parametric differentiation." Starting with the normalization integral, take the derivative of both sides with respect to  $\lambda$

$$-\frac{d}{d\lambda} \int_0^\infty \exp(-\lambda t) dt = -\frac{d}{d\lambda} \frac{1}{\lambda}.$$

The derivative can be brought inside the integral because the limits of the integral do not depend on  $\lambda$ . This brings down a  $-t$  from the exponential, resulting in the integral that we wish to evaluate

$$\int_0^{\infty} t \exp(-\lambda t) dt = \frac{1}{\lambda^2}.$$

Therefore, the average lifetime is just  $\langle t \rangle = 1/\lambda$ .

**53.**  $\langle t \rangle = \tau / \ln 2 = 2.197 \mu\text{s}$ .

**54.** (a) The desired equation is obtained by taking the logarithm (base 10) of both sides. (b) The table shows the result getting closer to  $e = 2.718281$ .

$\varepsilon$	$e$
0.1	2.593742
0.01	2.704814
0.001	2.716924
0.0001	2.718146
0.00001	2.718268

(c) Again, using the laws of exponents and logarithms, the equation derived in (a) can be written

$$\log_{10} e \approx \frac{1}{\varepsilon} \log_{10}(1 + \varepsilon) = \log_{10}(1 + \varepsilon)^{1/\varepsilon},$$

and taking the anti-logarithm of both sides gives

$$e \approx (1 + \varepsilon)^{1/\varepsilon},$$

where the approximation gets better in the limit  $\varepsilon \rightarrow 0$ . Of course, letting  $N = 1/\varepsilon$  results in the desired definition.

**56.** (a) When a particle can decay into two daughters, Eqs. (3.30) and (3.31) become

$$\begin{aligned} \frac{dN_1}{dt} &= -\lambda_2 N_1 - \lambda_3 N_1 = -(\lambda_2 + \lambda_3) N_1, \\ \frac{dN_2}{dt} &= +\lambda_2 N_1, \\ \frac{dN_3}{dt} &= +\lambda_3 N_1, \end{aligned}$$

which expresses the fact that both decay rates are proportional to the number of parent particles. Of course, the total rate is just the sum of the two rates,

$$\lambda = \lambda_2 + \lambda_3,$$

which means that

$$\tau = \frac{\ln 2}{\lambda_2 + \lambda_3}.$$

(b) Also, the fraction of each decay is related to the percentage of time the particle follows that “decay mode,”

$$\frac{\lambda_2}{\lambda} = 0.36, \quad \frac{\lambda_3}{\lambda} = 0.64$$

which means that

$$\lambda_2 = 0.36 \lambda = 0.36 \frac{\ln 2}{\tau} = \frac{0.36 \ln 2}{60.55 \text{ min}} = 6.87 \times 10^{-5} \text{ s}^{-1}.$$

Similarly,  $\lambda_3 = 1.22 \times 10^{-4} \text{ 1/s}$ .

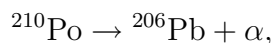
**57.** You can show that the reaction energy is just the difference in the mass excesses of the products minus the reactants,

$$Q = \left( \sum_i \Delta_i - \sum_f \Delta_f \right) c^2.$$

Using the NUBASE values for  $\Delta$  I get

$$Q = (-17,196.66 + 2,424.9156) \text{ keV} - (-20,200.6 + 8,071.3171) \text{ keV} = -2.6425 \text{ MeV}.$$

How much energy does the  $\alpha$  particles from polonium have? One of the decays is



which, again using NUBASE, gives  $Q = 5.497 \text{ MeV}$ . The kinetic energy of the  $\alpha$  particle is reduced by a factor of  $206/210$ , resulting in  $K_\alpha = 5.393 \text{ MeV}$ , which is plenty.

You'll find that all of the  $\alpha$  particles from polonium have about  $5 \text{ MeV}$  of kinetic energy, so we are not able to determine which isotope is responsible using this analysis.

**58.** Even though we don't know the number of  ${}^{235}\text{U}$  or  ${}^{238}\text{U}$  atoms in the Earth today or when the solar system was born, we do have information regarding their ratio. Today, it's just  $0.72/99.27$ , and after the supernova it was  $1.65$  (determined through modeling the nucleosynthesis that occurs during supernovae). Therefore, we have

$$\frac{N_5}{N_8} = \frac{N_{05}}{N_{08}} \frac{e^{-\lambda_5 t}}{e^{-\lambda_8 t}},$$

and we can solve for the only unknown,  $t$ . I get  $t = 6.54 \times 10^9$  years

**59.** (a) The energy required is  $Q = mc\Delta T = 544.7 \text{ J}$  for  $1.3 \text{ g}$  of water heated by  $100^\circ\text{C}$ . (b)  $E_0 = mc^2$  gives  $6.05 \times 10^{-10} \%$ . (c)  $3.66 \times 10^{10}$  decays/s. This is the definition of a "curie." (d)  $1.32 \times 10^{14}$  decays in one hour. (e) If we know how many decays in one hour (part d), and how much energy is produced (part a), we can calculate the energy per decay that must have been released:  $25.819 \text{ MeV}$ . (f) The reaction energy for the  $\alpha$  decay of the radium is  $Q = 4.863 \text{ MeV}$ . This is much less than what is needed. However, more energy is emitted because the radon decays into polonium, which then decays into astatine, lead, bismuth, thallium, and more radon, finally becoming the stable isotope  ${}^{206}\text{Pb}$ .

**60.** The table lists the conserved quantities both before and after the decay, where the  $x$  direction is the initial direction of travel of the mass  $M$ , and the  $y$  direction is perpendicular direction into which the decay products move.

	before	=	after
energy	$\frac{1}{2}MV^2 + Q$	=	$2 \left( \frac{1}{2} \left( \frac{M}{2} \right) v^2 \right)$
$p_x$	$MV$	=	$2 \left( \frac{M}{2} \right) v \cos \theta$
$p_y$	$0$	=	$\left( \frac{M}{2} \right) v \sin \theta - \left( \frac{M}{2} \right) v \sin \theta$

Note that I've added  $Q$  to the energy before the explosion because it is there as potential energy, waiting to be released chemically and turn into kinetic energy. (a) From the  $p_x$  equation,  $v \cos \theta = V$ . Using this to eliminate  $v$  from the energy equation, after some algebra I get

$$\cos^2 \theta = \frac{MV^2}{MV^2 + 2Q}.$$

(b) If  $Q = 0$  then  $\cos^2 \theta = 1$  and  $\theta = 0$ , as expected: the two pieces just keep moving forward in the initial direction. (c) Rewriting the equation as

$$\cos^2 \theta = \frac{1}{1 + \frac{2Q}{MV^2}},$$

and expanding the left-hand-side for small  $\theta$ , and expanding the right-hand-side for small  $Q$  (i.e.,  $2Q \ll MV^2$ ), I get

$$\left(1 - \frac{1}{2}\theta^2\right)^2 \approx 1 - \frac{2Q}{MV^2},$$

or, to lowest order in  $\theta$

$$\theta \approx \sqrt{\frac{2Q}{MV^2}}.$$

On the other hand, if  $Q$  is large, then  $\cos^2 \theta \approx MV^2/2Q \rightarrow 0$ , so that  $\theta \rightarrow 90^\circ$ , as expected. (d) For  ${}^8\text{Be}$ , the atomic mass is  $M = 8.005305 \text{ u} = 7456.9 \text{ GeV} \gg 100 \text{ keV}$ , so the non-relativistic approximation should be good. The  ${}^4\text{He}$  atomic mass is  $4.002603 \text{ u}$  (doubled results in  $8.005206 \text{ u}$ , which is *almost* the mass of  ${}^8\text{Be}$ ), and the mass difference is  $Q = 0.00099 \text{ u} = 92.2 \text{ keV}$ , which is just about equal to  $100 \text{ keV}$ , so neither of the approximations we developed in part (c) will work.<sup>25</sup>

$$\cos^2 \theta = \frac{100 \text{ keV}}{100 \text{ keV} + 92.2 \text{ keV}} = 0.52,$$

and  $\theta = 43.8^\circ$ .

**61.** Before the decay, the initial energy is  $\mathcal{M}c^2$  and the initial momentum is zero. After the decay, the total energy is

$$Mc^2 + mc^2 + \frac{1}{2}MV^2 + \frac{1}{2}mv^2,$$

and the zero momentum condition implies that  $MV = mv$ . The difference in the rest energies is just  $Q$  so that  $Q = MV^2/2 + K_\alpha$ . Since we want to solve for  $K_\alpha$ , it makes sense to eliminate  $V$  from these two equations, which gives

$$Q = \frac{1}{2}M \left(\frac{mv}{M}\right)^2 + K_\alpha = K_\alpha \left(\frac{m}{M} + 1\right).$$

A simple rearrangement gives the desired result. Second, if we assume that the mass number of the initial particle is  $A$ , i.e.,  $\mathcal{M} \approx Am_p$ , then the masses of the daughter particles are  $M \approx (A - 4)m_p$  and  $m \approx 4m_p$ . Plugging these in gives Eq. (3.40),

$$K_\alpha \approx \frac{A - 4}{A}Q.$$

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<sup>25</sup>Using the more accurate values of mass excess from Nubase results in  $91.8 \text{ keV}$ .

**63.** There are various values given for the mass of a banana and the amount of potassium that it contains, but a good approximation is that there is 450 mg of potassium in a medium-sized banana. Most natural potassium is  $^{39}\text{K}$  (93.3%) and  $^{41}\text{K}$  (6.7%). But there is 0.0117% radioactive  $^{40}\text{K}$  as well, or a fraction  $f = 1.17 \times 10^{-4}$ . The activity  $\mathcal{A}$  (decays per second) of the radioactive potassium in a banana is

$$\mathcal{A} = \lambda N_0 = \frac{\ln 2}{\tau} \frac{fm}{M_{40}} = \frac{\ln 2}{(1.251 \times 10^9 \text{ y})} \frac{(1.17 \times 10^{-4})(450 \text{ mg})}{(39.96 \text{ u})} \approx 14 \text{ s}^{-1}.$$

Most of the time, 89.28%,  $^{40}\text{K}$  decays via electron emission into  $^{40}\text{Ca}$ . The rest of the time, 10.72%, it decays via electron capture into  $^{40}\text{Ar}$ . Only 0.001% of the time does it emit a positron. So this reduces the above answer by  $10^{-5}$ , giving  $1.4 \times 10^{-4}$  positrons/sec, or, you'd have to wait about 7000 seconds (over an hour) between positrons.<sup>26</sup>

**65.** The mass of one  $^{14}\text{C}$  atom is

$$14 \text{ u} + \Delta = 14 \text{ u} + 3019.893 \text{ keV}/c^2 = 14.003242 \text{ u}.$$

The total number of radioactive carbon atoms is therefore

$$N = \frac{M_{tot}}{M_1} = \frac{7 \text{ kg}}{14.003242 \text{ u}} = 3.01 \times 10^{26}.$$

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<sup>26</sup>Engelkemeir, Flynn, and Glendenin made this measurement in 1962 (Phys. Rev. **126** 1818-1822, "Positron Emission in the Decay of  $\text{K}^{40}$ ." ) where they determined a ratio of  $\beta^+/\beta^- = 1.12 \times 10^{-5}$ , which gives a percentage of  $1.00 \times 10^{-3}\%$ .

# Chapter 4

## Introduction to Atomic Physics

*If we adopt Rutherford's conception of the constitution of atoms, we see that the experiments on absorption of  $\alpha$ -rays very strongly suggest that a hydrogen atom contains only one electron outside the positively charged nucleus.*

— Niels Bohr, 1913

### 4.1 Properties

As Ernest Rutherford showed (see Appendix B), atoms are composed of a small, massive, positively charged nucleus surrounded by several light, negatively charged electrons. Since the discovery of the neutron by James Chadwick [Nobel Prize, Physics, 1935] in 1932, the nucleus has been known to include positively charged protons (the nucleus of hydrogen) and electrically neutral neutrons. In its normal state each atom is electrically neutral, with the number of electrons equal to the number of protons in the nucleus — this number, known as the *atomic number*, determines the type of element. The number of neutrons is not constrained, but determines the isotope.



Since we know the constituent particles of the atom, we can enumerate their intrinsic properties, in the same manner as we have done for elementary particles as well as nuclei. These properties are mass, electric charge, and color, which are measures of how strongly a particle interacts via the gravitational force, the electric force, and the color force, respectively; spin and magnetic moment, which are inherently quantum; and size.

#### Color

White. The nucleus consists of colorless protons and neutrons and so must be colorless itself. Of course, nuclei can interact via the strong (color) and weak nuclear forces, but only if they are very close, since these forces are short range. In its normal state, however, the electrons surrounding the nucleus repel other atoms and do not allow any two nuclei to approach close enough for nuclear reactions to occur. It is true that some nuclei are

radioactively unstable and will decay regardless of any surrounding electrons, but this is a nuclear property (see Chapter 3) not an atomic property.

## Electric charge

Neutral. In its normal state, an atom has the same number of electrons and protons, and the electron and proton have an identical electric charge (although of opposite sign). Of course, an atom can be “ionized,” where one or more electrons are removed, so that the remaining “ion” is positively charged, but this is not an equilibrium atomic state. In addition, the negative electric charge due to the “cloud” of electrons does not necessarily exactly cancel the positive electric charge in the nucleus—that is, atoms and molecules sometimes have electric dipole moments. The residual electric force between otherwise neutral atoms is what is responsible for gases condensing into liquids, and also partially responsible for atoms bonding to form molecules (see Problem 36). The latter effect is very similar to the strong force (a residual color force) bonding nucleons into nuclei.

## Spin

Since the spin — intrinsic angular momentum — is a vector, the total spin of a system is just the vector sum of the individual spins. In the case of an atom, the individual spins are those of the nucleus along with each electron. However, we don’t know which direction the spin vector points, because of the limitations prescribed by the Heisenberg uncertainty principle—see Section 2.3. The  $z$  components of the individual spin vectors do just add, and therefore we can use the same technique as in Chapters 2 and 3 to determine whether the atom has integer spin or half integer spin (even though we can’t predict, at this point, what the actual value of the spin quantum numbers will be).

## Magnetic moment

While spin determines the allowed quantum states, due to the Pauli Exclusion Principle, the interaction between charged particles with spin is due to their magnetic dipole moments. Magnetic dipoles interact in a similar manner to electric dipoles, and they also experience forces and torques due to external magnetic fields. The Stern-Gerlach experiment (see Appendix C) is a prime example of how the spin of an atom is inferred from a measurement of its magnetic moment.

The net magnetic dipole moment of an atom consists of the vector sum of the constituent dipole moments. To determine the magnitude of the magnetic moment from a knowledge of the spin vector, we must know each particle’s “ $g$ -factor,” along with the typical magnitude of the magnetic moment. For example, nucleons have a much smaller magnetic moment than electrons. Recall — Secs. 2.4 and 3.6 — the electron magnetic moments are measured in units of “Bohr magnetons,”  $\mu_B$ , and nucleons in units of “nuclear magnetons,”  $\mu_N$ , where

$$\frac{\mu_N}{\mu_B} = \frac{m_e}{m_p} \approx \frac{1}{1836} \ll 1. \quad (4.1)$$



This means that, unless all the electron magnetic moments cancel, the contribution of the nucleus to the total magnetic moment can usually be ignored. This was the case in the Stern-Gerlach experiment, where 46 of the electrons in the silver atoms were paired up, so that their net magnetic moment was zero, and only the last, 47th, unpaired electron contributed to the magnetic moment of the silver atom. The magnetic moments of the nuclei of the two stable isotopes of silver are  $\mu = -0.113570\mu_N$  for  $^{107}\text{Ag}$ , and  $\mu = -0.1306905\mu_N$  for  $^{109}\text{Ag}$ , which means that they are far smaller than the magnetic moment of the single, unpaired electron  $\mu \approx -1.001\mu_B$ . See Problem 67 for a closer look at the silver nucleus.

## Mass

In Chapter 3 we assumed that the atomic mass was simply the nuclear mass plus the masses of any electrons. In fact, this is only approximately true because there is some binding energy that keeps the atom together

$$\begin{aligned} m_{atom} &= m_{nucleus} + Zm_e - \frac{B_{atom}}{c^2} \\ &\approx m_{nucleus} + Zm_e. \end{aligned} \quad (4.2)$$

In nuclear reaction calculations, the approximation is a good one because atomic binding energies  $B_{atom}$  are only about 10 eV (see Problem 10), but nuclear binding energies (and reaction energies) are on the order of  $10^6$  eV! So any error introduced by not considering the atomic binding energy is smaller than other uncertainties in a nuclear calculation.

In addition, while it is fairly easy to measure atomic binding energies by measuring their *ionization* energies (which are the same), it is very difficult to do so by measuring separately the atomic and nuclear masses and then solving Eq. (4.2). As you can see from Problem 10, both the nuclear and atomic masses need to be known to better than one part in  $10^8$  in order that a difference (a difference of two very nearly equal numbers) of 10 eV is statistically significant. This level of precision is the minimum required (for the hydrogen atom) — the problem gets worse with each nucleon added, so that it's practically impossible for heavy atoms.

## Size

As in the case with nuclei, we must be precise about what we mean by “size.” Let's take as an example the hydrogen atom. The nucleus, consisting of one proton, is about 1 fm in diameter. However, the electron “cloud” is spread out over a much larger volume. In fact, the quantum mechanical result (see Chapter 7) is that the electric charge density due to the electron is spherically symmetric and has the form

$$\rho(r) = \rho_0 e^{-2r/a_0}, \quad (4.3)$$

where  $a_0 \approx 0.05$  nm is the Bohr radius. Of course, the value of  $\rho_0$  is determined by the fact that the total charge must be equal to the electron charge. That is, the integral of the charge density over all space must be  $\int \rho dV = -e$ , where  $e \approx 1.6 \times 10^{-19}$  C (see

Problem 68). Even though the charge is spread out over an infinite volume (the charge *density* decays exponentially with distance from the proton, but never goes to zero), the Bohr radius is considered to be the “size” of the hydrogen atom. Why is this? There are at least three reasons, two of which won’t make any sense until you solve the Schrodinger equation for the hydrogen atom, so we will discuss them in Chapter 7. The simplest reason, however, which comes from observation, is evidenced by the way that atoms are packed in a solid. The geometrical array of atoms in a solid is arranged so that the distance between neighboring atoms is an equilibrium distance. If neighboring atoms were closer than this equilibrium distance their electron clouds would repel, and if they were further apart they would attract. It is logical, therefore, to call this equilibrium “nearest-neighbor” distance the size of the atom. As you might expect, this turns out to be approximately the Bohr radius.

Amazingly enough, using this idea, it turns out that all neutral atoms are approximately the same size. That is, they are each approximately spherical with a radius that varies between 0.3 nm and 0.7 nm (or around  $10a_0$ ). Why is this? Because the outermost electron “sees” a nucleus of charge  $+Z$  surrounded by  $Z - 1$  electrons, or a net charge of  $+e$ . This looks electrically like a proton, so that the outermost electron “thinks” it is in a hydrogen atom, and must orbit at the Bohr radius. Section 4.2.3 gives a slightly more sophisticated answer.

### Dimensional Analysis

“Your units are wrong!” cried the teacher.  
 “Your church weighs six joules — what a feature!  
 And the people inside  
 Are four hours wide,  
 And eight gauss away from the preacher.”

— David Morin

We can obtain an approximation for the size of an atom by using a technique called *dimensional analysis*.<sup>1</sup> To indicate how it works, let’s try to determine the period of a simple pendulum *without* actually solving the differential equation that results from Newton’s second law. The first step is to ask yourself “What quantities can the period possibly depend on?” In this case, I would include the properties of the pendulum, such as the string length  $\ell$ , the mass of the bob  $m$ , and the maximum amplitude  $\theta_0$ . The period also might depend on the strength of the restoring force, characterized by the acceleration due to gravity  $g$ . The second step is to write down the period as a product of these quantities raised to arbitrary powers

$$T = \ell^a m^b g^c \theta_0^d. \quad (4.4)$$

Now, at the very least the dimensions *must* match on both sides of any valid equation,

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<sup>1</sup>An excellent treatise on this technique is P. W. Bridgman, *Dimensional Analysis*, Yale University Press, 1922. It is also called “similitude.”

so I can express the dimensions of Eq. (4.4) as

$$[T]^1 = [L]^a [M]^b \left( \frac{[L]}{[T]^2} \right)^c, \quad (4.5)$$

where I've used  $[M]$ ,  $[L]$ , and  $[T]$ , for the dimensions of mass, length, and time, respectively. Notice that the parameter  $d$  doesn't appear because  $\theta_0$  is a dimensionless quantity, and will give us no dimensional information. Equating powers of each dimension results in three equations for the powers of mass, length, and time, respectively

$$\begin{aligned} b &= 0 \\ a + c &= 0 \\ -2c &= 1. \end{aligned} \quad (4.6)$$

Solving these equations results in  $a = \frac{1}{2}$ ,  $b = 0$ , and  $c = -\frac{1}{2}$ , and, therefore, the formula for the period must be of the form

$$T = f(\theta_0) \sqrt{\frac{\ell}{g}}. \quad (4.7)$$

Again, since the parameter  $\theta_0$  is dimensionless, the functional form of the dependence of  $T$  on  $\theta_0$  is arbitrary. A study of a simple pendulum shows that if  $\theta_0$  is small,  $f(\theta_0) \approx 2\pi$ , but that  $f$  increases as  $\theta_0$  increases.

A complete system of so-called “natural” units was devised in 1899 by Max Planck, in which  $\hbar$ ,  $G$  and  $c$  are set to unity. See the box on the next page for another view.

## Atomic Radius

Applying dimensional analysis to the hydrogen atom, an electron “orbiting” a proton, I hypothesize that the possible quantities that determine its size are its properties, such as  $e$ , a measure of the strength of the electric force between the proton and electron;  $m_e$ , a measure of the electron's inertia;<sup>2</sup> and  $\epsilon_0$ , also a measure of the strength of the attractive electric force. (It could be true that the gravitational force plays a role—if so, then we'd need to include  $G$ . But the gravitational force is very weak compared with the electric force, so we have good reason to ignore it.) If you include only these three parameters, you'll find that the resulting system of equations for the values of the exponents is overdetermined, i.e., there are no solutions (see Problem 69). However, if you include a fourth quantity,  $h$ , Planck's constant, then you'll find that the problem is solvable. The fact that a solution requires the inclusion of  $h$  means that the size of the atom is quantum mechanical in nature, and that without quantum mechanics, atoms would not have a well-defined size. In other words, it implies that quantum physics *must* play an important role in atomic structure. Our second step is now clear: the atomic radius  $r$  can be expressed as a product

$$r = e^a m_e^b \epsilon_0^c h^d. \quad (4.8)$$

To express this dimensionally, we need to introduce another dimension, charge, which I'll denote by  $[Q]$ . Since, in the SI system,  $\epsilon_0$  has the units  $C^2/Nm^2$ , and  $h$  has the units  $J s$ ,

<sup>2</sup>I'll make the approximation that the proton is very massive and therefore stationary.

**Electromagnetic mass.** The technique of dimensional analysis can be applied to a sphere of electric charge, such as the electron, in order to determine its “electromagnetic mass,” i.e., the mass that it has due to the electric field energy. It is another method to determine the classical electron radius that you calculated in Problem 8. Consider a sphere of electric charge  $q$  and radius  $r$ . What quantities might its mass depend on? Since the work necessary to assemble the charge depends on the electric force, the mass should depend on  $q$ ,  $r$ , and also  $\epsilon_0$  (which determines the strength of the electric force). If we set the mass equal to the product

$$m = q^a r^b \epsilon_0^f, \quad (4.9)$$

then the set of equations for the powers are four equations for only three unknowns,  $a$ ,  $b$ , and  $c$

$$\begin{aligned} -f &= 1 \\ b - 3f &= 0 \\ 2f &= 0 \\ a + 2f &= 0. \end{aligned} \quad (4.10)$$

The system is obviously overdetermined (e.g., the first and third equations are inconsistent). What does this mean physically? We originally guessed that since the sphere of charge was in empty space, the only possibilities were those that dealt with the electric force. But empty space *does* have properties, in the sense that the electric force is carried by photons, which travel at the speed of light. Therefore, we should include  $c$  in our list of possible dependencies, and use the formula

$$m = q^a r^b \epsilon_0^f c^d. \quad (4.11)$$

In Problem 71 you can show that the electromagnetic mass must take the form

$$m = \mathcal{D} \frac{q^2}{\epsilon_0 r c^2}, \quad (4.12)$$

where  $\mathcal{D}$  is an unknown dimensionless constant. This is identical to the result we found in Problem 8 for the classical electron radius.

I can express the dimensions of Eq. (4.8) as

$$[L]^1 = [Q]^a [M]^b \left( \frac{[Q]^2 [T]^2}{[M] [L]^3} \right)^c \left( \frac{[M] [L]^2}{[T]} \right)^d. \quad (4.13)$$

Again, equating powers of each dimension results in the set of equations for mass, length, time, and charge, respectively

$$\begin{aligned} b - c + d &= 0 \\ 2d - 3c &= 1 \end{aligned} \quad (4.14)$$

$$\begin{aligned} 2c - d &= 0 \\ a + 2c &= 0, \end{aligned}$$

allows us to determine that  $a = -2$ ,  $b = -1$ ,  $c = 1$ , and  $d = 2$ . Therefore the formula for the size  $r$  of the atom is

$$r = \mathcal{C} \frac{\epsilon_0 h^2}{m_e e^2}. \quad (4.15)$$

As is the case with all dimensional analysis, there is no way to determine the dimensionless numerical factor  $\mathcal{C}$ , but we'll see that if  $\mathcal{C} = 1/\pi$ , then  $r$  equals the Bohr radius. Interestingly, the fourth equation in Eq. (4.14) requires  $a = -2c$ , which means that  $e$  and  $\epsilon_0$  always occur as  $e^2/\epsilon_0$ . This makes sense because they both determine the strength of the electric force between two particles of charge  $e$ .

Have all possible dependencies been considered? For example, are relativistic effects important? Should we include the speed of light  $c$  in our analysis? Problem 70 investigates this possibility.

## 4.2 The Bohr Model

After 1911, when Ernest Rutherford convincingly demonstrated that the atom was comprised of a small, massive, positively charged nucleus, the question remained, “How do the electrons fit into the picture?” Niels Bohr decided to try to answer this question. One possible answer was that the atom is essentially a miniature solar system, with the electrons (point particles, as shown by Thomson) orbiting the nucleus due to the attractive Coulomb force. However, there was a problem. It was realized that if this “atomic solar system” were correct, atoms would not be stable, since Maxwell’s electromagnetic theory predicted that a charged particle radiated energy in the form of electromagnetic waves if it was accelerating. The resulting loss of orbital energy would cause the electron to spiral into the nucleus, implying that atoms should not exist!

How bad was the problem? Well, an electron with charge  $-e$ , moving uniformly in a circular orbit of radius  $r$  with speed  $v$  and orbital frequency  $\nu = v/2\pi r$  should radiate electromagnetic waves with a frequency equal to  $\nu$  and with a total power given by<sup>3</sup>

$$P = \frac{2}{3c^3} \frac{e^2}{4\pi\epsilon_0} \frac{v^4}{r^2}, \quad (4.16)$$

which is called the Larmor formula, named after Joseph Larmor (1857-1942) who first derived it in 1897. The single electron in the ground state of the hydrogen atom has an orbital radius equal to the Bohr radius,  $r = a_0 \approx 0.529 \text{ \AA}$ , and it moves with a speed

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<sup>3</sup>This is called “synchrotron” radiation, which is the name given to the radiation by charged particles accelerated in curved paths. This name comes from the machines which accelerate charged particles to high speeds in particle accelerators. It is to be compared with *brehmsstrahlung* radiation (which means “braking radiation” in German) which occurs when charged particles are accelerated in straight paths. The formula in Eq. (4.16) is the non-relativistic approximation, which is valid for electrons in hydrogen atoms. At relativistic speeds, the radiation is emitted not only at frequency  $\nu$ , but at multiples of  $\nu$  as well —  $2\nu$ ,  $3\nu$ , etc. Finally, most of the power is radiated in a direction perpendicular to the plane of the orbital circle.

$v \approx 2.2 \times 10^6$  m/s. Therefore, classically, we would expect the electron to emit synchrotron radiation of frequency  $6.58 \times 10^{15}$  Hz, which is in the extreme ultraviolet region of the spectrum. More important, the radiated power that the Larmor formula predicts is about 50 nW, which is small in macroscopic units, but converting to units more relevant to electrons in atoms, this power is about  $3 \times 10^{11}$  eV/s. Since the electron's orbital energy is on the order of a few eV, you can show that the electron would spiral into the proton in about  $10^{-11}$  s (see Problem 73).

If hydrogen atoms were this unstable, it was hard to see how the universe as we know it could exist. Bohr realized that a drastically new approach was needed, and that the solution would not be found in classical physics.

### 4.2.1 Angstrom and Balmer

The spectra of many elements had been observed and measured in the latter half of the 19th century.<sup>4</sup> In 1859, for example, Kirchhoff discovered sodium lines in the solar spectrum, and deduced that sodium must be present in the sun. Subsequently, in 1868, Ångström made the most accurate (up to that time) measurements of the visible spectrum of hydrogen. There are four lines in this part of the spectrum, and he measured their wavelengths to be

	$\lambda$
$H_\alpha$	$6562.1 \text{ \AA}$
$H_\beta$	$4860.7 \text{ \AA}$
$H_\gamma$	$4340.1 \text{ \AA}$
$H_\delta$	$4101.2 \text{ \AA}$

As you can see, the precision of Ångström's measurements was about  $10^{-5}$ , quite good for 1868. Now that these wavelengths were known, how are they to be interpreted? It is clear that discrete lines imply some kind of internal structure to the hydrogen atom, but what kind? One of the pastimes for physical scientists in the late 1800s was to try to determine formulas that could predict these spectral lines. Although just a formula does not necessarily *explain* anything, it can be a first step toward a deeper understanding.

In 1885, a schoolteacher, Johann Balmer, determined a formula that matched Ångström's wavelengths

$$\nu = Rc \left( \frac{1}{4} - \frac{1}{n^2} \right) \quad \text{where} \quad \begin{cases} n = 3 & \rightarrow H_\alpha \\ & 4 & H_\beta \\ & 5 & H_\gamma \\ & 6 & H_\delta \end{cases} \quad (4.17)$$

Here,  $c$  is the speed of light and  $R$  is Rydberg's constant, an experimentally determined quantity. Matching his formula to Ångström's measurements results in a determination of the Rydberg constant

$$R = 1.0972 \times 10^7 \text{ m}^{-1},$$

which is very close to the currently accepted value of  $1.097\,373\,156\,8525(73) \times 10^7 \text{ m}^{-1}$ . Balmer, in fact, did more than this. He interpreted the "4" in Eq. (4.17) as "2<sup>2</sup>" and

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<sup>4</sup>It might be helpful to review App. A.

thought that there might be other spectral lines if “2<sup>2</sup>” were replaced by other integers squared. So he generalized his formula to obtain

$$\nu = Rc \left( \frac{1}{n_2^2} - \frac{1}{n_1^2} \right), \quad (4.18)$$

where  $n_1 > n_2$ , and each choice of  $n_2$  results in a different “series.”<sup>5</sup> Although in principle there are an infinite number of series, there are currently six of them that have been named. They are listed in the following table.

if $n_2$	=	1	→	Lyman series (1906), UV
		2		Balmer series (1885), visible
		3		Paschen series (1908), IR
		4		Brackett series (1922), IR
		5		Pfund series (1924), IR
		6		Humphreys series (1953), far IR

In the table I’ve listed the names of the experimental discoverers, along with the date of discovery and the region of the spectrum that the spectral lines occur. The line in the table marks the time that Bohr developed his atomic model. He knew, of course, about Balmer’s generalized formula that explained Angstrom’s measurements, as well as the Lyman and Paschen measurements, and it seemed clear that more series would be observed as soon as the technology for observing in the far infrared improved.

Bohr’s goal was to explain Balmer’s generalized formula in the context of Rutherford’s nuclear atom *without* the problem of instability. In essence, he realized that the atom must exist in a “stationary state,” i.e., without the emission of radiation, and that when an atom made a transition between these stationary states it would emit (or absorb) a light quanta of a particular energy. Through the Planck/Einstein relation  $E = h\nu$  this meant that the emitted light had a particular frequency, giving rise to a spectral line. There were two problems with this proposal. First, how do atoms remain in a stationary state? Second, what is the process by which they make transitions between these states? Bohr was honest and stated that he didn’t know the answers to these questions; he specifically said, “...there obviously can be no question of a mechanical foundation of the calculations given in this paper.”<sup>6</sup> However, he said that what worked for Planck and the blackbody spectrum works here for the hydrogen spectrum: “...the essential point in Planck’s theory of radiation is that the energy radiation from an atomic system does not take place in the continuous way assumed in the ordinary electrodynamics, but that it, on the contrary, takes place in distinctly separated emissions...” Essentially, there must be something to this crazy business, otherwise it wouldn’t work so well.

<sup>5</sup>Actually, Balmer’s formula was for the wavelengths

$$\lambda = \frac{1}{R} \left( \frac{n_1^2 n_2^2}{n_1^2 - n_2^2} \right),$$

and Balmer, in comparing to Angstrom’s measurements, used  $n_2 = 2$  and found the quantity  $4/R = 364.56 \times 10^{-7}$  cm, which translates to  $R = 1.0972 \times 10^7$  m<sup>-1</sup>.

<sup>6</sup>Bohr, 1913.

There are (at least) three ways to derive the quantization condition for Bohr's model of the hydrogen atom.

1. The way Bohr did it. Since Bohr was the first to make the calculation, he made several arguments to make his model palatable to physicists of the time. He used as much classical physics as he could, and introduced Planck's and Einstein's quantization in an *ad hoc* manner, making sure, of course, that his Correspondence Principle was satisfied. This makes his method rather convoluted and difficult to follow.
2. Quantizing the electron's orbital angular momentum (remember that in 1913 electron spin was unknown). While this makes physical sense, it is actually wrong. It is true that the orbital angular momentum ( $\vec{L}$ ) of the electron *is* quantized, but it depends on a separate quantum number  $\ell$ , not the principal quantum number  $n$ , as Bohr's model suggests.
3. Forcing the de Broglie wavelength of the electron to fit in its confining box (the attractive force of the nucleus). While de Broglie's idea was not introduced until 1923, this is actually a more "modern" way of looking at the hydrogen atom. When any particle governed by the Schrodinger equation is confined in a finite volume, it is required that an integral number of wavelengths fit in that volume (simply because the Schrodinger equation is a wave equation). This requirement is identical to standing waves on a string, where the resonant frequencies are determined through the boundary conditions that the ends of the string must remain fixed, and therefore the allowed wavelengths are  $\lambda_n = 2L/n$ , where  $L$  is the length of the string and  $n = 1, 2, 3, \dots, \infty$ .

### 4.2.2 Bohr's approach

Niels Bohr's atomic model of 1913 is justly famous because he was able to pick out the essential pieces necessary to explain the emission lines of hydrogen. He made contradictory assumptions, but his intuition told him that this was necessary to make progress. Of course, even though his model correctly predicts the bound energy levels of the electron-proton system, it got the angular momentum wrong, and it could not predict the electron's wave function (that would have to wait another 12 years). This mixture of success and failure was typical of the old quantum theory, which was a hybrid of classical physics and *ad hoc* quantization rules. As stated above, the experimental evidence that Bohr used to develop his model was Balmer's generalized formula, Eq. (4.18), for the discrete wavelengths that constitute the spectrum of hydrogen. At that time Rydberg's constant  $R$  was simply an experimental constant, and there was no deeper understanding of its physical meaning.

The theoretical concept that Bohr used to explain Balmer's formula was the idea of Planck and Einstein that the energy of radiation was quantized  $E = h\nu$ . Planck's constant  $h$  was just another experimental constant that had been measured in two different ways: a fitting of the blackbody spectrum to Planck's formula, and a comparison of photoelectric effect data with Einstein's prediction. If  $h$  times the frequency  $\nu$  in Eq. (4.18) is interpreted as the energy of the radiation (light quanta) that the atom emits, it must be accompanied



by a loss of internal energy of the atom itself (physicists weren't ready to give up energy conservation just yet). That is, in the process of emitting a photon of frequency  $\nu$ , the atom is transformed from a state with energy  $E_1$  to another with energy  $E_2$

$$h\nu = E_1 - E_2. \quad (4.19)$$

Balmer's formula then requires that the atom is allowed to exist with one of a series of discrete energies, and it gives explicitly the possible values of that energy in terms of the integer  $n$ . That is, combining Eqs. (4.18) and (4.19) gives

$$E_1 - E_2 = \left(-\frac{Rch}{n_1^2}\right) - \left(-\frac{Rch}{n_2^2}\right), \quad (4.20)$$

which implies that the energy must take the form

$$E_n = -\frac{Rch}{n^2} + C. \quad (4.21)$$

Since the Balmer formula is a condition on only the difference in energies, Bohr could only determine the energy up to an additive constant  $C$ . While this turns out to be the correct expression for the energy levels, if you examine it closely, it does not really *explain* the Balmer formula. It simply replaces a quantization of wavelengths with a quantization of energies. It does not say *what* the energy of the atom is. In particular, Bohr wanted to predict the value of  $R$ . It turns out that the most illuminating way to determine  $R$  is to use method 3 from above. But in order to determine the allowed wavelengths of the electron, we need to know what the electron wavelength is, and for that we turn to Louis de Broglie.

### 4.2.3 The de Broglie wavelength

In 1923, Louis de Broglie proposed that, just as light waves have a particle nature (i.e., photons), electrons (particles) have a wave nature, and their wavelength is given by

$$\lambda = \frac{h}{p}. \quad (4.22)$$

Some physicists wondered why Einstein did not think of this, as all the elements were “under his nose,” so to speak. Einstein knew, of course, that a “light quanta” had an energy  $E = h\nu$ . In addition, from his special theory of relativity, he had developed the relationship between a photon's energy and its momentum<sup>7</sup>

$$E = pc. \quad (4.23)$$

A combination of these two gives the photon momentum  $p = h/\lambda$ , which Einstein also knew. In some sense, all de Broglie did was to turn this equation around. He asked the

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<sup>7</sup>This is the so-called “ultrarelativistic” approximation, which applies to particles moving close to the speed of light. Photons, of course, move *at* the speed of light, so by definition they are ultrarelativistic. For slow, nonrelativistic particles, the relationship is  $E = mc^2 + p^2/2m$ , where  $p^2/2m$  is the particle's kinetic energy. See Chapter 5 for the derivation of these relations.

question, “If waves with a wavelength  $\lambda$  have a momentum  $p$ , then why don’t particles with a momentum  $p$  have a wavelength  $\lambda$ ?” At that time there had been no hint that electrons had any wave properties at all. The wave-particle duality of light had been hotly debated since Einstein’s proposal in 1905, but de Broglie’s suggestion was out of the blue. As Abraham Pais relates

On September 10, 1923, [de Broglie] proposes that  $E = h\nu$  hold not only for photons but also for a ‘fictitious associated wave’ assigned to electrons. On September 24, he notes that accordingly one might anticipate diffraction phenomena for electrons... Einstein says of this work: ‘I believe it is a first feeble ray of light on this worst of our physics enigmas.’<sup>8</sup>

If we shine this ‘feeble ray of light’ on the hydrogen atom, it suggests a method to discretize (i.e., quantize) the electron’s orbit. In the case of standing waves on a string, the boundary conditions at the ends require that both ends be nodes, or that an integral number of half wavelengths fit on the string:  $L = n\lambda/2$ . For an electron in a circular orbit, there are no “ends,” but we *can* require that when the ‘fictitious associated wave’ makes one full circumference, it is in phase with itself. That is, we require that the electron *constructively* interfere with itself — if it *destructively* interfered with itself, there would be no electron! The mathematical statement of this condition is that if the electron is in a circular orbit with circumference  $C$ , then there must be an integer number of wavelengths in that same distance, or  $C = n\lambda$ , where  $n = 1, 2, 3, \dots, \infty$ . Since the circumference of a circle is just  $C = 2\pi r$ , we have

$$2\pi r = \frac{nh}{m_e v}. \quad (4.24)$$

This quantization condition turns out to be identical to method 2 above, since the electron’s orbital angular momentum is  $L = mvr$ , and solving Eq. (4.24) for  $mvr$  gives  $mvr = n\hbar$ . This means that the angular momentum is quantized, not in units of  $h$ , but in units of  $\hbar$ .<sup>9</sup>

While Eq. (4.24) is a quantization condition, it does not tell us the possible values of the radius, nor the possible values of the electron’s velocity. For that we need another relation between  $r$  and  $v$ , and this Bohr took from classical physics: Kepler’s third law as applied to the Coulomb force between the proton and electron. Recall that Newton was able to derive Kepler’s third law for a stable circular planetary orbit by realizing that the gravitational force of the Sun must be the cause of the planet’s centripetal acceleration. That is, the radial component of Newton’s second law

$$F_r = ma_r$$

applied to the planet gives

$$\frac{GM_\odot m}{r^2} = m \frac{v^2}{r}, \quad (4.25)$$

<sup>8</sup>Pais, *Inward Bound*, p 252. The “diffraction phenomenon for electrons” that de Broglie predicted was first demonstrated by Clinton Davisson and Lester Germer in 1927.

<sup>9</sup>As we have already seen, both orbital and spin angular momentum is quantized in units of  $\hbar$ , but we will see below that this quantum number  $n$  here determines the energy and radius of the electron’s “orbit.” Angular momentum has a different quantum number; so in this respect the Bohr model is incorrect.

where  $M_\odot$  is the mass of the Sun, and  $m$  is the mass of the planet. This can be rearranged to give Kepler's third law:  $T^2 \propto r^3$ . In the present case, where the force is the Coulomb force between a nucleus of charge  $Ze$  and an electron of charge  $-e$ , the radial component of Newton's second law applied to the electron gives

$$\frac{1}{4\pi\epsilon_0} \frac{Ze^2}{r^2} = m_e \frac{v^2}{r}, \quad (4.26)$$

where I've assumed that the nucleus is infinitely massive, and therefore stationary.<sup>10</sup> Treating Eqs. (4.24) and (4.26) as two equations in two unknowns ( $r$  and  $v$ ), the unknowns can be solved for, with the result that the orbital radius and velocity are both separately quantized

$$r_n = \frac{n^2}{Z} \frac{\epsilon_0 h^2}{\pi m_e e^2}, \quad (4.27)$$

$$v_n = \frac{1}{n} \frac{Ze^2}{2\epsilon_0 h}. \quad (4.28)$$

The implications of these equations are extremely profound. First, the orbital radius, the *size* of the atom, agrees with the result of our dimensional analysis in Eq. (4.15), with the dimensionless factor  $C = n^2/Z\pi$ . Second, for the "ground state" ( $n = 1$ ) of hydrogen ( $Z = 1$ ), the orbital radius is equal to the "Bohr radius,"

$$r_1 \equiv a_0 = \frac{\epsilon_0 h^2}{\pi m_e e^2} \approx 0.529 \text{ \AA}. \quad (4.29)$$

**Atomic size.** As alluded to earlier, all neutral atoms have approximately the same size. I argued that this is because the outermost electron is shielded from the full nuclear charge by the other electron and therefore only "sees" a single positive charge  $+e$ . In fact, this shielding is not perfect, as you might have suspected, and in addition, as you try to place more and more electrons into a small space they tend to repel each other. A correct, quantum-mechanical calculation results in the fact that the average radial distance of the outermost electron is approximate  $r_{ave} \approx na_0$ , where  $n$  is the principle quantum number of that last electron. Since many electrons can fit in each shell, the largest value of  $n$  in stable elements is 5. [Cesium ( $Z = 55$ ) and above have 2 electrons in the  $6s$  subshell, but not until mercury ( $Z = 80$ ) are the  $4f$  and  $5d$  subshells filled, so effectively  $n = 5$  is the highest electron level.]

Third, and most important, a knowledge of  $r_n$  and  $v_n$  allows us to calculate the energy of each of these stationary states, and to compare the resulting prediction with Eq. (4.21). The total energy of the electron is just the sum of its kinetic energy plus potential energy

$$E = K + U = \frac{1}{2} m_e v^2 - \frac{1}{4\pi\epsilon_0} \frac{Ze^2}{r}. \quad (4.30)$$

<sup>10</sup>The factor of  $Z$  in Eq. (4.26) allows us to treat all "hydrogen-like" atoms, that is, atoms with only one electron, all other electrons having been removed.

Inserting the quantized values of  $r_n$  and  $v_n$  results in a quantized value for the total energy

$$E_n = \frac{Z^2}{n^2} \left( -\frac{m_e e^4}{8\epsilon_0^2 h^2} \right), \quad (4.31)$$

where the expression in parentheses is just  $E_1 \approx -13.6$  eV, as claimed in Chapter 1. More important, however, is that a comparison with Eq. (4.21) results in a *theoretical prediction*

**Landau levels.** (This box is optional.) This method of quantizing a classical trajectory can also be used when studying charged particles executing cyclotron motion in an applied magnetic field. A particle of charge  $q$  in a magnetic field of strength  $B$  travels in a circle in a plane perpendicular to the magnetic field direction. (For simplicity we'll ignore any motion along the magnetic field — in any case, this motion is not quantized.) Just as with an electron orbiting the proton in hydrogen, Eq. (4.24) is the requirement that an integral number of de Broglie wavelengths fit on the circumference of the circular path. The second, classical, equation needed that relates  $r$  and  $v$  is just the analogue to Eq. (4.26),

$$qvB = m_e \frac{v^2}{r}, \quad (4.32)$$

which states that the magnitude of the Lorentz force ( $q\mathbf{v} \times \mathbf{B}$ ) causes the centripetal acceleration. Combining these two results in the allowed values for  $r$  and  $v$ , just as in Eqs. (4.27) and (4.28),

$$r_n = \sqrt{\frac{n\hbar}{qB}}, \quad (4.33)$$

$$v_n = \frac{\sqrt{n\hbar qB}}{m}. \quad (4.34)$$

And following the hydrogen atom derivation, we can calculate the allowed energy levels

$$E_n = \frac{1}{2} m v_n^2 = \frac{1}{2} \hbar \omega_c n. \quad (4.35)$$

Just as with the Bohr derivation, this is not quite correct. The reason is that we have used a mix of classical and quantum concepts, which results in the right idea, but with incorrect details. We'll see in Chapter 6, that the correct answer is identical to the quantum harmonic oscillator energy levels,  $E_n = \frac{1}{2} \hbar \omega_c (n + \frac{1}{2})$ . This means that cyclotron motion is in some sense identical to harmonic oscillation.

for  $R$ . Letting  $Z = 1$  (for hydrogen), I get<sup>11</sup>

$$R_\infty = \frac{m_e e^4}{8\epsilon_0^2 h^3}. \quad (4.36)$$

<sup>11</sup>The subscript  $\infty$  in Eq. (4.36) indicates that the mass in the numerator is the electron mass.

If you evaluate the right hand side of Eq. (4.36), the resulting value matches the experimental value of  $R$ , and this gave Bohr confidence in his model. The important physical fact is that all the quantities on the right hand side of Eq. (4.36) were measured in different types of experiments:  $\epsilon_0$  was measured by Coulomb using charged pith balls,  $h$  was measured by comparing Planck's formula with blackbody radiation,  $c$  was measured by Michelson and Morley, and  $e$  was measured by Thomson following an electron in a magnetic field. The fact that all these disparate physical constants come together to equal a constant that shows up in atomic spectra is solid evidence for Bohr's theory, and also indicates that atomic spectra must be quantum and electric in nature.

#### 4.2.4 Ionized helium and deuterium

Bohr's model, of course, could not explain the spectrum of other atoms with more than one electron. This did not stop Arnold Sommerfeld (among others) from trying to generalize Bohr's model by allowing the electrons to orbit in elliptical trajectories rather than circles. He also attempted to include relativistic effects by allowing the electron's speed to approach  $c$ . These extensions met with only limited success, however, because ultimately the spectra of heavier elements were much more complex than that of hydrogen, and a complete explanation would have to wait until 1926 when a true quantum mechanics was developed. In fact, in 1923 Arnold Sommerfeld stated, "All attempts made hitherto to solve the problem of the neutral helium atom have proved to be unsuccessful."<sup>12</sup>

In any case, one example to which Bohr's model *could* be applied successfully was singly-ionized helium.  $\text{He}^+$ , also known as He II, consists of only one electron orbiting an  $\alpha$  particle. While it was not known exactly what the composition of the  $\alpha$  particle was, it was clear that it was four times as massive as the hydrogen nucleus, and had twice the positive electric charge. In the observation of the spectral lines, the Rydberg constant  $R$  is an easily measurable quantity, and Eq. (4.31) implies that  $R(Z)$ , the Rydberg constant for any single-electron atom with nuclear charge  $Ze$ , is given by

$$R(Z) = Z^2 \frac{m_e e^4}{8\pi^2 \epsilon_0^2 h^3}. \quad (4.37)$$

Comparing the results for helium and hydrogen *should* give the ratio 4, because for helium is  $Z = 2$ . However, the experimental value of the ratio was found to be

$$\frac{R(\text{He})}{R(\text{H})} = 4.0016. \quad (4.38)$$

While this is a small discrepancy, it was well within the experimental precision. Bohr quickly realized his mistake: by assuming that the nucleus remained stationary he had effectively ignored the mass of the electron, and instead of  $m_e$  in Eqs. (4.31) and (4.37), what was needed was the reduced mass (see Appendix F)

$$\mu = \frac{M m_e}{M + m_e}, \quad (4.39)$$

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<sup>12</sup>Quoted in Pais, *Inward Bound*, page 215.

where  $M$  is the nuclear mass. Of course, in the limit  $m_e \ll M$ , the reduced mass is approximately  $\mu \approx m_e$ . For hydrogen

$$\mu_H = \frac{m_p m_e}{m_p + m_e} = m_e \left( \frac{1}{1 + \frac{m_e}{m_p}} \right) \approx (0.999455)m_e, \quad (4.40)$$

and for helium

$$\mu_{He} \approx (0.999864)m_e, \quad (4.41)$$

so that the ratio of the Rydberg constants is

$$\frac{R(\text{He})}{R(\text{H})} = \frac{4\mu_{He}}{\mu_H} = 4.00163, \quad (4.42)$$

exactly in agreement with experiment. After this success, critics of the Bohr model quickly dropped their opposition.

Deuterium is another case where the theoretical prediction required the use of the reduced mass. A stable isotope of hydrogen, deuterium has a natural abundance on Earth of only 0.0115%. In 1931, Harold Urey [Nobel Prize, Chemistry, 1934] was able to produce hydrogen with a high concentration of the  $A = 2$  isotope, which he then identified spectroscopically.<sup>13</sup> You can show (see Problem 81) that the spectral lines of deuterium are slightly shifted compared to those of hydrogen. The spectrum from a sample of hydrogen that contains both isotopes shows both sets of lines, and the relative intensity of the lines reveals information about the relative concentration of the two isotopes. The discovery of deuterium and other stable isotopes of low  $Z$  elements were a strong impetus to definitively determine the structure of the nucleus — recall the struggle with the proton-electron model in Section 2.3.2. Urey also looked for spectral lines from tritium, but didn't see any because its short half life meant that its natural abundance on Earth is very low.



### 4.3 The periodic table

Does Bohr's model explain the periodic table? No, but it comes close. To explain the periodic table would require a complete quantum mechanical analysis of the hydrogen atom that we will study in Chapter 7. However, I can state some results and give qualitative reasons for them.

First, Bohr got the energy levels (of hydrogen, at least) correct. Equation (4.31), with the modification that  $m_e$  be replaced by the reduced mass  $\mu$ , correctly gives the allowed energies of all one-electron atoms as a function of the principle quantum number  $n$ . We shall see in Chapter 7, however, that when a particle is confined in a three-dimensional box (which is essentially what an electron attracted to a proton is) there must be three

<sup>13</sup>The fact that  $^1\text{H}$ , also known as "protium," is lighter than deuterium means that protium evaporates faster than deuterium from hydrogen that has been liquefied, and therefore the remaining hydrogen has a higher concentration of deuterium.

quantum numbers that define the state of the particle — one for each dimension. In the case of a spherically symmetric atom, the other two quantum numbers—which we have seen before—are  $\ell$  and  $m_\ell$ , which determine the magnitude of the orbital angular momentum and its  $z$ -component, respectively.<sup>14</sup> In addition, each electron has another “degree of freedom,” in that it can vary its spin quantum number  $m_s$  to be either  $\pm\frac{1}{2}$ , i.e., it can be either spin up or down. (Recall that for each electron  $s = \frac{1}{2}$  is an unchanging fundamental property.) This structure holds for all atoms, even those with more than one electron: each electron is characterized by four quantum numbers,<sup>15</sup> and no two electrons can have the same set (so sayeth Pauli).

What are the possible allowed values for these quantum numbers? Bohr was again correct in showing that  $n$  could take on integer values from 1 through  $\infty$ . However, for each value of  $n$ , the (orbital) angular momentum quantum number can take on any integer value as long as  $\ell < n$ . The reason for this comes from the solution of the Schrodinger equation for a hydrogen atom, so it is essentially a mathematical requirement about the behavior of the solution. It can be understood qualitatively, however, by looking at well-known results for the classical orbits in a  $1/r$  (gravitational) potential. There, the total energy of the particle (or planet) is determined by the semi-major axis of the elliptical orbit. The angular momentum depends on the eccentricity of the orbit, with the minimum value of zero corresponding to a straight line orbit, and the maximum value occurring for a circular orbit. In the quantum case, for a given energy (determined by  $n$ ) there will be a maximum amount of angular momentum (determined by  $\ell$ ), which is just what the inequality  $\ell < n$  states. Finally, as we have seen, the  $z$ -component of a vector cannot be larger than the vector itself, so that restricts the third quantum number to the values  $|m_\ell| \leq \ell$ . The second restriction on  $m_\ell$ , the fact that neighboring values must differ by 1, comes from the requirement that the solution to the Schrodinger equation must be well-behaved.

Putting all this together, along with the requirements of the Pauli exclusion principle (no two electrons with the same four quantum numbers), tells us how many electrons can fit in each “subshell,” where a subshell is defined as all the quantum states with the same  $n$  and  $\ell$ . These subshells are listed in Table 4.1. For example, the lowest energy level,  $n = 1$ , has only one subshell, because  $\ell = 0$  and  $m_\ell = 0$  are required, but since  $m_s$  can be either  $\pm\frac{1}{2}$ , this means that two electrons can fit in this subshell, called the “1s” subshell. Each electron is in a different quantum state—as required by Pauli—because one has the quantum numbers  $(1,0,0,\frac{1}{2})$  and the other has  $(1,0,0,-\frac{1}{2})$ .

Another result that Bohr could not predict is that as soon as there is more than one electron in the atom, the energy depends not only on the quantum number  $n$  but also on  $\ell$  (see the box on page 8). That is, the  $2s$  subshell ( $\ell = 0$ ) has a lower energy than the  $2p$  subshell ( $\ell = 1$ ), which is why it is filled first.<sup>16</sup> This means that the third electron (i.e., lithium) occupies the  $2s$  subshell, and it is not until boron ( $Z = 5$ ) that the fifth electron occupies the  $2p$  subshell. The reason for this is that the Coulomb potential is a special case. Only when  $U \sim 1/r$  are the energy levels independent of  $\ell$ . A second

<sup>14</sup>This means that Bohr was incorrect in believing that  $n$  also determined the magnitude of the electron’s orbital angular momentum.

<sup>15</sup>The four quantum numbers are usually listed in the following order  $(n, \ell, m_\ell, m_s)$ .

<sup>16</sup>The pre-quantum notion that particles like to occupy the lowest energy state is still valid.

$n$	$\ell$	subshell	# $e^-$	noble gas	$Z$
1	0	1s	2	He	2
2	0	2s	2		
2	1	2p	6	Ne	10
3	0	3s	2		
3	1	3p	6	Ar	18
3	2	3d	10		

Table 4.1: Allowed numbers of electrons in each subshell, denoted by  $n\ell$ , where  $n = 1, 2, 3, \dots$ , and  $\ell = 0, 1, 2$ , denoted by  $s, p, d$ , respectively. The last column lists the atomic “magic numbers,” which result in especially stable electronic configurations. The higher atomic magic numbers are 36 (Kr), 54 (Xe), 86 (Rn), 118 (Uuo).

electron, however, experiences a potential due to both the nucleus and the “cloud” of the first electron, so the net potential is not proportional to  $1/r$  (see Problem 83). This special nature is also true classically. A key result from classical mechanics is Bertrand’s theorem, which states that for central force laws of the form  $F_r \sim -r^n$  there are only two values of  $n$  for which all orbits are closed (meaning that the orbiting particle returns to the same locations). These two values are  $n = -2$  (gravitational force) and  $n = 1$  (Hooke’s law spring force).<sup>17</sup>

## 4.4 Moseley’s Law

*Had the European War no other result than the snuffing out of this young life, that alone would make it one of the most hideous and most irreparable crimes in history.* — Robert Millikan, referring to Henry Gwyn-Jeffreys Moseley<sup>18</sup>

Until 1914, the atomic number of an element had no physical meaning. It was just the listing of the known elements in order of atomic weight, as shown in the table below.

<i>element:</i>	H	He	Li	Be	B	C	N	O	...
<i>atomic “number”:</i>	1	2	3	4	5	6	7	8	...
<i>atomic weight:</i>	1	4	7	9	11	12	14	16	...

With some pairs of elements, like cobalt and nickel, and tellurium and iodine, it was not clear whether they should be placed in order of their atomic weight or in an order that better fit their chemical properties. Mendeleev guessed correctly that chemical properties were more important than atomic weight. In addition, it was not known how many (if any) missing elements there were. For example, only in the previous twenty years had the

<sup>17</sup>A large quantity of research is aimed at understanding the relationship between the properties of classical orbits and quantum states.

<sup>18</sup>After his groundbreaking scientific work in 1913 and 1914, Moseley joined the Royal Engineers after the outbreak of World War I, and died in 1915 at age 27 in Gallipoli, Turkey.



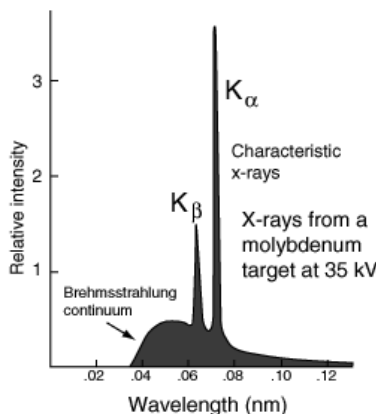


Figure 4.1: X-ray spectrum of molybdenum. The continuum X-ray spectrum is produced by electrons with 35 keV of kinetic energy (i.e., it was accelerated through a potential difference of 35 kV). This means that if it gives up all its kinetic energy to the bremsstrahlung X-rays, those photons would have a wavelength of  $\lambda = hc/E = 0.035$  nm. This is just the lower cutoff of the spectrum. In addition, secondary X-rays (also called characteristic X-rays) from molybdenum are seen in the spectrum. In particular, the  $K_\alpha$  and  $K_\beta$  lines are seen, corresponding to the  $2 \rightarrow 1$  and  $3 \rightarrow 1$  transitions. From HyperPhysics, <http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/xrayc.html>

entire column of noble gases been discovered. In addition, after the discovery of helium<sup>19</sup> in 1895 it was speculated that there were two elements *between* hydrogen and helium — popular suggestions were “nebulium” and “coronium,” two other elements thought to exist only in the sun. It was the work of Henry Moseley in 1913-1914 that correctly revealed the atomic number of all elements up to gold ( $Z = 79$ ), and also determined the number of missing elements. He did it with X-rays.

X-rays had been known since 1895, when Roentgen realized that radiation was coming from the collecting plate of his vacuum tube. As we understand it today, the cathode rays (electrons) emitted *bremsstrahlung* (braking radiation) as they slowed down when they hit the target. These high-energy photons are emitted with a continuum of wavelengths and are called “primary X-rays.” The maximum energy such an X-ray photon could have is if the electron lost all its kinetic energy (for an electron to be annihilated, and therefore give up its rest energy also, positrons would have to be present). A typical X-ray photon has an energy of 1 keV, which means that the voltage across Roentgen’s vacuum tube must have been at least 1 kV.



In 1897 it was found that when these primary X-rays impinged upon another substance, that substance gave off “secondary X-rays.” A typical X-ray spectrum showing

<sup>19</sup>An unknown spectral line had been observed in the solar corona in 1868 by Pierre Janssen during a solar eclipse. This new element was postulated to exist only on the sun, so it was called helium. It was discovered on Earth in 1895 by Cleve and Langer in Sweden: they dissolved uranium-laden rock in acid and collected the gas bubbles that escaped. The spectrum of this gas positively identified it as helium.

both types is shown in Fig. 4.1. Slowly, over the course of the next fifteen years, different substances were found to emit different types of secondary X-rays. One of the differences was that each substance emitted X-rays with a discrete set of wavelengths (rather than the primary X-rays that were a continuum). It was similar to the optical spectrum of a gas, in which each element had a different “spectral fingerprint.” In the case of X-rays, however, there was some regularity among the elements,<sup>20</sup> and Moseley discovered this regularity. It took Moseley and his predecessors several years to unravel the details of the X-ray spectra, especially given that X-ray diffraction was discovered only in 1912 by Max von Laue [Nobel Prize, Physics, 1914]. The history of these investigations is interesting and informative, but is best left for later. For now, I will explain Moseley’s work as we currently understand it.



When a primary X-ray encounters a neutral atom, it can be absorbed and eject an orbiting electron from that atom. When it does so, another orbiting electron from the same atom can “drop down” and fill the hole left behind. This transition between energy levels is accompanied by the emission of a photon of the appropriate energy. In the specific case where a  $1s$  electron is ejected (i.e., from the  $K$  shell) and an  $n = 2$  electron makes a transition to fill the empty hole, the photon that is emitted is called a  $K_\alpha$  X-ray. This is similar to the  $H_\alpha$  line of the hydrogen spectrum. If we assume that Bohr’s model holds (at least approximately) for this many-electron atom, then the transition is from  $n = 2$  to  $n = 1$ , and Eqs. (4.18) and (4.37) predict the frequency of the emitted X-ray to be

$$\nu = Z_{\text{eff}}^2 \frac{3R_\infty c}{4}. \quad (4.43)$$

Here I’ve ignored the reduced mass (the approximation inherent in the use of  $R_\infty$  gets better as the nuclear mass increases) and  $Z_{\text{eff}}$  is the effective nuclear charge that the transitioning electron sees. Because this electron is originally in the  $n = 2$  energy level, and there is still one electron left in the  $1s$  subshell, that electron shields the full nuclear charge  $Ze$  and makes it appear to be  $(Z - 1)e$ , which means that  $Z_{\text{eff}} \approx Z - 1$ .

What Moseley first did was to plot  $\sqrt{\nu}$  versus  $Z$  for ten elements between calcium ( $Z = 20$ ) and zinc ( $Z = 30$ ), and his results are shown in Fig. 4.2. The prediction given by Eq. (4.43) is, of course, a straight line

$$\sqrt{\nu} = \sqrt{\frac{3R_\infty c}{4}} (Z - 1), \quad (4.44)$$

which matches the data quite well. Two quantitative checks can be made. First, that the slope of the line is  $\sqrt{3R_\infty c/4} = 4.967 \times 10^7 \sqrt{\text{Hz}}$ . Second, that the  $Z$  intercept is unity. A least squares fit to Moseley’s data (Problem 84) shows that the measured slope matches the predicted slope to within 0.5%, and that the  $Z$  intercept is 1.13. Although this is not exactly unity, it was within the experimental precision. Moseley knew that this represented shielding of the nuclear charge, but at that time Bohr thought that for elements with  $Z > 7$ , there were four electrons in the  $K$  shell. Remember, Bohr had just

<sup>20</sup>Only for hydrogen and the alkalis did the optical spectra show some regularity.

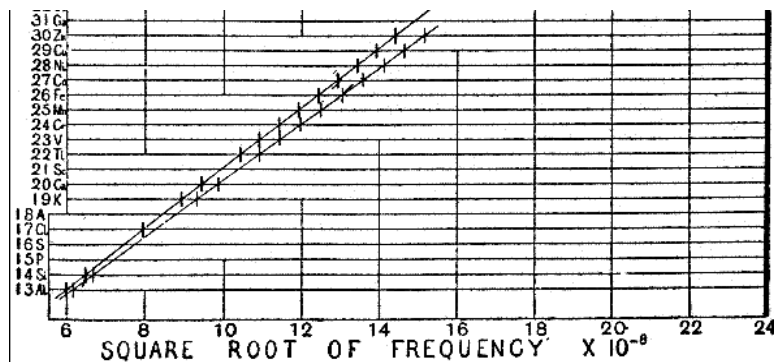


Figure 4.2: Moseley's plot of the frequency of the  $K_\alpha$  and  $K_\beta$  X-rays from various elements. The  $K_\beta$  photons come from a transition between  $n = 3$  and  $n = 1$ , and therefore are of a slightly higher energy and frequency; in particular, the factor  $\frac{3}{4}$  in the expression for the  $K_\alpha$  line is replaced by  $\frac{8}{9}$ . The experimental points fall on a straight line to within 0.5%. A portion of Figure 3 from Moseley, 1913.

developed his hydrogen model the previous year and no one had been able to apply it successfully to other elements. Also, spin and the Pauli exclusion principle were ten years away.

After the success of these initial experiments, Moseley then went on to determine  $Z$  for all known elements up to uranium ( $Z = 92$ ), and was able to conclusively show that seven were “missing.” These seven were protactinium ( $Z = 91$ , discovered 1917), hafnium ( $Z = 72$ , discovered 1923), rhenium ( $Z = 75$ , discovered 1925), technetium ( $Z = 43$ , discovered 1937), francium ( $Z = 87$ , discovered 1939), astatine ( $Z = 85$ , discovered 1940), and promethium ( $Z = 61$ , discovered 1945). Some of these, for example technetium, had been suspected previously, due to the chemical properties of the surrounding elements, but Moseley was the first to show that there was a definite hole in the periodic table. (See App. J for a list of other discoveries.)

## Problems

**67.** In this problem, you will develop an argument to show that the contribution of the nucleus to the magnetic moment of silver is negligible, which is why Stern and Gerlach were able to ignore it. Silver has two stable isotopes,  $^{107}\text{Ag}$  and  $^{109}\text{Ag}$ , whose nuclei contain 47 protons and 60 (or 62) neutrons. (a) Using what you know about how nucleonic spins add, can you guess what the total spin of these nuclei are? Of course, you don't know how to calculate the exact value of the spin, but you can make an educated guess. Back up your guess with a logical argument. (b) Now that you know the spin quantum number, what is your best guess for the magnetic moment? Consider which of the nucleonic magnetic moments cancel and which do not. Again, back up your answer with an argument.

**68.** Obtain the correct value for  $\rho_0$  by integrating the charge density in Eq. (4.3) over all space and setting the integral equal to  $-e$ . Remember that in spherical coordinates the volume element is  $dV = r^2 dr \sin \theta d\theta d\phi$ .

**69.** Attempt to obtain the size of the hydrogen atom using dimensional analysis assuming that  $r$  depends only on  $e$ ,  $m_e$ , and  $\epsilon_0$ . What mathematical problems do you encounter?

**70.** Attempt to obtain the size of the hydrogen atom using dimensional analysis assuming that  $r$  depends on  $e$ ,  $m_e$ ,  $\epsilon_0$ , as well as  $h$  and  $c$ . Do you encounter any mathematical problems in this case?

**71.** Use dimensional analysis to show that the mass of a sphere of charge in empty space takes the form of Eq. (4.12) if you assume dependences of the form of Eq. (4.11).

**72.** Show that the classical orbital frequency of an electron orbiting a positive charge  $Ze$  is

$$\nu = \sqrt{\frac{Ze^2}{4\pi\epsilon_0} \frac{1}{4\pi^2 m_e r^3}}.$$

Use a similar technique to that used in calculating gravitational orbits: solve the radial component of Newton's Second Law using the fact that the force of attraction causes the centripetal acceleration.

**73.** How quickly does an electron orbiting a proton spiral into the proton? In other words, how long should hydrogen atoms survive, from a classical point of view? The electron radiates energy at a rate given by Eq. (4.16), its orbital radius  $r$  decreases, and to answer the question above we will need to know the rate at which  $r$  changes. Since the chain rule gives

$$P \equiv \frac{dE}{dt} = \frac{dE}{dr} \left( -\frac{dr}{dt} \right),$$

if you can obtain the relationship  $E(r)$ , then you can obtain a differential equation involving  $dr/dt$ , which can then be solved for  $r(t)$ . The minus sign denotes the fact that as the electron radiates,  $r$  decreases with time. HINT: use the result of the previous problem.

**74.** If an accelerating charged particle emits electromagnetic radiation, shouldn't an accelerating massive particle emit gravitational radiation? The answer is yes. (a) Starting with the Larmor formula in Eq. (4.16), replace the electric quantities,  $1/4\pi\epsilon_0$  and  $e$ , with gravitational quantities,  $G$  and  $m$ , and obtain a "gravitational Larmor formula." (b) Calculate the power that the Earth emits in the form of gravitational waves. How long will it take for the Earth to spiral into the Sun?

**75.** In 1871, George Stoney guessed (incorrectly) that the frequencies of three of the lines that Ångström had measured for hydrogen (see page 100) were harmonics of a fundamental vibration frequency  $\nu_1$ . That is, he proposed the wavelength ratios

$$\lambda_1 : \lambda_\alpha : \lambda_\beta : \lambda_\delta = 1 : \frac{1}{20} : \frac{1}{27} : \frac{1}{32}$$

where  $\lambda_1$  is a wavelength that corresponds to a fundamental frequency  $\nu_1$ . (a) Determine the frequency  $\nu_1$  of the "fundamental vibration." (b) Why did Stoney not include the  $H_\gamma$  line in his list?

**76.** Show that the speed of an electron in a hydrogen atom, in the Bohr orbit labeled by  $n$ , is given by  $\alpha c/n$ , where  $\alpha$  is the fine structure constant.

**77.** Consider a particle of mass  $m$  in a stable circular orbit about a point where the central force is of the form  $F_r = -k/r^2$ . (a) Show that  $U = -2K$ , where  $U$  is the particle's

potential energy and  $K$  its kinetic energy. (b) Show that the total energy  $E = K + U$  of the particle can be written in two ways

$$E = -K = \frac{1}{2}U.$$

**78.** Consider an electron in state  $n$  of the hydrogen atom. (a) Assuming that it is in a circular orbit with radius  $r_n$  (i.e., the Bohr model), derive a formula for its orbital frequency  $\nu$ . (This is the frequency at which classical theory would predict it to radiate.) (b) Derive a formula for the frequency of emitted radiation when the electron makes a transition from state  $n$  to state  $n - 1$ . (c) Evaluate (and compare) these two formulas numerically when  $n = 10, 100, 1000$ .

**79.** Obtain an approximate analytic expression for the frequency of emitted radiation when an electron makes a transition from state  $n$  to state  $n - 1$ , in the limit where  $n \rightarrow \infty$ . (You might find the binomial expansion or the Taylor series helpful.) Compare this with your formula from part (a) of Problem 78. (This problem and Problem 78 are an example of Bohr’s Correspondence Principle. That is, the classical prediction—in this case the radiation by a circulating electron—must equal the quantum prediction in the limit when the quantum number is large.)

**80.** Rederive the Bohr model in the case where the two objects are attracted not electrically, but gravitationally. Since both the Coulomb force and Newton’s Universal law of gravitation are inverse square laws, the derivation follows the same steps. Apply your “Bohr-gravity” formula to the case of the Sun and the Earth. What is the quantum number  $n$  of the state that the Earth is in? Should Bohr’s Correspondence Principle apply?

**81.** (a) Assuming Bohr’s model, calculate the wavelengths of the Balmer lines  $H_\alpha$  through  $H_\delta$  in ordinary hydrogen ( $^1\text{H}$ ) and in deuterium ( $^2\text{H}$ ) (HINT: You’ll need to take into account the reduced mass.) What is the difference between the wavelengths? Do you agree with Urey’s calculations of  $\Delta\lambda$  below from his 1932 discovery paper? (His units are Ångstroms.)

Line	$H_\alpha$	$H_\beta$	$H_\gamma$	$H_\delta$
$\Delta\lambda$ calc.	1.793	1.326	1.185	1.119
$\Delta\lambda$ obs.				
Ordinary hydrogen	—	1.346	1.206	1.145
1st sample	—	1.330	1.199	1.103
2nd sample	1.820	1.315	1.176	—

(b) Calculate the wavelengths of the Balmer lines in tritium ( $^3\text{H}$ ). Do you think these would have been detectable by Urey? (c) Do you think that Urey’s measurements above definitively prove the existence of deuterium?

**82.** Fill in the next four rows of Table 4.1. Refer to a periodic table to determine which subshells are filled first.

**83.** The electric potential that an electron “sees” due to a proton (at the origin) is  $e/(4\pi\epsilon_0 r)$ . This is the potential that is used, in quantum mechanics, to solve the Schrodinger equation and calculate the wave function  $\psi$  for that electron. It is, of course, related to the electric field generated by the nucleus  $E_r = e/4\pi\epsilon_0 r^2$ . This is good for

hydrogen, but what happens when we consider helium? A simple model, which is too simplistic but gives a feeling for the complexity inherent in a “three-body problem,” is to assume that the first electron is in the ground state  $n = 1$ . Then the second electron experiences not only the electric field due to the nucleus (with charge  $+2e$ ) but also the field due to the first electron. Of course, the first electron is affected by the second electron, and so the ground state is modified, but we’ll ignore that for this problem. Here, I’m asking you to calculate the electric field  $E_r(r)$  due to the first electron in the ground state. The charge density of the first electron in the ground state can be found from the Schrodinger equation (Chapter 7) is<sup>21</sup>

$$\rho(r) = \frac{-e}{\pi} \left( \frac{Z}{a_0} \right)^3 e^{-2Zr/a_0},$$

where  $a_0 = 4\pi\epsilon_0\hbar^2/me^2$  is the Bohr radius. Your task in this problem is to calculate  $E_r$  due to this  $\rho$ . The simplest method, since the system is spherically symmetric, is to use Gauss’s Law to obtain the radial component of the electric field. Of course, for use in the Schrodinger equation, the potential  $V$  must be obtained from an integration of  $\vec{E}$ .

**84.** Moseley’s first measurements of the wavelengths of the  $K_\alpha$  lines of different elements are given in the following table.

element	$Z$	$\lambda$
Ca	20	3.368 $\times 10^{-8}$ cm
Ti	22	2.758 ”
V	23	2.519 ”
Cr	24	2.301 ”
Mn	25	2.111 ”
Fe	26	1.946 ”
Co	27	1.798 ”
Ni	28	1.662 ”
Cu	29	1.549 ”
Zn	30	1.445 ”

Use linear regression techniques (i.e., a least-squares analysis) to obtain the slope and  $Z$  intercept of the best fit line. How well does it match the predicted values?

## Solutions

**67.** (a) Both isotopes of silver have odd  $A$ , so that means the nuclei have half integer spin, but we don’t know *which* half integer. However, using the fact that all even-even nuclei have zero spin, we might guess that the 60 (or 62) neutrons pair up and contribute zero to the total spin. Also, 46 of the 47 protons might also pair up, leaving only one unpaired proton. Thus, the spin of the entire nucleus would be equal to the spin of this

<sup>21</sup>If you are curious, the wave function of the first electron is

$$\psi_{100} = \frac{1}{\sqrt{\pi}} \left( \frac{Z}{a_0} \right)^{3/2} e^{-Zr/a_0},$$

and the charge density is given by  $\rho = -e|\psi|^2$ .

proton, which is  $s = \frac{1}{2}$ . In fact, this is the case: both nuclei are spin  $\frac{1}{2}$  (see NUBASE). This doesn't have to be the case. Consider  $^{59}\text{Co}$  ( $Z = 27$ ,  $N = 32$ ), which has a nuclear spin of  $\frac{7}{2}$ . This means that it's not true that all pairs of protons and pairs of neutrons have oppositely aligned spin, but it's a reasonable first guess. (b) If all the pairs of spins add to zero, then their magnetic moments must cancel, so that means that the magnetic moment of the nucleus would simply be the magnetic moment of the unpaired proton, which is  $\mu_z = 2.793 \mu_N$ . The measured values are  $-0.113570 \mu_N$  for  $^{107}\text{Ag}$ , and  $-0.1306905 \mu_N$  for  $^{109}\text{Ag}$ . Obviously, there is something more complicated going on.

**68.** The angular integrals are just  $4\pi$

$$\int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi = (2)(2\pi) = 4\pi,$$

so we have to solve

$$4\pi \int_0^\infty r^2 \rho(r) dr = -e$$

for  $\rho_0$ . Since  $\int_0^\infty r^n e^{-ar} dr = \frac{n!}{a^{n+1}}$ , the left hand side of the above equation is

$$4\pi \rho_0 \int_0^\infty r^2 e^{-2r/a_0} dr = 4\pi \rho_0 \left( \frac{2!}{(2/a_0)^3} \right),$$

which results in

$$\rho_0 = \frac{-e}{\pi a_0^3} \approx -3.44 \times 10^{11} \frac{\text{C}}{\text{m}^3}.$$

**69.** Our guess for the form of  $r$  is

$$r = e^a m_e^b \epsilon_0^c.$$

As in the text, equating powers of dimensions on each side of the equation gives a set similar to Eq. (4.14), except that all the values of  $d$  are zero

$$\begin{aligned} b - c &= 0 \\ -3c &= 1 \\ 2c &= 0 \\ a + 2c &= 0, \end{aligned}$$

Immediately we see that the second and third equation are not consistent. In fact, the entire system is overdetermined, as there are too many equations for the number of free variables. The set of equations is not linearly independent.

**70.** Now we have to add a term to our guess for  $r$

$$r = e^a m_e^b \epsilon_0^c h^d c^e.$$

This means that equating dimensions, as in Eq. (4.13), results in

$$[L]^1 = [Q]^a [M]^b \left( \frac{[Q]^2 [T]^2}{[M][L]^3} \right)^c \left( \frac{[M][L]^2}{[T]} \right)^d \left( \frac{[L]}{[T]} \right)^e.$$

Now, however, we have an underdetermined system, which means that there's not enough information to solve for all 5 exponents.

$$\begin{aligned} b - c + d &= 0 \\ 2d - 3c + e &= 1 \\ 2c - d - e &= 0 \\ a + 2c &= 0, \end{aligned}$$

The last equation again tells us that  $e$  and  $\epsilon_0$  must be grouped as  $e^2/\epsilon_0$ . The first equation minus the sum of the second and third results in  $b = -1$ , as before. But I then obtain only two equations for the final three exponents,  $c = d - 1$  and  $c - e = 1$ . This system is underdetermined. A more likely solution to the problem of introducing relativistic effects is to assume that the radius is

$$r = e^a m_e^b \epsilon_0^c h^d f\left(\frac{v}{c}\right),$$

where  $f$  is a dimensionless function of  $\beta = v/c$ . This will result in Eq. (4.15) multiplied by  $f$ , and other considerations will be needed to determine the functional form of  $f$ .

**71.** With the addition of the factor of  $c^d$ , Eqs. (4.10) become

$$\begin{aligned} -f &= 1 \\ b - 3f + d &= 0 \\ 2f - d &= 0 \\ a + 2f &= 0 \end{aligned}$$

These four equations with four unknowns can be easily solved:  $a = 2$ ,  $b = -1$ ,  $f = -1$ , and  $d = -2$ , which results in Eq. (4.12).

**72.** For a stable circular orbit Newton's second law, stating that the electric force causes the centripetal acceleration, is

$$\frac{1}{4\pi\epsilon_0} \frac{Ze^2}{r^2} = \frac{mv^2}{r},$$

where  $Ze$  is the charge of the central nucleus, and  $m$  and  $e$  are the mass and charge of the orbiting electron. Given that the orbital velocity can be written  $v = 2\pi r\nu$ , simply solving for  $\nu$  gives the answer.

**73.** From the previous solution, Newton's second law (which is just Kepler's third law in this case) can be written in the form

$$v^2 r = \frac{1}{4\pi\epsilon_0} \frac{Ze^2}{m}.$$

When  $Z = 1$  and  $r = 0.0529$  nm, this results in an orbital speed of  $v/c = 0.0073$ , which is nonrelativistic. So at least in the electron's initial (Bohr) radius, it is appropriate to use the nonrelativistic Larmor formula. Of course, as  $r$  decreases,  $v$  must increase, so that when  $r = 2.82$  fm, the orbital speed must be equal to  $c$  so that Larmor's formula



is no longer valid. However, it will still be useful to give an order-of-magnitude estimate of the “collapse time” for the spiralling electron. For stable orbits you can show that the potential and kinetic energies are related by  $U = -2K$ , and that the total energy of the orbiting object is  $E(r) = K + U = \frac{1}{2}U = -Ze^2/8\pi\epsilon_0 r$ . (See Problem 77.) This means that

$$\frac{dE}{dr} = \frac{Ze^2}{8\pi\epsilon_0 r^2}.$$

We needed to obtain  $E$  as a function of  $r$  only, to make sure that we were differentiating with respect to the entire dependence on  $r$ . Using the stable circular orbit speed  $v$ , expressing  $P$  in terms of  $r$ , and noting that  $dr/dt$  must be negative if the electron is radiating, I obtain

$$\frac{e^2}{6\pi\epsilon_0 c^3} \frac{1}{r^2} \left( \frac{Ze^2}{4\pi\epsilon_0 m r} \right)^2 = \frac{Ze^2}{8\pi\epsilon_0 r^2} \left( -\frac{dr}{dt} \right).$$

Algebraic manipulation gives the simple ordinary differential equation

$$\frac{dr}{dt} = -\frac{A}{r^2} \quad \text{where } A = \frac{Ze^4}{12\pi^2\epsilon_0^2 m^2 c^3}$$

which can be integrated using separation of variables to give

$$r(t) = \sqrt[3]{r_0^3 - 3At}.$$

If the electron starts out at an orbital radius of  $r_0$ , then  $r \rightarrow 0$  when  $t = t_{collapse} = r_0^3/3A$ . This “collapse time” can be written in a form that allows straightforward calculation

$$t_{collapse} = \frac{r_0^3}{4cZ} \left( \frac{4\pi\epsilon_0}{e^2} \right)^2 (mc^2)^2 = \frac{1}{Z} \left( \frac{r_0}{1 \text{ fm}} \right)^3 1.05 \times 10^{-25} \text{ s},$$

where I’ve used the fact that  $e^2/4\pi\epsilon_0 = 1.44 \text{ MeV fm}$ , and  $mc^2 = 0.511 \text{ MeV}$ . For an electron in the ground state of the hydrogen atom,  $r_0 = 0.529 \text{ \AA}$  and  $Z = 1$ , the collapse time is  $1.55 \times 10^{-11} \text{ s}$ . Hydrogen atoms would not last long.

**74.** (a) Making the substitution of gravitational constants for electric constants results in a power formula

$$P_G = \frac{2}{3c^3} Gm^2 \frac{v^4}{r^2}.$$

(b) For the Earth,  $m = 6 \times 10^{24} \text{ kg}$  and  $v \approx 30 \text{ km/s}$ . Plugging these in, I get  $P_G = 2.3 \times 10^{12} \text{ W}$ . While this may seem like a large value, the Earth has a lot of energy, and if you calculate the collapse time as in the previous problem, you get  $1.4 \times 10^{29} \text{ s}$  which is  $4.6 \times 10^{21} \text{ years}$ , much longer than the age of the universe.

Actually, the Larmor formula is due to “dipole” radiation, which exists because there are two kinds of electric charge, and the electric dipole moment of the hydrogen atom oscillates as the electron orbits the proton. However, there is only *one* kind of mass, and for nonrelativistic speeds the gravitational waves are “quadrupole” radiation, which is due to the fact that the moment of inertia of the Sun-Earth system is oscillating as the Earth

orbits the Sun. The correct formula is smaller than the Larmor formula by a factor that is on the order of  $\sim v^2/c^2$ , and can be written<sup>22</sup>

$$P = \frac{32Gm^2\omega^6r^4}{5c^5}.$$

This calculation gives 200 W for the power radiated by the Earth, and a correspondingly longer collapse time.

**75.** There are several things that you can do with Angstrom's measurements. First, as he did, you can calculate an average value for  $R$ , the Rydberg constant. If you calculate a different value for each line, then the average and standard deviation are  $R = 1.09722(2) \times 10^7 \text{ m}^{-1}$ . (a) If you assume, as Stoney did, that the  $H_\alpha$  line has a frequency that is the 20th harmonic of the "fundamental" frequency, then you obtain for that frequency  $\nu_1 = 22,842,722,451,654 \text{ Hz}$ . Doing the same for  $H_\beta$  and  $H_\delta$ , taking the average and the standard deviation, I get  $\nu_1 = 2.28431(3) \times 10^{13} \text{ Hz}$ , which translates to an electromagnetic wavelength of  $13.1 \mu\text{m}$ . This is in the far infrared part of the spectrum. Of course, radio waves hadn't been utilized for commercial purposes in 1871, but one question that might be asked is, what is the significance of this fundamental frequency? What does it tell us about the atomic structure? (b) The  $H_\gamma$  line does not fit into this scheme, because it is 30.24 times the fundamental frequency. So it's not an exact harmonic.

**76.** The orbital speed of an electron in state  $n$  is

$$v_n = \frac{1}{n} \frac{Ze^2}{2\epsilon_0 h} = \frac{Z}{n} \left( \frac{e^2}{4\pi\epsilon_0} \right) \left( \frac{1}{\hbar c} \right) c = \frac{Z}{n} \alpha c,$$

The terms in parentheses are commonly encountered products. The first you saw in Problem 34 and is  $e^2/4\pi\epsilon_0 \approx 1.44 \text{ MeV fm}$ , and the second is listed on page 9 and is  $\hbar c \approx 197 \text{ MeV fm}$ . The ratio of these is just the fine structure constant  $\alpha \approx 1/137$ .

**77.** This analysis shows that a similar structure exists for both Newtonian gravity and electrostatics, since they are both represented by inverse square force laws.

(a) The radial component of Newton's 2nd Law for the particle in orbit says  $F_r = ma_r$ , which is

$$\frac{k}{r^2} = m \frac{v^r}{r},$$

or

$$\frac{k}{r} = mv^2.$$

This is a condition between  $v$  and  $r$  for circular orbits. The kinetic energy is, of course,  $K = mv^2/2$ , and the potential energy  $U$ , obtained from  $F_r = -dU/dr$ , is  $U = -k/r$ . The orbital condition above can thus be expressed as

$$-U = 2K.$$

This is a simplified version of the "virial theorem."

<sup>22</sup>Weinberg, *Gravitation and Cosmology*, page 272.

(b) The total energy is  $E = K + U$ , so that using the virial theorem to eliminate  $K$  gives

$$E = K + U = \left(-\frac{U}{2}\right) + U = +\frac{U}{2}.$$

Similarly, eliminating  $U$  results in  $E = -K$ .

**78.** (a) The orbital frequency of an object in a circular orbit is just  $\nu = v/2\pi r$ . In state  $n$ , we have obtained the speed  $v_n$  and the orbital radius  $r_n$  from Bohr's model, so we can simply plug those in. I get for this classical frequency

$$\nu_C = \frac{Z^2}{n^3} \frac{m_e e^4}{4\epsilon_0^2 h^3} = \frac{Z^2}{n^3} \frac{\alpha^2 m_e c^2}{h} = Z^2 \frac{\alpha^2 m_e c^2}{2h} \left(\frac{2}{n^3}\right),$$

where I've written the formula in three ways. First, in terms of fundamental constants; second, combining those constants into the fine structure constant  $\alpha$ ; and third, a form convenient for comparison in part (c). The numerical value of the coefficient of  $Z^2/n^3$  is  $6.6 \times 10^{15}$  Hz. So in the ground state of hydrogen ( $Z^2/n^3 = 1$ ) the electron "should" classically radiate light with a wavelength 45.6 nm, which is in the extreme ultraviolet.

(b) The energy of state  $n$  can be written

$$E_n = -\frac{Z^2}{n^2} \frac{\alpha^2 c^2 m_e}{2}.$$

The frequency of light emitted due to a transition is given by  $E_n - E_{n-1} = h\nu$  so that the Bohr frequency is

$$\nu_B = Z^2 \frac{\alpha^2 c^2 m_e}{2h} \left(\frac{1}{(n-1)^2} - \frac{1}{n^2}\right).$$

(c) Rather than comparing the frequencies,  $\nu_C$  and  $\nu_B$ , we only really need to compare the terms in parentheses, as the rest of the factors are identical. In the following table I list  $n$ , the term in parentheses from  $\nu_C$  divided by 2, that is  $1/n^3$ , the term in parentheses from  $\nu_B$  divided by 2, and the fractional error between the two,  $(\nu_C - \nu_B)/\nu_B$ :

$n$	classical	Bohr	fractional error
$10^1$	$10^{-3}$	$1.173 \times 10^{-3}$	$-1.474 \times 10^{-1}$
$10^2$	$10^{-6}$	$1.015 \times 10^{-6}$	$-1.497 \times 10^{-2}$
$10^3$	$10^{-9}$	$1.002 \times 10^{-9}$	$-1.500 \times 10^{-3}$
$10^4$	$10^{-12}$	$1.000 \times 10^{-12}$	$-1.500 \times 10^{-4}$

**79.** This is a straightforward continuation from the last problem. Expanding one of the terms in the expression for  $\nu_B$  for large  $n$ , I get

$$\frac{1}{(n-1)^2} = (n-1)^{-2} = \frac{1}{n^2} \left(1 - \frac{1}{n}\right)^{-2} \approx \frac{1}{n^2} \left(1 + \frac{2}{n}\right).$$

From this is subtracted  $1/n^2$  so that the factor in parentheses becomes  $(2/n^3)$ , which exactly matches the expression for  $\nu_C$ . Conclusion: in the limit of large  $n$ , the classical and quantum predictions for the frequency emitted by the hydrogen atom are identical. This is Niels Bohr's "Correspondence Principle."

**80.** The equation for a stable circular orbit in a gravitational system (i.e., Kepler's third law) is  $v^2 r = GM$ , where  $M$  is the mass of the central object. When the electric force replaces the gravitational force,  $G$  is replaced by  $1/4\pi\epsilon_0$ , and  $M$  is replaced by  $Ze$ . In addition, however, the mass of the secondary object does not cancel in Newton's second law, and an additional factor of  $e/m$  is needed, where  $e$  and  $m$  are the charge and mass of the secondary object. Hence, in the Bohr model results for  $E_n$ ,  $r_n$ , and  $v_n$ , the following replacement is needed to describe a gravitating system

$$\frac{Ze^2}{4\pi\epsilon_0 m_e} \rightarrow GM.$$

For the energy and radius, I get

$$E_n = -\frac{G^2 M^2 m^3}{2\hbar^2 n^2}, \quad r_n = \frac{\hbar^2}{GMm^2} n^2.$$

For the Earth-Sun system,  $M = M_\odot = 2 \times 10^{30}$  kg and  $m = M_\oplus = 6 \times 10^{24}$  kg. I get that the "Bohr radius" is  $r_1 \equiv a_0 \approx 10^{-137}$  m, which is extremely small. Since the Earth-Sun distance is  $1.5 \times 10^{11}$  m, this means that the Earth is in the  $n \approx 10^{74}$  quantum state. Yes, Bohr's Correspondence Principle definitely applies!

**81.** Bohr's model assumes that the nucleus is infinitely massive, and does not move as the electron orbits. However, we know from gravitation theory that both the satellite and the central object orbit their common center of mass. In this case, the "two-body problem" can be reduced to a "one-body problem" by a transformation of coordinates that introduces the reduced mass of the system  $\mu$ . (See Appendix F.) For a large nuclear mass  $M$  and a small electron mass  $m_e$  the reduced mass can be written

$$\frac{1}{\mu} = \frac{1}{m_e} + \frac{1}{M} = \frac{1}{m_e} \left(1 + \frac{m_e}{M}\right).$$

Bohr, of course, retained only the first term, but for this problem you need to calculate it exactly. (a) For hydrogen, the proton-electron mass ratio is 1836.152 673, and the deuteron-electron mass ratio is 3670.482 958 (both from the CODATA sheet), which means that for  $^1\text{H}$ ,  $m_e/M = 5.446 \times 10^{-4}$ , and for deuterium,  $m_e/M = 2.724 \times 10^{-4}$ . Therefore, the  $H_\alpha$  line has a wavelength of

$$\lambda = \frac{36}{5R_\infty} \frac{m_e}{\mu} = \frac{36}{5R_\infty} \left(1 + \frac{m_e}{M}\right) = 656.112 \text{ nm} \left(1 + \frac{m_e}{M}\right).$$

In the infinite nuclear mass limit (this is the meaning of the subscript on  $R_\infty$ ), the wavelength should be 656.112 nm. However, with the correction for a proton nucleus, I get 656.470 nm, and for a deuteron nucleus I get 656.291 nm. This is a difference of 0.179 nm, which agrees with Urey's result, and Ångstrom should have been able to detect it if he had been looking for it. However, the abundance of deuterium is very low (only 0.0115% on Earth), so in a typical sample of hydrogen the line strength would be very weak and possibly not visible with his apparatus. (b) For tritium, use the fact that  $M \approx 3m_p$ .

**82.** Examining a periodic table that lists the electronic configurations of each neutral atom, you can see that the subshells are not filled in the order expected, but that some  $s$

subshells ( $\ell = 0$ ) are of lower energy than the previous  $p$  or  $d$  subshells. The following table lists the approximate order of filling. In some cases, there are exceptions. For example, chromium (Cr,  $Z = 24$ ) *should* have its valence electrons in the subshells  $4s^23d^4$ , following vanadium perfectly. However, one of the  $4s$  electrons fills the  $3d$  subshell instead, and it is not until manganese that the “proper” order is restored. The table below ignores these slight discrepancies.

$n$	$\ell$	subshell	# $e^-$	noble gas	$Z$
1	0	$1s$	2	He	2
2	0	$2s$	2		
2	1	$2p$	6	Ne	10
3	0	$3s$	2		
3	1	$3p$	6	Ar	18
4	0	$4s$	2		
3	2	$3d$	10		
4	1	$4p$	6	Kr	36
5	0	$5s$	2		
4	2	$4d$	10		
5	1	$5p$	6	Xe	54
6	0	$6s$	2		
4	3	$4f$	14	lanthanides	
5	2	$5d$	10		
6	1	$6p$	6	Rn	86
7	0	$7s$	2		
5	3	$5f$	14	actinides	
6(?)	2	$6d(?)$	10		
7(?)	1	$7p(?)$	6	Uuo	118

Note that each noble gas has a filled  $p$  subshell as its last filled subshell.

**83.** Gauss’s law,  $\epsilon_0 \int \vec{E} \cdot d\vec{A} = \int \rho dV$ , under the usual assumption of spherical symmetry, results in

$$\begin{aligned} E_r(r) &= \frac{1}{4\pi\epsilon_0 r^2} \int_0^r \rho(r') 4\pi r'^2 dr' \\ &= \frac{-eZ^3}{\pi\epsilon_0 r^2 a_0^3} \int_0^r r'^2 e^{-2Zr'/a_0} dr'. \end{aligned}$$

Changing the integration variable from  $r'$  to  $s' = 2Zr'/a_0$  results in the integral

$$E_r = \frac{-e}{8\pi\epsilon_0 r^2} \int_0^s s'^2 e^{-s'} ds'$$

where  $s = 2Zr/a_0$ . This integral can be done “by parts” to give

$$\int_0^s s'^2 e^{-s'} ds' = -(s'^2 + 2s' + 2)e^{-s'} \Big|_0^s = 2 - (s^2 + 2s + 2)e^{-s}.$$

The radial electric can now be written in its full glory, replacing the physical variable  $r$

$$E_r(r) = \frac{-e}{4\pi\epsilon_0 r^2} + \frac{e}{4\pi\epsilon_0 r^2} \left\{ \frac{2Z^2 r^2}{a_0^2} + \frac{2Zr}{a_0} + 1 \right\} e^{-2Zr/a_0}.$$

The first term is the field of a point electron at the origin, and the second gives the reduction of that field due to the spread out “cloud” of charge. As  $r \rightarrow \infty$  the second term vanishes, leaving the field of a point charge. This result nicely illustrates Newton’s shell theorems, which state that when you are outside of a spherical mass or charge distribution, it appears as if all the mass or charge is concentrated at the origin. Clearly, this electric field is not the Coulomb electric field, so that the quantum energy levels of the second electron will not depend on  $n$  alone, but also on  $\ell$ .

EXTRA: Quantum mechanics expresses not the forces, but the potential energy. So for this case, the electric potential can be obtained from the integration  $V = \int \vec{E} \cdot d\vec{\ell}$ , and the simplest method is to integrate radially from  $\infty$  to  $r$ . To evaluate this, we’ll need the following integrals

$$\begin{aligned} \int_{\infty}^r \frac{dr}{r^2} &= -\frac{1}{r} \\ \int_{\infty}^r e^{-\beta r} dr &= -\frac{e^{-\beta r}}{\beta} \\ \int_{\infty}^r \frac{e^{-\beta r}}{r} dr &= -E_1(\beta r) \\ \int_{\infty}^r \frac{e^{-\beta r}}{r^2} dr &= \text{can you evaluate this?} \end{aligned}$$

where  $E_1$  is the exponential integral, defined by

$$E_1(z) \equiv \int_z^{\infty} \frac{e^{-t}}{t} dt,$$

which is a function that cannot be expressed in terms of other elementary functions, but whose properties have been tabulated.<sup>23</sup> In these formulas,  $\beta = 2Z/a_0$ . For those of you who are mathaholics, the exponential integral is related to the incomplete Gamma function  $\Gamma$ ,

$$E_1(z) = \Gamma(0, z).$$

The incomplete Gamma function is defined as

$$\Gamma(a, z) \equiv \Gamma(a) - \int_0^z e^{-t} t^{a-1} dt,$$

and the reason it is called “incomplete” is because the gamma function is the full integral

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt.$$

The Gamma function is related to factorials

$$\Gamma(n+1) = n!,$$

where  $n$  is an integer.<sup>24</sup>

**84.** Using standard linear regression formulas for  $\sqrt{\nu}$  versus  $Z$ , I obtain a slope of  $4.9912 \times 10^7 \sqrt{\text{Hz}}$ , which is 0.48% larger than the current theoretical value. Flipping the axes and applying linear regression to a plot of  $Z$  versus  $\sqrt{\nu}$  (because we want the “ $y$ ” intercept), I get a  $Z$  intercept of 1.13, very close to unity.

<sup>23</sup>See, e.g., <http://mathworld.wolfram.com/ExponentialIntegral.html>.

<sup>24</sup>See, e.g., <http://mathworld.wolfram.com/GammaFunction.html>

# Interlude

Now that we have obtained a brief overview of the physics that the twentieth century has to offer, it's time to pause a moment and reflect on the different epistemology that we have encountered. What kinds of things can we say “for certain,” and what questions are not “askable”? Since all of our intuition has been built in a macroscopic world of balls and blocks and inclined planes, what language must we use when we talk about the atomic and subatomic worlds? How do we interpret the equations that govern this world? Does the interpretation matter?

Two quotes capture much of the new way of thinking. The first is the quote on page 10 by Linus Pauling on the wave-particle duality. I recommend that you re-read it at this time. In fact, a re-reading of all of Chapter 1 is a good idea. Another quote, this one concerning the dynamics of quantum phenomena, is by Abraham Pais, and lays out how we must think about processes such as photon emission and absorption, nuclear decay, etc.

*At a moment which cannot be predicted an excited atom makes a transition to its ground state by emitting a photon. Where was the photon before that time? It was not anywhere; it was created in the act of transition.*

*At a moment which cannot be predicted a beta-radioactive nucleus decays into another nucleus, an electron, and a neutrino. Where were the electron and neutrino before that time? They were not anywhere; they were created in the act of beta-disintegration.*

*An atom absorbs a photon and goes into an excited state. Where is the photon after the absorption? It is not anywhere; it is extinct, annihilated.*

*Is there a theoretical framework for describing how particles are made and how they vanish? There is: quantum field theory. It is a language, a technique, for calculating the probabilities of creation, annihilation, scatterings of all sorts of particles: photons, electrons, positrons, protons, mesons, others, by methods which to date invariably have the character of successive approximations. No rigorous expressions for the probability of any of the above-mentioned processes has ever been obtained.<sup>25</sup>*

Pais describes how creation and annihilation occur — it is nothing but the philosophy of the “exchange particle” view that I described in Section 2.2. That is, there are no “forces”

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<sup>25</sup>Pais, *Inward Bound*, page 324-5.

or “fields,” but particles interact locally, and sometimes that interaction involves changes in momentum and energy, and sometimes creation and annihilation.

Actually, and this is now a preview of Chapters 6 and 7, in this third formulation, we *do* talk about fields (this is why it is called quantum field theory), but in a different manner. That is, it is not that particles *have* fields (such as the electric field of an electron), but that particles *are* fields. Now, though, a field is a function that describes the probability of finding the particle at a particular location — this is the wave function of quantum mechanics that we will consider in Chapter 7. This field formulation is what allows us to calculate the probabilities that Pais was talking about.

If the best we can do is to describe particles using this probabilistic formulation it begs the question: where was the particle *before* we measured its location? We will discuss the philosophy of quantum mechanics after you have learned the mathematical formalism, but for now there are two quotes by Niels Bohr and one by Asher Peres that succinctly state some of the concepts that we will wrestle with in Chapters 6 and 7 concerning what it means to make a measurement, and what happens in between measurements.

“No phenomenon is a phenomenon until it is an observed phenomenon.”

“Nothing exists until it is measured.”

“Unperformed experiments have no results . . . We then discuss . . . a comparison of the results of experiments which were actually performed, with those of hypothetical experiments which could have been performed but were not. It is shown that it is *impossible to imagine* the latter results in a way compatible with the results of actually performed experiments . . . and quantum mechanics.” — Asher Peres

*Portrait of the theoretical physicist as a young boy (by Leon Lederman):*

Mother: “Johnny, what are you doing?”

Johnny: “I’m drawing a picture of God.”

Mother: “Don’t be silly. Nobody knows what God looks like.”

Johnny: “They will when I’m finished.”



# Chapter 5

## Introduction to Special Relativity

Einstein's special theory of relativity is a description of kinematics and dynamics in four-dimensional "spacetime," and in particular, how to describe the motion of objects from different points of view, that is, by observers in different *reference frames*. It is essentially a reformulation of Newtonian mechanics, correct for all particle velocities. In fact, Newtonian mechanics turns out to be a special, limiting case, valid when velocities are small compared with the speed of light  $c$ .

Since its development by Einstein over a century ago, a virtual cottage industry has existed, consisting in the delineation and resolution of a multitude of counterintuitive paradoxes. Even though everyday Newtonian physics is contained within special relativity, the consequences of the principle of the constancy of the speed of light introduces profound effects that we do not experience in everyday life.

Common paradoxes, such as the "twin paradox," where a person's rate of aging depends on their speed of travel, and the "pole and barn paradox," where the observations of a pole fitting inside a barn depends on the observer's frame of reference, will be covered in this chapter. More exotic effects, such as the appearance of a cube that is moving near the speed of light, are beyond the scope of this book.

### Einstein's own words

Einstein was led to his theory of relativity by the observation of certain electromagnetic effects. Specifically, he asked himself the following question, "How does a magnet induce current in a nearby conductor?" The induced current depends only on the *relative* motion of the magnet and conductor, but the physical explanation — in terms of a magnetic field or an electric field — depends on which object is "actually" moving. This apparent contradiction led Einstein to abandon Newton's concepts of "absolute time" and "absolute space," and to replace the Galilean transformation between two reference frames, moving with respect to each other, with the Lorentz transformation.

As with any other piece of fundamental physics, it is always useful to go first to the primary source; in this case, here are the first two paragraphs of Einstein's seminal paper on this subject in 1905, "On the electrodynamics of moving bodies."

It is known that Maxwell's electrodynamics — as usually understood at the present time — when applied to moving bodies, leads to asymmetries which

do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise — assuming equality of relative motion in the two cases discussed — to electric currents of the same path and intensity as those produced by the electric forces in the former case.

Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the “light medium,” suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that, as has already been shown to the first order of small quantities, the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We will raise this conjecture (the purport of which will hereafter be called the “Principle of Relativity”) to the status of a postulate, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity  $c$  which is independent of the state of motion of the emitting body. These two postulates suffice for the attainment of a simple and consistent theory of the electrodynamics of moving bodies based on Maxwell’s theory for stationary bodies. The introduction of a “luminiferous ether” will prove to be superfluous inasmuch as the view here to be developed will not require an “absolutely stationary space” provided with special properties, nor assign a velocity-vector to a point of the empty space in which electromagnetic processes take place.<sup>1</sup>

The “unsuccessful attempts” were, of course, the experiments by Michelson and Morley. Albert A. Michelson [Nobel Prize, Physics, 1907] performed his first experiment measuring the speed of light in different directions in 1881 while working in Helmholtz’s laboratory in Berlin, and of course got a null result. That is, the velocity of the Earth as it orbited the Sun did not affect the speed of light as it moved through the ether. Then, in 1887, when he was at the Case School of Applied Science in Cleveland, he joined forces with Edward Morley, a chemist from Western Reserve College, and they again got a null result. The experiment has been performed many times since then, always with the



<sup>1</sup>From a translation by W. Perrett and G.B. Jeffery, *The Principle of Relativity*, 1923.

same result. Although Einstein alludes to these experiments, they were not the main motivation for his development of special relativity. As he states, the motivation was the asymmetry in Maxwell's equations, along with the fact that Lorentz had already shown that transformations of Maxwell's equations were inconsistent with Newton's Laws and Galilean Relativity (see Sec. 5.3.2).

The standard way to develop the principles of special relativity is to follow Einstein. This method starts with Einstein's two postulates mentioned above, and derives time dilation and length contraction using a light ray as a clock. Here, I want to take a different approach, one that is due to Peter Scott and Bill Burke.<sup>2</sup> It starts with an experimental fact and deduces what theoretical conclusions are needed to explain the observation. The experimental fact that we will use is the observation of muons from cosmic rays. Appendix G describes the process of muon creation and observation, and constitutes good background reading at this time.

Before the muon observations can be analyzed, however, I need to introduce a graphical device known as a "spacetime diagram," which is essential to the understanding of special relativity.

## Spacetime diagrams

An extremely useful way to conceptually understand (as well as make quantitative calculations) special relativity is the *spacetime diagram*. A spacetime diagram is nothing more than a one-dimensional position-time graph, familiar from elementary mechanics, with two simple changes: (i) an interchange of the  $x$  and  $t$  axes, and (ii) a rescaling of the spatial axis so that it has dimensions of time. That is, rather than  $x$  versus  $t$ , a spacetime diagram is a plot of  $t$  versus  $x/c$ . The trajectory of a particle is called its "world line." A particle with constant speed  $v$  therefore has a world line that makes an angle  $\theta$  with the positive  $t$  axis that is given by  $v = c \tan \theta$ . Velocities are also scaled by the speed of light,  $c$ , so that it is common to refer to  $\beta \equiv v/c$ , instead of  $v$ . Importantly, a photon (that is, light) has a world line that makes a  $45^\circ$  angle with the positive  $t$  axis. In other words, it has a slope of unity. Figure 5.1 depicts these features.

Why do I scale distances so that they have dimensions of time? It turns out that in relativity, space and time are treated on equal footing, so the fabric in which events occur is called *spacetime*. Because they have equal status, it makes sense to measure them in the same units. In a similar vein, it would be strange to measure north-south distances in meters and east-west distances in feet. From this viewpoint, the speed of light is just a conversion factor between our usual distance units and our usual time units. For example, a "light-year" is a unit of length equal to the distance that light travels in one year. Hence, the speed of light can be written as

$$c = 3 \times 10^8 \text{ m/s} = 1 \frac{\text{light-year}}{\text{year}}. \quad (5.1)$$

Therefore, if  $x = 5$  light-years, say, then

$$\frac{x}{c} = \frac{5 \text{ light-years}}{1 \text{ light-year/year}} = 5 \text{ years}. \quad (5.2)$$

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<sup>2</sup>Scott and Burke, *Special Relativity Primer*.

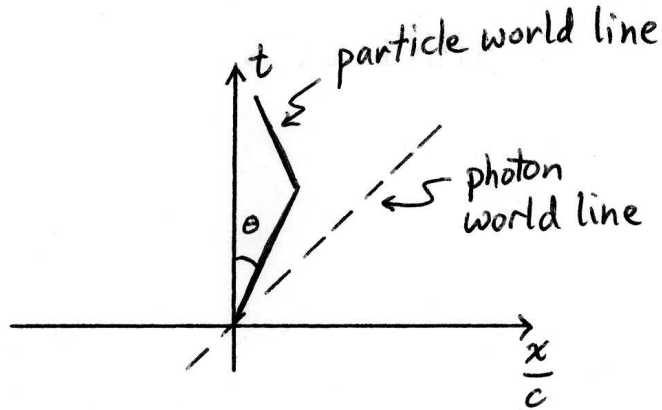


Figure 5.1: Spacetime diagram depicting the world line of a photon (dashed line) and the world line of a massive particle (solid line). The particle is shown starting at the origin, moving to the right and then moving back to the left. The speed of any particle can be determined by the angle its world line makes with a vertical line parallel to the  $t$  axis. That is,  $v = c \tan \theta$ .

Why do I scale velocities to the speed of light? Since it turns out that  $c$  is a cosmic speed limit, it makes sense to compare all speeds to  $c$ . More fundamentally, however, we'll see that it is impossible to determine how fast you are traveling in an absolute sense. You can only tell how fast you are traveling with respect to another object. In addition, all observers, regardless of their motion, measure light to travel at the same speed,  $c$ . For these reasons, the speed of light is the natural speed in the universe. If you tell someone how fast you are moving, and they reply with the query, "Compared to what?", the speed of light is the only possible comparison.

Both of these rescalings compel some physicists to set  $c \equiv 1$  so that it does not appear in any equations. However, in this chapter I will leave it explicitly in the equations, but it is important to remember that it is simply a conversion factor.

It is possible to portray two spatial dimensions (and time, of course) using a perspective portrayal of three dimensions on two-dimensional paper, as shown in Fig. 5.2. All the essential physics, however, can be shown in a two-dimensional spacetime diagram with just one spatial dimension.

## 5.1 Time dilation

An investigation into the phenomenon of time dilation cuts to the heart of what special relativity is, and how it differs from classical (i.e., Galilean) relativity. The first concept to be very clear about is that of a **CLOCK**. We will need to assume that our clocks tick *uniformly* and *homogeneously*. What this means is that their ticking rate does not depend on time nor on the clock's location. Of course, since a clock is how we actually measure time, we cannot tell if it were not uniform, so we just have to assume it.

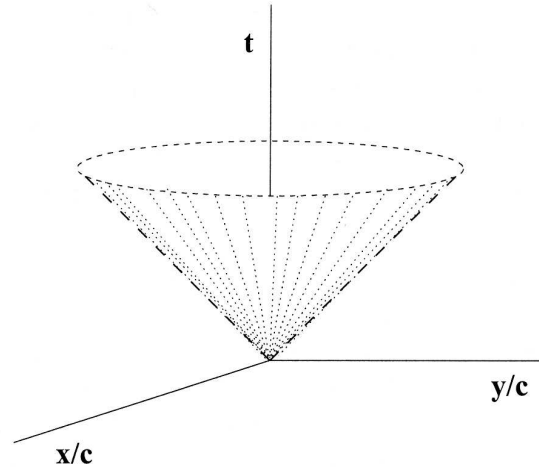


Figure 5.2: Spacetime diagram with two spatial dimensions,  $x$  and  $y$ . Since light travels at speed  $c$ , the set of all possible photon world lines that pass through the origin forms a cone with its vertex at the origin. This is called the “light cone.” All massive particles must have world lines that are “within” this cone, or closer to the  $t$  axis. From Scott and Burke, *Special Relativity Primer*, Figure 4.

For our first thought experiment (*Gedanken* experiment), a muon will act as our clock.<sup>3</sup> The average lifetime of a muon,<sup>4</sup> as measured by the muon itself, is  $t_0 = 2.2 \mu\text{s}$ . Even though this is just an average, because some muons exist for longer time intervals and some shorter, it will be convenient to assume that they all live for exactly  $2.2 \mu\text{s}$  before decaying into an electron. Our thought experiment is given by the following scenario.

Imagine that we have a number of identical clocks (muons). They are all created simultaneously at the origin, and subsequently they move away from the origin (in both the positive and negative  $x$  directions) at different constant speeds. (One spatial dimension is all that is needed for this experiment.) Let’s now follow one of the muons. If it moves at speed  $v$ , it will have traveled a distance  $x = vt$  by the time it decays at time  $t$ , as measured by another clock that remains at the origin. How is  $t$  (the spacetime coordinate) related to  $t_0$  (the muon’s lifetime)?

“What?” you say. “Isn’t it obvious? They are equal,  $t = t_0$ .” This is exactly what Newton and Galileo would say. But they are wrong! Experiments show that

$$t_0^2 = t^2 - \frac{x^2}{c^2}. \quad (5.3)$$

A plot of the spacetime locations  $(x/c, t)$  of the decays of each of the muons results in a hyperbola, shown in Fig. 5.3. It seems rather strange that a moving clock, as represented

<sup>3</sup>Refer to Chap. 2 for a discussion of muons and to App. G for a description of their production in cosmic rays.

<sup>4</sup>The precise value is  $2.197\,019(21) \mu\text{s}$ , but in this chapter, we only need two significant digits.

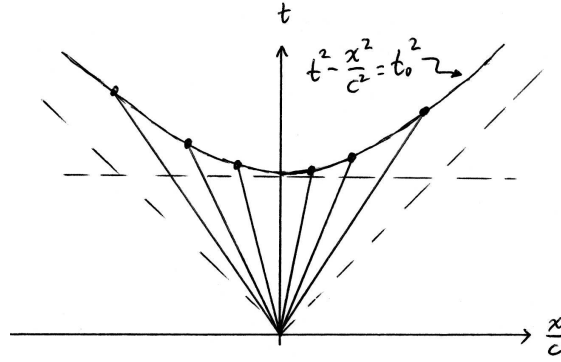


Figure 5.3: The locus of points in spacetime at which muons decay after moving a distance  $x/c$  at speed  $v$ . The horizontal dashed line is the Newtonian prediction  $t = t_0$ , and is a good approximation when  $v \ll c$ . The diagonal dashed lines are, of course, the world lines of photons, i.e., the light cone.

here by a muon, and a stationary clock would measure different time intervals between the same two events (in this case the creation and decay of the muon), but as Scott and Burke point out, “The hyperbola represents a description of the experimental data for real clocks, and is not subject to dispute.”<sup>5</sup> This means that our task as physicists is not to mold this result into our pre-existing notions of space and time, but to *explain* this observation. In this case it requires a completely new and different conception of spacetime.

## Muon experiments

As explained in App. G, particles from outside the Earth’s atmosphere (called cosmic rays) collide with the molecules in the atmosphere and create many secondary particles, some of which are muons. The altitude at which most of the muons are created in this process is near 20 km, and these muons are subsequently observed on the ground. Let’s assume that the muons are of very high energy, and that they travel at almost the speed of light. If this is true, then the maximum distance they could travel before decaying is

$$d = v t_0 \approx c t_0 = (3 \times 10^8 \text{ m/s})(2.2 \mu\text{s}) = 660 \text{ m}, \quad (5.4)$$

which means that they would not reach the ground! They would decay, on average, far above the ground.

However, since they *are* observed reaching the ground, and since they do *not* travel faster than the speed of light  $c$ , we can determine what their average lifetime must be — as measured by someone on the ground — in order to explain the observations. A clock on the ground measures their lifetime to be

$$t \approx \frac{d}{c} = \frac{20 \text{ km}}{c} \approx 6.6 \times 10^{-5} \text{ s} = 30 t_0. \quad (5.5)$$

<sup>5</sup>Scott and Burke, *Special Relativity Primer*.

This means that cosmic ray muons that are moving close to the speed of light — relative to the ground — survive about 30 times longer than stationary muons in the laboratory. This is what is meant by *time dilation*.

## Proper time and the spacetime interval

Can we draw a general conclusion from this result? Yes. First, it is important to distinguish the two times that we have been discussing. The first,  $t$ , is called the *coordinate time*. It is the time measured by a clock tied to the coordinate system  $(x/c, t)$  in which we measure the location of events. The second,  $t_0$ , is called the *proper time*.

The *proper time* is the elapsed time between two events as measured by a clock that is at the *same* spacetime location as both events.

That is, a clock always measures its own proper time, because it is always at the origin of its own frame of reference, the origin of its own coordinate system.

You may have noticed that in special relativity an “event” is elevated to a special status. An event occurs at a specific location and specific time, i.e., at a specific point in spacetime. The proper time interval between two events is given by the “spacetime interval.” For example, if a firecracker’s fuse is lit at  $(x_1/c, t_1)$  and the firecracker explodes at  $(x_2/c, t_2)$ , then a clock attached to the firecracker will record a proper time interval  $\Delta t_0$  between the two events given by

$$(\Delta t_0)^2 = (t_2 - t_1)^2 - \frac{(x_2 - x_1)^2}{c^2} = (\Delta t)^2 - \frac{(\Delta x)^2}{c^2}. \quad (5.6)$$

This quantity,  $(\Delta t)^2 - (\Delta x)^2/c^2$ , is called the *spacetime interval*.<sup>6</sup> A clock tied to the coordinate system will record a time interval  $\Delta t$  (called coordinate time) and a space interval  $\Delta x$  between the two events. This definition is consistent with the muon experiment above. A clock tied to the ground measures a coordinate time interval of  $6.6 \times 10^{-5}$  s between muon creation and decay as well as a spatial distance of 20 km between the two events. A clock attached to the muon, however, recorded a proper time interval of  $2.2 \mu\text{s}$  with no spatial displacement.

## The relativistic factor $\gamma$

More information can be gleaned from Eq. (5.6) by noting that, if the muon is traveling at a constant speed, which we are assuming, the coordinate distance it travels  $\Delta x$  is simply its speed  $v$  times the coordinate time interval  $\Delta t$ . Replacing  $\Delta x$  by  $v\Delta t$  in Eq. (5.6) and rearranging to solve for  $\Delta t$  gives<sup>7</sup>

$$\Delta t = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \Delta t_0 = \frac{1}{\sqrt{1 - \beta^2}} \Delta t_0 \equiv \gamma \Delta t_0, \quad (5.7)$$

<sup>6</sup>It is also common to denote  $(\Delta s)^2 = (\Delta x)^2 - c^2(\Delta t)^2$  as the spacetime interval. This differs from our definition by a factor  $-c^2$ , and thus has the same invariant character.

<sup>7</sup>This is the correct application of the constant velocity relation that we misapplied in Eq. (5.4).

where  $\gamma$  is the relativistic factor introduced back in Chapter 1. This equation states the general conclusion that stationary clocks observe moving particles to survive longer than expected, which means that stationary clocks observe the moving clocks to “run slow.” The moving clocks thus experience a *time dilation*.

With this result, and our determination from Eq. (5.5), we can deduce how fast the muon was actually moving (relative to the ground). Since the muon’s lifetime must increase by a factor of 30 in order to be observed at the ground, Eq. (5.7) states that this factor of 30 is nothing more than the relativistic factor  $\gamma$ . Setting  $\gamma = 30$  and solving for  $\beta$  gives

$$\beta^2 = 1 - \frac{1}{\gamma^2} = 1 - \frac{1}{900}, \quad \text{or} \quad \beta \approx 0.9994. \quad (5.8)$$

Our typical cosmic ray muon therefore travels at 99.94% of the speed of light.

## Inertial reference frames and the invariance of the interval

The spacetime interval between two events, as defined by Eq. (5.6), is a quantity that is invariant. That is, all observers measure the same value for  $\Delta t_0$ , regardless of the fact that they may measure different values for  $\Delta t$  and  $\Delta x$ . We have been talking about stationary reference frames (e.g., the frame tied to the Earth) and moving reference frames (e.g., the frame tied to the muon), but there really is no way to tell who is moving and who is not. We could just as correctly take the muon as stationary and the Earth as moving. On the other hand, it *is* possible to determine if a reference frame is accelerating, and this brings us to another definition:

An *inertial reference frame* is one in which Newton’s first law holds true.

That is, when you perform an experiment in which you release an object at rest, if it remains at rest then your reference frame is inertial. If, however, your object does not remain at rest, but accelerates, then you might conclude that there is a mysterious force acting on it (in accordance with Newton’s second law). Of course, there is no force, but rather your reference frame is accelerating, and is therefore not inertial. A frame that is moving at constant velocity is a natural choice for an inertial frame.<sup>8</sup> If we now consider two inertial reference frames that are in relative motion, and we investigate how each of them measure events, we’ll find that the spacetime locations of events (i.e., positions and times) will be different, but observers in the two frames will measure the same spacetime interval between two events. That is, if there are two events, then one observer will measure the coordinates  $(x_1/c, t_1)$  and  $(x_2/c, t_2)$  as before. But the other observer (who is in what we will call the “primed frame”) measures the coordinates  $(x'_1/c, t'_1)$  and  $(x'_2/c, t'_2)$ . Although  $x_1 \neq x'_1$  in general, and the other coordinates will also have different values in the two frames, it will always be true that

$$(\Delta t)^2 - \frac{(\Delta x)^2}{c^2} = (\Delta t')^2 - \frac{(\Delta x')^2}{c^2}. \quad (5.9)$$

---

<sup>8</sup>Free fall in a gravitational field is *also* an inertial reference frame. But including gravity leads us into the domain of general relativity, which is beyond the scope of this book.



This invariance is very similar to the measurement of the distance between two spatial locations using two different coordinate systems. While the  $x$  and  $y$  coordinates of the two points as measured in the two systems will, in general, be different, the Euclidean *distance* between the two points,  $\Delta L$ , is invariant — measurements in *any* coordinate system will give the same result

$$(\Delta L)^2 = (\Delta x)^2 + (\Delta y)^2. \quad (5.10)$$

## Minkowski Spacetime

The minus sign in Eqs. (5.6) and (5.9) has far reaching consequences. First, it indicates that instead of the Euclidean geometry of flat space, spacetime is described by a hyperbolic geometry, also known as *Minkowski spacetime*, after **Hermann Minkowski (1864-1909)**.<sup>9</sup> The fact that time was now inextricably linked to space was poetically described by Minkowski in 1908



*The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.*

Second, in the standard distance formula of Euclidean geometry, the plus sign implies that the *shortest* distance between two points is a straight line. This can be proved by integrating the differential form of Eq. (5.10)

$$dL = \sqrt{(dx)^2 + (dy)^2} \quad \Rightarrow \quad L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \quad (5.11)$$

To actually evaluate  $L$  it is necessary to know  $y(x)$ , the shape of the curve connecting the two points. A technique from advanced calculus, called “calculus of variations,” can be used to show that when  $y(x)$  is a straight line, then  $L$  is a minimum. Similar mathematical steps can be used to show that the world line that joins two spacetime events with a constant slope results in the *largest* proper time. That is, integrating the differential form of Eq. (5.6) gives

$$dt_0 = \sqrt{(dt)^2 - \left(\frac{dx^2}{c^2}\right)} \quad \Rightarrow \quad \Delta t_0 = \int_1^2 \sqrt{1 - \left(\frac{1}{c} \frac{dx}{dt}\right)^2} dt. \quad (5.12)$$

Again, in order to evaluate the proper time interval  $\Delta t_0$  that elapses on a clock traveling from event 1 to event 2, the function  $x(t)$  must be known. The two integrals in Eqs. (5.11) and (5.12) are *not* path independent. This property is investigated mathematically in Problem 92 for a simple case where  $x(t)$  consists of piecewise straight line segments. However, the fact that the largest elapsed proper time comes about when  $x(t)$  is a straight line is a general result, and is shown graphically in Fig. 5.4. The “straight line” path

<sup>9</sup>Minkowski did not win a Nobel prize, but he probably would have had there been one in mathematics.

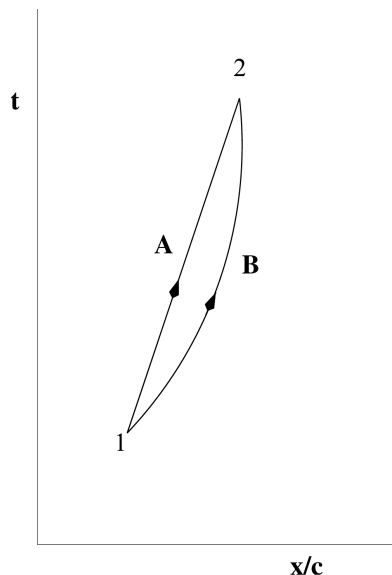


Figure 5.4: The “straight line” path (A) between events 1 and 2 results in a longer elapsed proper time than the curved path (B). That is,  $\tau_A > \tau_B$ . Any other path that appears to denote a longer “distance” between 1 and 2 will result in an even shorter elapsed proper time. From Scott and Burke, *Special Relativity Primer*.

between events 1 and 2 is indicative of a constant velocity,  $dx/dt = v$ , and results in a proper time for which Eq. (5.12) can be easily integrated to obtain  $\Delta t_0 = \Delta t/\gamma$ . Here,  $\Delta t$  is the elapsed coordinate time, and this result confirms what we have already deduced in Eq. (5.7).

## 5.2 Length contraction

An unescapable conclusion that is intimately connected to time dilation is the fact that objects are measured to be shorter when they are moving with respect to the measuring reference frame. Let’s take the example of the cosmic ray muon again. In the rest frame of the Earth, the distance between the ground and the height at which the muon is created is 20 km. However, in the reference frame of the muon, the Earth is moving while the muon is stationary. In addition, the muon lives for only  $2.2 \mu\text{s}$ , which means that even if the Earth is moving at close to the speed of light (from the muon’s point of view) only 660 m will have passed by in those  $2.2 \mu\text{s}$  — this is the same calculation as in Eq. (5.4). Therefore, the 20 km object (the atmosphere) must have *contracted* to 660 m, again from the muon’s point of view.

More rigorously, Newton would say that since the muon’s lifetime is  $\Delta t_0$  it must travel a distance  $v\Delta t_0$ . On the other hand, Einstein would say that moving clocks run slow, so that the muon travels a distance  $v\Delta t = v\gamma\Delta t_0 > v\Delta t_0$ . In other words, the distance

traveled by the muon — measured in the Earth’s frame — is<sup>10</sup>

$$\Delta x = \gamma v \Delta t_0, \quad (5.13)$$

which is 20 km. From the muon’s point of view the Earth is moving at speed  $v$ , which means that the distance the Earth has traveled is  $v\Delta t_0 = \Delta x/\gamma < \Delta x$ . This means that the muon thinks that the distance from the top of the atmosphere to the ground is only 660 m, and therefore it is possible for it to cover that distance in only 2.2  $\mu\text{s}$ .

## How to measure lengths

How do you actually measure the length of a moving object? Before we consider this problem, we need to decide how to measure the length of a *stationary* object. Consider a rod that is at rest in a particular coordinate system. You can mark the location at one end of the rod with an event in spacetime (position *and* time), and then wander over to the other end of the rod and do the same. But because the rod is at rest, there is no time dilation, which means that the spatial distance between the two events will not depend on the temporal distance between the two events. This length that you have just measured is called the *proper length*,  $L_0$ .

The *proper length* of an object is the length measured in a reference frame in which the object is at rest.

How do you measure the length of a *moving* rod? In this case, you don’t have the luxury of marking the position at one end of the rod, and then wandering over to mark the position of the other end of the rod at a later time. The rod has moved in the time between the two events, so the distance you have measured is not the length of the rod. To account for this motion, there are two methods that can be used. First, you can measure the positions of the two ends of the rod *at the same time*. This means that you need a partner to mark the location of one end while you mark the location of the other. However, the two events, while they are simultaneous in your frame, they are *not* simultaneous in the frame that is moving with the rod.<sup>11</sup> To relate your length measurement,  $L$ , to the proper length,  $L_0$ , therefore, requires a transformation from your frame to the moving frame. This transformation is called the *Lorentz transformation*, and a correct analysis of this measurement method must wait until Section 5.3.2.

The second method is to remain at one location and mark the two times at which the two ends of the moving rod pass your location, and then multiply by the relative velocity of the two frames. This method requires the use of the invariance of the spacetime interval as stated in Eq. (5.9). The two events of interest are event A, when the front end of the

<sup>10</sup>This can also be obtained directly from the invariance of the spacetime interval. Replacing  $\Delta t$  in Eq. (5.6) with  $\Delta x/v$  results in

$$(\Delta t_0)^2 = \frac{(\Delta x)^2}{v^2} \left(1 - \frac{v^2}{c^2}\right)$$

which can be rearranged to give Eq. (5.13).

<sup>11</sup>As we will see below, the simultaneity of events depends on which reference frame is making the observation.

rod passes you at  $t = t_A$ , and event B, when the back end of the rod passes you at  $t = t_B$ . The spatial distance between the two events is, of course,  $\Delta x = 0$  (they occur at the same spatial location), and the temporal distance (what you measure with your clock) is  $\Delta t = t_B - t_A$ . By definition,

$$\Delta t = \frac{L}{v}. \quad (5.14)$$

What about the same quantities as measured by an observer traveling with the rod? Of course, the spatial distance is just the proper length,  $\Delta x' = L_0$ , because that observer does not care how much time passes between the two events, and the temporal distance in that moving frame,  $\Delta t'$ , is the length of the rod (this time the proper length) divided by the relative velocity,  $\Delta t' = L_0/v$ . Inserting these values into Eq. (5.9) gives

$$\left(\frac{L}{v}\right)^2 - 0^2 = \left(\frac{L_0}{v}\right)^2 - \frac{L_0^2}{c^2}, \quad (5.15)$$

and solving for  $L$  gives

$$L^2 = L_0^2 \left(1 - \frac{v^2}{c^2}\right) \quad \text{or} \quad L = \frac{L_0}{\gamma}. \quad (5.16)$$

Since  $\gamma > 1$ , the length you measure is always *less* than the proper length. The rod's length has *contracted*. For the muon, the proper length of the distance it travels is  $L_0 = 20$  km, but its relativistic factor is  $\gamma = 30$  so that the muon perceives the distance traveled as  $L = 660$  m.

Which observer is correct? The muon or the one tied to the Earth? They both are correct! Each observer records a set of internally consistent measurements, and possessing a knowledge of only those measurements would give neither observer cause for alarm. There is absolutely nothing strange when observers look only at their own measurements. It is only when two observers, in two different inertial reference frames, compare their measurements do they realize that they have obtained different values for the temporal and spatial intervals between events. A closer look reveals, however, that they still measure the same value for the spacetime interval. To understand their results and how they differ, and to be able to predict the measurements of other observers, a knowledge of how to transform the results is needed.

### 5.3 Transformations between reference frames

A transformation between two coordinate systems, or reference frames, is simply a relation between the position and time coordinates of an event as measured by observers in the two frames using the two different coordinate systems. The form of the transformation is determined by the geometry of the space and time in which the measurements are made. The Galilean, or Newtonian, transformation assumes that both time and space are *absolute*. In addition, it is assumed that space itself is flat, or Euclidean. At speeds small compared to the speed of light, this is approximately true, and since  $c$  is so large compared with everyday motion, it is not surprising that Galileo and Newton obtained this approximation to the correct transformation, called the ‘‘Lorentz transformation.’’ Because the method

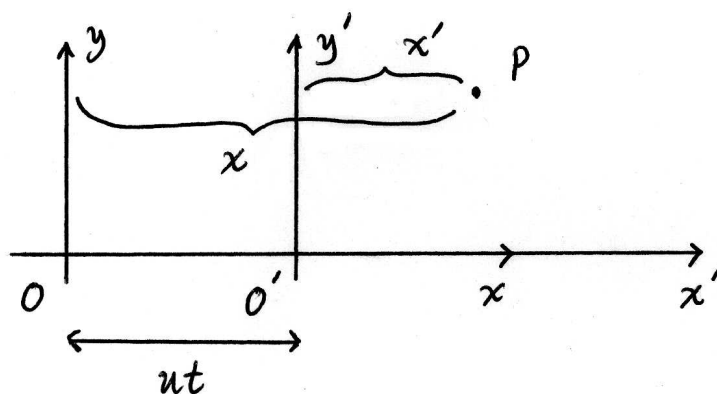


Figure 5.5: Geometry of a transformation between two inertial coordinate systems. Reference frame  $O$  is stationary, while reference frame  $O'$  moves with speed  $u$  (relative to  $O$ ) in the positive  $x$  direction. At  $t = t' = 0$  both origins coincide, the  $y$  and  $y'$  directions are the same, and the  $z$  and  $z'$  directions are the same. In the figure, the  $z$  and  $z'$  axes are suppressed for clarity.

of obtaining the transformation equations is identical for both Galileo and Lorentz, I will develop the Galilean transformation in some detail so that any assumptions we make are clearly stated. Then, when I derive the Lorentz transformation equations, some of those assumptions will have to be modified, but the method will be exactly the same.

### 5.3.1 Galilean transformation

If two different observers measure the spatial location and the time of the same event, it is useful to know how the measurements of the two observers compare. That is, how can we *transform* the quantitative results of one observer to obtain those of the second observer? Galileo and Newton asked this question, and they were able to answer it in the following manner.

Galileo developed his understanding through his study of projectiles in the uniform gravitational field near the Earth's surface. He realized that any trajectory was parabolic, and, more important, that the horizontal and vertical motions were independent. This meant that if one observer saw a parabola, another observer, moving with the same (constant) horizontal velocity of the projectile, would see the projectile simply rise and fall, as if it had initially been thrown straight upward.

Consider Fig. 5.5, where an event  $P$  takes place at a specific position and time. An observer in the unprimed frame  $O$  (sometimes called the “lab frame”) measures the event to take place at  $(x, y, z)$  and time  $t$ . An observer in the primed frame  $O'$  (sometimes called the moving frame or the “rocket frame,” which is moving with speed  $u$  in the positive  $x$  direction relative to the lab frame<sup>12</sup>) measures the same event at  $(x', y', z')$  and time  $t'$ . Both of these reference frames are inertial. How are the sets of coordinates related? The

<sup>12</sup>I use  $u$  for the relative speed of the two frames because the point  $P$  may be moving, and its speed will be denoted by  $v$  and  $v'$  as measured by observers in the two frames.

first, and most fundamental underlying idea is that

$$t = t'. \quad (5.17)$$

This relation means that time is absolute — all observers, regardless of their frame of reference, experience the same time. As Newton put it

*Absolute, true and mathematical time, of itself, and by its own nature, flows uniformly on, without regard to anything external...*<sup>13</sup>

This proposition, which seems obvious, turns out not to be true, which we have discovered by measuring muon lifetimes. The second relation, also obvious, is that the coordinates perpendicular to the relative motion of the two coordinate systems are equal

$$y = y' \quad (5.18)$$

$$z = z'. \quad (5.19)$$

This will also be true relativistically, and can be shown quite easily by having someone in the  $O'$  frame drag a piece of chalk along a wall that is stationary in the  $O$  frame. This line on the wall *must* be the same distance from the  $x$  axis that the chalk is from the  $x'$  axis.

Finally, what about the  $x$  and  $x'$  coordinates? Since  $x'$  is the distance between the  $y'$  axis and point  $P$  (at time  $t'$ ), and the distance between the  $y$  and  $y'$  axes increases uniformly with time, it must be true that

$$x = x' + ut'. \quad (5.20)$$

This just expresses the fundamental notion that the total length of a straight line is the sum of the lengths of the segments that make up the line. Of course, we could write  $x = x' + ut$  since  $t = t'$  by Eq. (5.17), but Eqs. (5.17)–(5.20) take the form of a transformation, in which the unprimed variables  $(x, y, z, t)$  are expressed as function of the primed variables  $(x', y', z', t')$ . In this way, knowing the space and time coordinates of an event as measured in one coordinate system, we can predict the coordinates of the same event as measured in another coordinate system.

How do velocities of objects, as measured by the two observers, transform? We should obtain, of course, the “relative velocity” formula that you have learned in elementary mechanics. If point  $P$  labels the position of an object that happens to be moving, then we can take its position as a function to time to be a series of events, each with a spatial and temporal location in each frame of reference. If, for example, the velocity components are known in  $O'$ , then the Galilean transformation equations can be differentiated to give the velocity components in the  $O$  frame.

To show how this works, let’s differentiate Eq. (5.18) with respect to  $t$ . This, of course, will give us the  $y$ -component of velocity in the  $O$  frame

$$\frac{dy}{dt} = \frac{dy'}{dt} = \frac{dy'}{dt'}, \quad (5.21)$$

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<sup>13</sup>Newton, *Principia*, 1687.

where the second equality holds because *time is absolute*. The left-hand-side of Eq. (5.21) is the definition of  $v_y$ , and the right-hand-side is the definition of  $v'_y$ . A similar analysis results in  $v_z = v'_z$ .

The  $x$  component of velocity is slightly trickier. Equation (5.20) differentiated gives

$$\frac{dx}{dt} = \frac{dx'}{dt} + \frac{d}{dt}(ut') = \frac{dx'}{dt} + u, \quad (5.22)$$

where again, absolute time has been invoked. This last result can be written more concisely

$$v_x = v'_x + u, \quad (5.23)$$

which is nothing but the relative velocity formula.

The final transformation is that of the accelerations. Because the relative velocity of the two frames  $u$  is not a function of time, the accelerations are identical. A time derivative of Eq. (5.23) gives

$$a_x = a'_x, \quad (5.24)$$

with a similar transformation for the other coordinates. Since the accelerations are the same, and the force on an object is the same, then Newton's second law takes the same form in all inertial reference frames! This is just Einstein's Principle of Relativity.

### 5.3.2 Lorentz transformation

How did Lorentz develop his transformation? Why was he unhappy with the Galilean transformation? Because, although Eqs. (5.17)–(5.20) correctly showed that Newton's laws had the same form in all reference frames, when he applied the Galilean transformation to Maxwell's equations of electrodynamics, the form of Maxwell's equations were modified, which was not in accordance with Einstein's Principle of Relativity. Even before Einstein's work in 1905, therefore, it was realized that Newton's laws of dynamics were not consistent with Maxwell's equations. So Lorentz developed a transformation between two coordinate frames that were moving at a constant velocity relative to each other, with the requirement that the form of Maxwell's equations was invariant (a slightly different requirement than the Galilean transformation). In other words, Lorentz proved a "theorem of corresponding states."<sup>14</sup> This theorem says that if  $\vec{E}$  and  $\vec{B}$  are the electromagnetic fields in a coordinate system  $(\vec{x}, t)$  that is *at rest relative to the ether*, then in a second coordinate system  $(\vec{x}', t')$  moving with velocity  $\vec{u}$  relative to the first, then, to first order in  $u/c$ ,  $\vec{E}'$  and  $\vec{B}'$  are the same functions of  $(\vec{x}', t')$  as  $\vec{E}$  and  $\vec{B}$  are of  $(\vec{x}, t)$ , if the coordinates transform as Eqs. (5.25) and (5.26), below.

What is the ether? It was believed that electromagnetic waves must propagate in some medium — just like sound waves propagate in a gas or solid — and that medium was called the "luminiferous æther," or ether for short. The ether was supposed to be at rest with respect to Newton's absolute space, and it was thought that its existence allowed a measurement of the absolute velocity of light — with respect to the ether. However, Einstein showed in 1905 that it was possible to understand the Lorentz transformation in

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<sup>14</sup>Pais, *Subtle is the Lord*, page 124.

a “relative” manner, such that the speed of light was constant in all reference frames, and no ether was required.

A simple method to obtain the Lorentz transformation is to assume a linear transformation between the primed and unprimed coordinates. The correct transformation might not be linear, but one should always make the simplest attempt first. The simplest “guess” is therefore

$$\frac{x}{c} = A \frac{x'}{c} + B t' \quad (5.25)$$

$$t = C \frac{x'}{c} + D t', \quad (5.26)$$

and our task is to determine  $A$ ,  $B$ ,  $C$ , and  $D$ . Of course, Eqs. (5.18) and (5.19) must still hold due to the chalk-on-the-wall argument. In addition, we require that in the limit of low speeds, i.e., when  $u/c \rightarrow 0$ , the Lorentz transformation should be approximated by the Galilean transformation in Eqs. (5.17) and (5.20).

We can obtain  $B$  and  $D$  from the muon thought experiment by assuming that the muon remains at the origin (at rest, of course) of the primed frame,  $O'$ . Further mathematical simplification occurs if the muon is created at time  $t = t' = 0$ , when the two origins are co-located. This makes the coordinates of event 1 (muon creation)

$$\begin{aligned} x_1 = 0 & \quad x'_1 = 0 \\ t_1 = 0 & \quad t'_1 = 0 \end{aligned} \quad (5.27)$$

The coordinates of event 2 (muon decay) are, and I’ll leave off the subscript 2 in the interest of notational simplicity,

$$x' = 0 \quad (5.28)$$

$$t' = t_0 = 2.2 \mu\text{s}, \quad (5.29)$$

because in its own frame, the muon doesn’t move and lives for  $2.2 \mu\text{s}$ . What about the coordinates of event 2 in the unprimed frame? The time, as we discovered above, is dilated,<sup>15</sup>  $t = \gamma_r t'$ , and the position is simply the position of the origin  $O'$  after a time  $t$ :  $x = ut = u\gamma_r t'$ , or

$$\frac{x}{c} = \beta_r \gamma_r t', \quad (5.30)$$

where

$$\gamma_r \equiv \frac{1}{\sqrt{1 - \beta_r^2}} \quad \text{and} \quad \beta_r \equiv \frac{u}{c}, \quad (5.31)$$

are the usual relativistic quantities for the *relative* velocity of the two frames. These two facts mean that

$$B = \beta_r \gamma_r \quad (5.32)$$

$$D = \gamma_r. \quad (5.33)$$

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<sup>15</sup>Recall that we used the invariant interval to get

$$t'^2 = t^2 - \frac{x^2}{c^2} = t^2 - \frac{u^2}{c^2} t^2 = \frac{t^2}{\gamma_r^2}.$$



What about  $A$  and  $C$ ? The muon thought experiment doesn't give us any information because  $x' = 0$  for all time, so the coefficients of  $x'$  can take on any values and the transformation in Eqs. (5.25) and (5.26) will still correctly describe the muon creation and decay. One way to determine  $A$  and  $C$  is to enforce the invariance of the interval. (Remember, the fact that the interval is invariant is an experimental fact, so it must be true.) Starting with the partially complete transformation equations that we have just determined

$$\frac{x}{c} = A \frac{x'}{c} + \beta_r \gamma_r t' \quad (5.34)$$

$$t = C \frac{x'}{c} + \gamma_r t', \quad (5.35)$$

we can insert these expressions into the right-hand-side of

$$t'^2 - \frac{x'^2}{c^2} = t^2 - \frac{x^2}{c^2}. \quad (5.36)$$

Note that this is identical to Eq. (5.9) since the coordinate values for event 1 are all zero. The final step is to require that the coefficients of both  $x'$  and  $t'$  match (see Problem 95). The final result is the complete Lorentz transformation

$$\frac{x}{c} = \gamma_r \frac{x'}{c} + \beta_r \gamma_r t' \quad (5.37)$$

$$t = \beta_r \gamma_r \frac{x'}{c} + \gamma_r t'. \quad (5.38)$$

A final check is to make sure that these reduce to the Galilean transformation in the limit of small velocities. Approximating the relativistic factor by  $\gamma_r \approx 1 + \beta_r^2/2$ , and keeping terms of order  $\beta_r$  or smaller, Eqs. (5.37) and (5.38) become

$$\frac{x}{c} \approx \frac{x'}{c} + \frac{ut'}{c} \quad (5.39)$$

$$t \approx \frac{ux'}{c^2} + t'. \quad (5.40)$$

The first equation is identical to Eq. (5.20), but the second equation has an extra term. What has happened? Is the small velocity limit incorrect? No, because we can show that  $ux'/c^2$  is actually of order  $\beta^2$  (and therefore should be ignored) in the following way. The position  $x'$  (if the object is moving slow compared with the speed of light) can be approximated by  $x' \approx v'_x t'$ , where  $v'_x$  is its speed as measured in the primed frame. Using this replacement, Eq. (5.40) can be written

$$t \approx t' \left( 1 + \frac{uv'_x}{c^2} \right) = t' (1 + \beta_r \beta'_x) \approx t', \quad (5.41)$$

where  $\beta'_x \equiv v'_x/c$ . Since the second term has two powers of velocity,  $\beta_r \beta'_x$ , it is of order  $\beta^2$ , as claimed. (Notice that the notation has become more cumbersome, but it is necessary because there are several velocities that must be distinguished.)

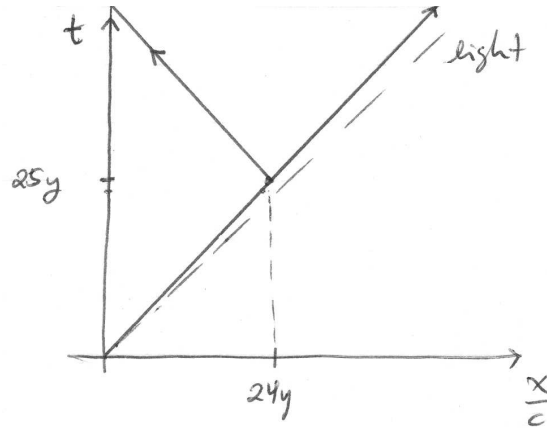


Figure 5.6: The spacetime diagram depicting the twin paradox. Paul’s world line is just the  $t$  axis — he remains at the origin. Peter’s world line consists of two segments, each inclined at an angle  $\theta$  with the  $t$  axis, where  $\tan \theta = v/c = 0.96$ , or  $\theta = 43.83^\circ$ .

## 5.4 Paradoxes

There are several situations in which the special relativistic result appears initially incorrect or inconsistent. A closer look, however, reveals that there is no inconsistency, and that the relativistic result is correct and can be confirmed by experiment.

### 5.4.1 The twin paradox

The twin paradox is usually stated as follows:

On their twenty-first birthday, Peter leaves his twin Paul behind on the earth and goes off in the  $x$  direction for seven years of his time at  $(24/25) = 0.96$  the speed of light, then reverses direction and in another seven years of his time returns at the same speed. (a) What is Peter’s age on his return? (answer: 35 years) (b) How old is Paul at the moment of reunion? (answer: 71 years)

When Peter returned from his fourteen years of traveling he was still young enough to learn some relativity. But the more he studied the more puzzled he became. He and his brother Paul, being in relative motion, “each should see the other’s clocks running slow.” This simple slogan, put in Paul’s mouth, made it easy enough to understand why Peter’s clocks—and Peter’s aging process—ran slow, so that Peter was the younger of the two on his return. “But if the slogan is valid,” Peter asked. “then would not *I*—if I had investigated—have found *Paul’s* clocks running slow? So how did he age more than I?” What is the way out of Peter’s difficulties?<sup>16</sup>

<sup>16</sup>Taylor and Wheeler, *Spacetime Physics*, pages 71, 94.

Figure 5.6 shows the world lines for both Peter and Paul. The seeming paradox is resolved by noting that the world lines of the two twins are not identical. In fact, in this inertial reference frame, Paul's world line is straight, while Peter's world line is curved. Given that they start and end at the same spacetime location, it's a direct consequence of the hyperbolic geometry that the clock which traveled the straight line (Paul's) recorded the greatest elapsed time. (See the discussion on page 135.) Another way of "proving" that Peter was the twin that left and came back is the fact that he must have accelerated. As we have already discussed, it is trivial to decide if your reference frame is accelerating or not. The fact that Peter accelerated means that his inertial reference frame on the outbound trip was different from his inertial reference frame on the inbound trip. Finally, besides the observations of cosmic ray muons, there is concrete experimental verification of this resolution of the paradox: clocks have been flown around the Earth, traveling fast by terrestrial standards, but slow compared with the speed of light, and their elapsed time has been compared with stationary clocks. Result: the moving clocks run slow.

Let's analyze the twins' aging mathematically using our knowledge of time dilation and length contraction effects. (A similar case is in Problem 92.) Peter's age is quite simple, since the problem states that he travels for a total of 14 years *of his time*. This means he must have aged 14 years and so is 35 years upon his return. Paul's age is slightly trickier. The 14 years that Peter's clock measured we can take to be the "proper time" of the journey. Peter thus acts like the cosmic ray muon, traveling at 96% of the speed of light. It is necessary first to calculate the relativistic factor  $\gamma$  for Peter. Since  $\beta = 24/25$ , you can show that  $\gamma = 25/7 \approx 3.57$ .<sup>17</sup> Equation (5.7) then tells us how the coordinate time  $t$  compares with the proper time  $t_0$

$$\Delta t = \gamma \Delta t_0 = \frac{25}{7} 14 \text{ years} = 50 \text{ years.} \quad (5.42)$$

Therefore Paul is 71 years old.

What is the distance that Peter traveled? From Paul's point of view — Paul is in the "stationary" reference frame, so he measures the proper length — his brother traveled

$$\frac{\Delta x}{c} = \frac{L_0}{c} = \frac{u\Delta t}{c} = \frac{24}{25} \times 25 \text{ years} = 24 \text{ years} \quad (5.43)$$

each way for a total of 48 years. Or, converting to meters gives  $2.27 \times 10^{17}$  m each way. How far did Paul *think* he traveled? Since Paul was moving past the 'object,' he saw it length-contracted by a factor  $\gamma$ , which means that he thought his trip distance was  $L/c = L_0/c\gamma = 24 \text{ years}(7/25) = 6.72 \text{ years}$ , or  $6.35 \times 10^{16}$  m each way. This is just like the muon that observed the 20-km distance between the top of the atmosphere and the ground to contract to 660 m.

Can this twin paradox analysis be applied to astronauts on the Space Shuttle or the ISS? Do those astronauts (since they are accelerating, they take the place of Peter in our story) age more slowly than their friends they left behind on Earth? Yes, but by how much? Assuming they are in LEO (low Earth orbit, altitude  $\approx 300$  km), their speed (relative to the Earth) is  $7.7 \times 10^3$  m/s. This gives them  $\beta = 2.58 \times 10^{-5}$  and a relativistic

<sup>17</sup>I have used the fact that a 7-24-25 triangle is a right triangle.

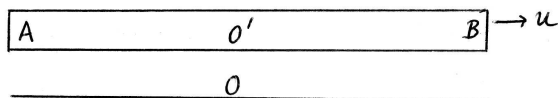


Figure 5.7: The geometry of Einstein’s simultaneity *Gedanken* experiment.

factor of  $\gamma = 1 + 3.32 \times 10^{-10}$ . The time dilation effect means that their clocks run more slowly than clocks on Earth by this factor, or, for a mission of 1 year, the astronaut ‘twin’ is younger by 10 ms!<sup>18</sup> (See Problem 98.)

### 5.4.2 Einstein’s *Gedanken* experiment on simultaneity

One of the perplexing new truths of special relativity is the fact that events that are simultaneous in one frame of reference are not simultaneous in another frame. This, of course, cuts at the heart of Newton’s concept of absolute time. If there truly is a cosmic timekeeper, making sure that time everywhere in the universe flows “uniformly on,” then any event should be observed simultaneously by all observers. However, the fact that the speed of light is not infinite, but finite (albeit large), implies directly that information takes a finite amount of time to propagate from the event to the observer. Since each observer is a different distance from the event, and perhaps moving relative to the event as well, they each will observe that the event occurred at a different time. If there are two events, they may be simultaneous for some observers and distinct for others.

In 1917, Einstein developed a *Gedanken* experiment that explored and elucidated this strange notion of non-simultaneity. He envisioned a train moving to the right with speed  $u$ , with three men riding on the train, one at the front (point  $B$ ), one at the rear (point  $A$ ), and one in the middle. There is also a man on the train platform (i.e., in a “stationary” frame of reference). He then asked the following question

Are two events which are simultaneous *with reference to the railway embankment [platform]* also simultaneous *relatively to the train*? We shall show directly that the answer must be in the negative.<sup>19</sup>

The specific event geometry is as follows (see Fig. 5.7): The two men at  $A$  and  $B$  flash lights toward the center of the train. At the instant that the man at the center of the train (the origin of the  $O'$  coordinate system) passes the man on the platform (the origin of the  $O$  coordinate system), the two flashes of light reach both men. Since by definition that is the origin of both time coordinates also, it means that at  $t = t' = 0$  the two flashes of light arrive at  $O$  and  $O'$  from  $A$  and  $B$ .

Who emitted their signal first,  $A$  or  $B$ ?

The answer depends on who you ask. From the point of view of the man on the train, in the  $O'$  frame, everyone is at rest,  $A$  and  $B$  are equidistant and therefore they emitted

<sup>18</sup>I used the approximation  $\gamma \approx 1 + \beta^2/2$  and  $\gamma^{-1} \approx 1 - \beta^2/2$ , where  $\beta^2/2 = 3.32 \times 10^{-10}$ . Multiplying this by 1 year  $\approx \pi \times 10^7$  s gives 10 ms.

<sup>19</sup>Einstein, *Relativity*, page 25. The italics are Einstein’s.

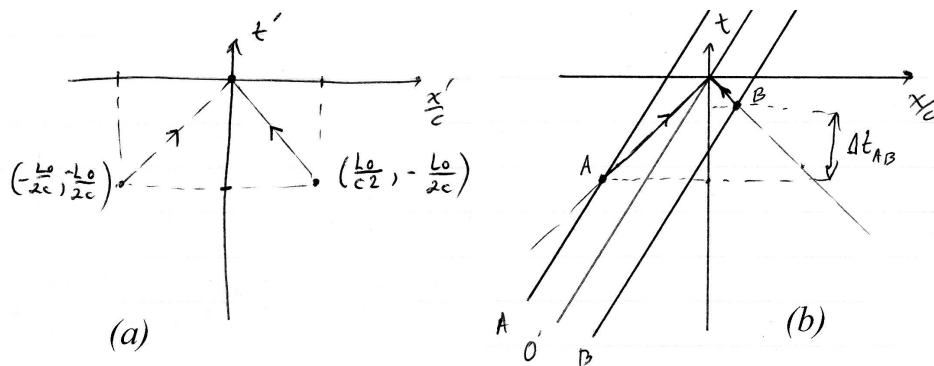


Figure 5.8: The two spacetime diagrams depicting the train and light rays from both reference frames. (a) The moving frame  $O'$ . In this frame, the emission of the light rays is simultaneous. (b) The stationary frame  $O$ . Here,  $A$  and  $B$  label the world lines of the ends of the train, and  $O'$  labels the world line of the center of the train. In this frame, the man at the rear of the train ( $A$ ) emits his flash of light before the man at the front of the train ( $B$ ).

the light *simultaneously*. From the point of view of the man on the platform, in the  $O$  frame, the flashes must have been emitted *before*  $O'$  reached  $O$ , and since at that time  $B$  was closer to  $O$  than  $A$  was,  $A$  must have sent the signal first.<sup>20</sup>

Who *really* emitted their signal first?

The Newtonian worldview states that there is a reality that is independent of any observation, so regardless of the fact that different observers make different measurements, there must be a way to definitively state what the reality of the situation is. Unfortunately, there is not. As Comstock states

We are, therefore, forced to the conclusion that, unless we discard one of the two relativity postulates, the simultaneity of two distant events means a different thing to two different observers if they are moving with respect to each other.<sup>21</sup>

We can solve this problem quantitatively with the tools we now have at our disposal. Figure 5.8(a) shows the spacetime diagram as viewed in the  $O'$  frame. The light signals meet at  $(0, 0)$ , and since they must have traveled along the past light cone from points that were equidistant, they were emitted at the same time

$$t'_A = -\frac{L_0}{2c} \quad t'_B = -\frac{L_0}{2c}, \quad (5.44)$$

where  $L_0$  is the proper length of the train, and  $A$  and  $B$  denote the times that the flashed were emitted by the two sources. What about the observer on the ground? Figure 5.8(b) depicts the situation from his point of view. As can be seen graphically,  $A$  indeed

<sup>20</sup>An observer in a third frame of reference,  $O''$ , moving to the right *faster* than the train would conclude that  $B$  sent his signal first.

<sup>21</sup>D. H. Comstock, "The principle of relativity," *Nature* **31**, 767-772 (1910).

emitted his signal first. By how much? Since we know the coordinates of each of the events (emission of light signals) in the  $O'$  frame, we can use the Lorentz transformation equations to calculate the coordinates of those same two events in the  $O$  frame.

**Emission of light by  $A$ :** The coordinates of this event in the primed frame are  $t' = -L_0/2c$ ,  $x'/c = -L_0/2c$ . Of course they are identical because they are on a line that passes through the origin  $O'$  and has a slope of 1. The Lorentz transformation gives

$$\frac{x}{c} = \gamma_r \left( -\frac{L_0}{2c} \right) + \beta_r \gamma_r \left( -\frac{L_0}{2c} \right) = \left( -\frac{L_0}{2c} \right) \gamma_r (1 + \beta_r) \quad (5.45)$$

$$t = \beta_r \gamma_r \left( -\frac{L_0}{2c} \right) + \gamma_r \left( -\frac{L_0}{2c} \right) = \left( -\frac{L_0}{2c} \right) \gamma_r (1 + \beta_r). \quad (5.46)$$

Of course they have the same values for the same reason as the other reference frame. Most important, however, is the fact that the quantity  $\gamma_r(1 + \beta_r)$  is greater than unity (when  $u > 0$ ), and it is equal to unity when  $u = 0$ . The proof is straightforward since  $\gamma_r \geq 1$  and  $\beta_r \geq 0$ . This means that it always takes longer for the flash from  $A$  to arrive for the man on the platform than it does for the man on the train.

**Emission of light by  $B$ :** Using the coordinates of this event in the primed frame,  $t' = -L_0/2c$ ,  $x'/c = L_0/2c$ , gives the time of this event in the unprimed frame as

$$t = \left( -\frac{L_0}{2c} \right) \gamma_r (1 - \beta_r). \quad (5.47)$$

You can show (Problem 104) that the factor  $\gamma_r(1 - \beta_r) \leq 1$ , the equality holding when  $u = 0$ , as before. Our final result, therefore, is that the time difference between the emission of the flashes from  $A$  and  $B$  as viewed by the man on the platform is

$$\Delta t_{AB} = \left( \frac{L_0}{c} \right) \gamma_r \beta_r > 0, \quad (5.48)$$

which is positive definite. The man on the platform always thinks  $A$  emitted the flash first.

This result can be obtained in another, slightly more elegant manner. The Lorentz transformation equations are not just valid for coordinates of an event, they are also valid for intervals between events. In particular, for this interval between the flashes of light we have from the time equation, Eq. (5.38),

$$\Delta t = \gamma_r \Delta t' + \gamma_r \beta_r \frac{\Delta x'}{c}. \quad (5.49)$$

Since the observer on the train sees the flashes to be simultaneous,  $\Delta t' = 0$ , and the spatial separation to be just the proper length of the train,  $\Delta x' = L_0$ , the transformation equation gives the same result as Eq. (5.48),  $\Delta t_{AB} = L_0 \gamma_r \beta_r / c$ .

## 5.5 Addition of velocities

One of the most profound consequence of special relativity is that no object with a nonzero mass can travel faster than (or even at) the speed of light  $c$ . Most important, this result is independent of the frame of reference! Specifically, it means that velocities can not add the way Galileo thought they should — Eq. (5.23). That equation states that if you are in a “stationary” frame of reference, standing on the side of the highway, for example, watching your friend drive by (in a moving frame of reference) at speed  $u = 60$  mph, and if your friend throws a ball forward at speed  $v'_x = 10$  mph (relative to the car), then you will observe the ball traveling at a speed  $v_x = 70$  mph (relative to the ground). That statement (that velocities just add) is so patently obvious it is no wonder that Galileo believed it to be true. It turns out, though, that if you measure carefully, you will observe the speed of the ball to be slightly *less* than 70 mph. If we take our example to the extreme and consider the situation where your friend’s relative speed is  $0.9c$ , and he throws the ball at  $0.9c$  forward, it can’t be true that you observe the ball traveling at  $1.8c$ .

To determine the correct relative velocity formula we must differentiate the Lorentz transformation Eqs. (5.37) and (5.38) in the same manner that we did with the Galilean transformation in order to obtain Eq. (5.23). The differentiation process is a little bit trickier this time, since time is not absolute. The velocities in the two frames are defined as

$$v_x \equiv \frac{dx}{dt} \quad \text{and} \quad v'_x \equiv \frac{dx'}{dt'}, \quad (5.50)$$

where  $v'_x$  must be a derivative of  $x'$  with respect to  $t'$ . That is, to determine the velocity of an object as measured in a particular frame, the *time* in that frame must be used, and Eq. (5.38) shows that  $t \neq t'$ . The simplest method is to differentiate Eq. (5.37) with respect to  $t'$

$$\frac{1}{c} \frac{dx}{dt'} = \frac{\gamma_r}{c} \frac{dx'}{dt'} + \beta_r \gamma_r = \frac{\gamma_r}{c} v'_x + \beta_r \gamma_r. \quad (5.51)$$

The left-hand-side is *not*  $v_x$  because the derivative is with respect to  $t'$ , not  $t$ . However, we can use the chain rule to obtain

$$v_x = \frac{dx}{dt} = \frac{dx}{dt'} \frac{dt'}{dt}, \quad \text{or} \quad \frac{dt}{dt'} v_x = \frac{dx}{dt'}, \quad (5.52)$$

which means that we need to evaluate  $dt/dt'$ , which is the derivative of Eq. (5.38)

$$\frac{dt}{dt'} = \frac{\beta_r \gamma_r}{c} \frac{dx'}{dt'} + \gamma_r = \frac{\beta_r \gamma_r}{c} v'_x + \gamma_r. \quad (5.53)$$

Now, inserting Eqs. (5.51) into the second form of the right-hand-side of Eq. (5.52), and (5.53) into the left-hand-side, gives

$$\beta_x \left( \frac{\beta_r \gamma_r}{c} v'_x + \gamma_r \right) = \frac{\gamma_r}{c} v'_x + \beta_r \gamma_r, \quad (5.54)$$

where  $\beta_x = v_x/c$ . Dividing by  $\gamma_r$ , solving for  $\beta_x$  and simplifying results in

$$\beta_x = \frac{\beta_r + \beta'_x}{1 + \beta_r \beta'_x}, \quad (5.55)$$

where  $\beta'_x = v'_x/c$ . Multiplying through by a factor of  $c$  gives a more familiar result

$$v_x = \frac{u + v'_x}{1 + \frac{uv'_x}{c^2}}. \quad (5.56)$$

Now the logic is clear: if both speeds,  $u$  and  $v'_x$ , are small compared with  $c$ , then the denominator is approximately unity, and the velocities add in a Galilean manner. However, as the speeds increase and become a sizable fraction of  $c$ , the extra term in the denominator keeps the “sum” from exceeding  $c$ . For example, if  $\beta_r = 0.5$  and  $\beta'_x = 0.5$  (your friend is driving at half the speed of light and throws a ball at half the speed of light) then the speed of the ball as observed by you is only  $0.8c$ . Another important example is that if your friend “throws a photon,” which means  $v'_x = c$ , then no matter what the value of  $u$  is, Eq. (5.56) gives  $v_x = c$  (you should show this). Photons travel at the speed of light *as measured by any observer*, confirming Einstein’s second postulate.

## 5.6 Relativistic dynamics

In the preceding sections, we have focused on constant velocity kinematics. In that restricted case, when you are measuring phenomena in your own reference frame, e.g., velocities of particles, you don’t see any relativistic effects. However, we have seen that when comparing your measurements of length and time with those of an observer in another reference frame you will detect that something is amiss. Each of you will conclude that the same laws of physics apply, but not agree on the detailed distances and times. However, we have not yet determined the form of those laws. They can’t be the ones we are familiar with (i.e.,  $\vec{F} = m\vec{a}$ ) because we have just shown that particles can’t travel faster than  $c$ , and  $\vec{F} = m\vec{a}$  implies that by applying a constant force to a particle we should be able to accelerate it to any velocity whatsoever (if we apply the force for a long enough time). But we know that is incorrect. So we must use the relativistically correct laws.

It turns out that the familiar form of Newton’s 2nd Law,  $\vec{F} = m\vec{a}$  does *not* hold relativistically. The relativistically correct form is the one in which Newton first stated it

$$\vec{F} = \frac{d\vec{p}}{dt} \quad (5.57)$$

where

$$\vec{p} = \gamma m \vec{v}. \quad (5.58)$$

Why is this definition of momentum the correct definition? Because we still require that momentum must be conserved. If you analyze high velocity collisions between particles, the requirement of momentum conservation will lead you to Eq. (5.58). Some authors define a “relativistic mass” to equal to  $\gamma m$ , where  $m$  is the “rest mass.” This implies that the mass of an object increases as its speed increases. Currently, however, the more rigorous viewpoint is that the mass  $m$  is an invariant property of a particle, and that it is simply the momentum that increases as the speed increases so that it takes a larger and



larger force to change a particle's momentum (as its speed becomes close to the speed of light).<sup>22</sup>

What about energy? You would think (and you'd be correct) that we should require that energy still be conserved, too. For this to be true, the energy of a particle must take the form

$$E = \gamma mc^2. \quad (5.59)$$

This is the total energy, kinetic plus rest,  $E = E_0 + K$ , where  $E_0 = mc^2$  is the rest energy of the particle, as we've seen many times before. (See Problem 101 for a proof.)

Actually, there is another requirement that both the energy and momentum must satisfy (besides conservation), and that is they must reduce to the nonrelativistic forms when  $v \ll c$ . Since  $\gamma \rightarrow 1$  in that limit, it is clear that  $\vec{p} \rightarrow m\vec{v}$  in the low-velocity limit. If we let  $\gamma = 1$  in Eq. (5.59), we would obtain simply  $E = E_0$ . This is, of course, true when  $v = 0$ , but what if  $v$  is small, but not zero? As we saw in Chapter 1, we can expand  $\gamma$  for small values of  $\beta$ ,

$$\gamma = (1 - \beta^2)^{-1/2} \approx 1 + \frac{\beta^2}{2}. \quad (5.60)$$

Therefore, the total energy is approximately

$$E \approx \left(1 + \frac{\beta^2}{2}\right) mc^2 = mc^2 + \frac{1}{2}mv^2 \quad (5.61)$$

which is just the rest energy plus the (nonrelativistic) kinetic energy.

In contrast with elementary mechanics, where the mass and velocity of a particle are considered to be the fundamental quantities (from which the energy and momentum can be calculated), the fundamental quantities describing a particle in relativistic dynamics are its total energy  $E$  and its momentum  $\vec{p}$ . As we derived in Chapter 1, the relationship between the energy and the momentum obtained from their definitions is

$$E^2 = (pc)^2 + (mc^2)^2. \quad (5.62)$$

In the nonrelativistic limit,  $pc \ll mc^2$ , the energy can be approximated as

$$E = \sqrt{(pc)^2 + (mc^2)^2} = mc^2 \sqrt{1 + \left(\frac{p}{mc}\right)^2} \approx mc^2 + \frac{p^2}{2m}. \quad (5.63)$$

We now have three nonrelativistic approximations that are all essentially equivalent. The first is that of low speed,  $v \ll c$ ; the second is low kinetic energy,  $K \ll E_0$ , and the last is low momentum,  $p \ll mc$ .

In the ultrarelativistic regime, on the other hand (large kinetic energy and large momentum), the energy of a particle can be approximated as

$$E \approx pc. \quad (5.64)$$

For photons, of course, this is an exact equality,  $E = pc$ , not just an approximation, since photons have zero rest mass. Even though they have zero rest mass, they do have energy and momentum.<sup>23</sup>

<sup>22</sup>See L. B. Okun, "The concept of mass," Phys. Today, **42**, 31-36 (June 1989), for a more complete discussion.

<sup>23</sup>Recall the discussion in Chapter 4 regarding the wavelength and frequency of light, and their relation to a photon's energy and momentum.

## Kaon decay

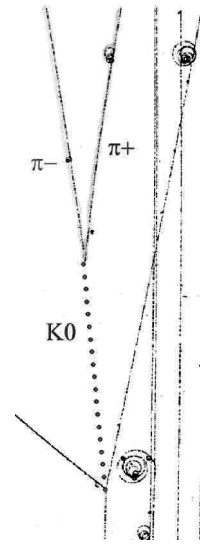
The classic situation in which the conservation of relativistic energy and momentum must be applied is that of Compton scattering, described in App. H. There, a photon interacts with a free electron, and one way of explaining the experimental results is to assume that the photon is a massless particle with energy  $E = h\nu$ , and since it is ultrarelativistic, it has momentum  $p = E/c$ . The experiment is not consistent with the picture of an electromagnetic wave interacting with a classical electron. (However, see the discussion at the end of App. E concerning what minimum assumptions are needed to predict the experimental results.)

Another process which illustrates the necessity of the relativistic conservation equations is that of kaon decay. A neutral kaon  $K^0$  is a heavy meson ( $E_0 = 498$  MeV) that decays into two pions<sup>24</sup> with a mean lifetime of  $9 \times 10^{-11}$  s



The rest energies of the pions are 140 MeV each, so that in the frame of reference in which the kaon is at rest, the reaction energy of the decay is  $Q = 218$  MeV. In this frame, the momentum of the kaon is zero, so that the total momentum of the pions must also be zero: they travel away from the location of the decay with equal kinetic energies ( $K = 109$  MeV each) and oppositely directed momenta.

The laboratory is not usually in this reference frame, however, since the kaon is typically the result of another interaction and therefore is moving when it decays. (The figure to the right shows a bubble chamber with tracks of various particles. Since the  $K^0$  is neutral it does not ionize particles in the chamber and hence leaves no track, but its existence is inferred from the presence of the two pions.) To illustrate how either frame can be used to analyze such a decay, let's consider the situation where a kaon has 325 MeV of kinetic energy. This value is chosen because it is neither nonrelativistic nor ultrarelativistic, and no approximations can be made. The full relativistic expressions must be used. (See Problem 60 in Chapter 3 for a nonrelativistic version of this problem.)



**Laboratory frame:** Without any loss of generality, let's take the kaon's velocity to be in the positive  $x$  direction. To make the analysis simple, we'll assume that the resulting pions also move in the  $x$  direction (allowing them to have a nonzero  $y$  component of velocity is only slightly more difficult algebraically). Referring to Fig. 5.9, the  $K^0$  has an energy  $E$  and momentum  $p$ , while the resulting pions  $\pi^\pm$  have energy and momenta  $E_\pm$  and  $p_\pm$ .

The first step is to analyze the motion of the kaon. Its relativistic factor is

$$\gamma = \frac{E}{E_0} = \frac{823 \text{ MeV}}{498 \text{ MeV}} = 1.65 \quad (5.66)$$

<sup>24</sup>The neutral kaon is a linear combination of the quark-anti-quark pairs  $d\bar{s}$  and  $s\bar{d}$ . The combination I'm discussing here is called "K-short," because its lifetime is short. Another combination, "K-long," typically decays into three pions with a mean lifetime of  $5 \times 10^{-8}$  s.

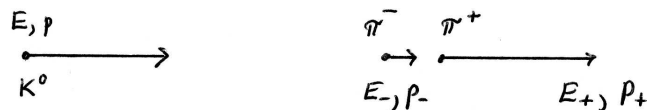


Figure 5.9: Schematic of the kaon decay in the laboratory frame. On the left is the situation *before* the decay, where the  $K^0$  is traveling to the right with an energy  $E$  and momentum  $p$ . On the right are the particles *after* the decay, with the pions also moving in the  $x$  direction with energies  $E_{\pm}$  and momenta  $p_{\pm}$ .

and hence its speed is  $\beta = 0.796$ . We know its total energy  $E = E_0 + K = 823$  MeV, but its momentum must be obtained from Eq. (5.62)

$$pc = \sqrt{E^2 - (Mc^2)^2} = 655 \text{ MeV}, \quad (5.67)$$

where  $Mc^2 = 498$  MeV.

The next step, just like in elementary mechanics, is to apply the conservation laws and obtain the pion energies and momenta. Since the reaction is one dimensional, there is only one component of momentum to worry about, and hence there are two conservation equations

$$E = E_+ + E_- \quad (5.68a)$$

$$pc = p_+c + p_-c. \quad (5.68b)$$

It appears that there are four unknowns,  $E_{\pm}$  and  $p_{\pm}$ , but the energy and momentum of each the pions are related by Eq. (5.62), which gives us two more equations. Expressing the energies in terms of their respective momenta allows us to replace Eq. (5.68a) with

$$E = \sqrt{p_+^2c^2 + (mc^2)^2} + \sqrt{p_-^2c^2 + (mc^2)^2} \quad (5.69)$$

where  $mc^2 = 140$  MeV. The simplest way to solve these two equations for the two unknowns,  $p_{\pm}$ , is to solve Eq. (5.68b) for  $p_+$  and substitute that into Eq. (5.69). Solving then for  $p_-$  results in the quadratic equation for  $p_-c$  (see Problem 105)

$$(p_-c)^2 - (pc)(p_-c) + \left( \frac{E^2(mc^2)^2}{(Mc^2)^2} - \frac{(Mc^2)^2}{4} \right) = 0, \quad (5.70)$$

where  $(Mc^2)^2 = E^2 - p^2c^2$  is the square of the rest energy of the kaon.

Evaluating this equation for the current initial conditions of the kaon gives

$$p_-c = 667.9 \text{ MeV}, \quad -12.7 \text{ MeV} \quad (5.71)$$

Which solution is the one we want? Both of them! Since Eq. (5.69) is symmetric in  $p_+$  and  $p_-$ , we would have obtained the same result had we solved for  $p_+$ . This means that one of the pions is emitted traveling forward with momentum 667.9 MeV/c, and one is emitted backward with momentum 12.7 MeV/c, for a net momentum of 655 MeV/c, the initial momentum of the kaon. It is always a good idea to confirm energy conservation as well. The energies of two pions can be found from Eq. (5.62), 682.4 MeV and 140.6 MeV. These add, of course, to 823 MeV, which was the initial energy of the kaon.

**Kaon rest frame:** A simpler way to analyze this decay is in the rest frame of the kaon. Let's make a Lorentz transformation to the 'primed' frame that is moving with the kaon. The speed of the primed frame relative to the unprimed (laboratory) frame is just the speed of the kaon,  $\beta_r = 0.796$ . In this primed frame, the analysis is simple, as mentioned above: the kaon has no kinetic energy, the pions each have a total energy  $E' = E_0 + K' = 249$  MeV, and since they move in opposite directions, their momenta in the  $x'$  direction is

$$p'_{\pm}c = \pm\sqrt{E'^2 - (mc^2)^2} = \pm 205.9 \text{ MeV}. \quad (5.72)$$

In order to be able to transform *back* to the laboratory frame, and calculate the energy and momentum of the pions as measured by an observer in *that* frame, we need to calculate the relativistic factor  $\gamma'$  and the velocity  $\beta'_x$  of each pion as measured in the primed frame,

$$\gamma' = \frac{E'}{E_0} = \frac{249 \text{ MeV}}{140 \text{ MeV}} = 1.78 \quad (5.73)$$

and

$$\beta'_x = \pm\sqrt{1 - \frac{1}{\gamma'^2}} = \pm 0.827. \quad (5.74)$$

Now that we have the velocities in the primed frame, we can use Eq. (5.55) to solve for the velocities of the pions in the *unprimed* frame

$$\frac{v_{\pm}}{c} = \frac{\beta_r + \beta'_x}{1 + \beta_r\beta'_x} = \frac{0.796 \pm 0.827}{1 + (0.796)(\pm 0.827)} = \begin{cases} +0.979 \\ -0.090 \end{cases} \quad (5.75)$$

As expected, one pion has a large positive velocity, and the other has a small negative velocity.

We have shown that although the energies and momenta are measured to be different by observers in different frames of reference, Newton's laws (e.g., conservation of energy and momentum) still hold. Key to showing this is the Lorentz transformation, which was obtained from Einstein's two postulates: relativity and the constancy of the speed of light.

### Collateral Reading

- "Zur Elektrodynamik bewegter Körper," by Albert Einstein, *Ann. Phys.* **17** 891 (1905). An English translation, "On the electrodynamics of moving bodies," can be found in two places: First, *Einstein's Miraculous Year*, edited by John Stachel, and second, *The Principle of Relativity*, online at <http://www.fourmilab.ch/etexts/einstein/specrel/www/>
- *Spacetime Physics*, 2nd ed. by Edwin Taylor and John Wheeler, Freeman 1992. (ERAU: QC 173.65 T37 1991)
- J.C. Hafele and R. E. Keating, "Around the world atomic clocks," *Science* **177**, 166-168, 168-170 (1972).

## Problems

**85.** Calculate, in seconds (to 3 significant digits) (a) your height, (b) the distance between the student village and Lehman building, (c) the circumference of the Earth, (d) 1 A.U.

**86.** Show that if  $\beta = 1 - \epsilon$ , and  $\epsilon \ll 1$ , then

$$\gamma \approx \frac{1}{\sqrt{2\epsilon}}.$$

**87.** At their fastest, the protons in the beam of the Large Hadron Collider (LHC) at CERN travel with velocities  $\beta = 0.999\,999\,991$ . Calculate their  $\gamma$  and the total energy of one proton (in both MeV and J). How many protons must be in the beam for it to have 1 J of energy?

**88.** A clock moves from (0,0) to (6 s, 10 s) as measured by a stationary clock (i.e., coordinate time). What is the clock's speed? What is the elapsed time on the clock?

**89.** When Einstein was a boy, he mulled over the following puzzler: A runner looks at himself in a mirror that he holds at arm's length in front of him. If he runs at nearly the speed of light, will he be able to see himself in the mirror? Analyze this question in terms of relativity. Draw spacetime diagrams in both the lab frame and the runner's frame depicting the world lines of the runner, the mirror, and light.

**90.** As measured by an observer on the Earth, a spacecraft runway on Earth has a length of 3600 m. (a) What is the length of the runway as measured by a pilot of a spacecraft flying past at a speed of  $4.0 \times 10^7$  m/s relative to the Earth? (b) An observer on Earth measures the time interval from when the spacecraft is directly over one end of the runway until it is directly over the other end. What result does she get? (c) The pilot of the spacecraft measures the time it takes him to travel from one end of the runway to the other end. What value does he get?

**91.** How fast must an object move before its length appears to be contracted to one-half its proper length?

**92.** For this exercise, assume that velocity of light is 5 miles per hour. Dave starts from home at 6 am and walks down a long straight road at 1 mile per hour. His friend Erin starts (from the same home) 9 hours later (at 3 pm) and follows Dave, walking at 2 miles per hour. Draw their world lines (to scale) on a suitable spacetime diagram, and determine graphically the coordinates of the event E: Dave and Erin meet. Their dog Fido leaves home just when Erin does, pursuing Dave at 4 miles per hour, meets Dave, reverses direction and returns to Erin (also at 4 miles per hour), reverses to Dave, etc., until the event E. How far does Fido walk? Now Dave, Erin and Fido each carry ordinary clocks, all of which have been synchronized at 6 am, the moment when Dave leaves. What are the readings of each of the 3 clocks at the event E, when they are all back together again? (Partial answer: Erin's clock reads 11:15 pm.)

**93.** You wish to travel to a distant star that is 200 light years from Earth. What must your speed be if you want to age only 10 years during the round trip?

**94.** Invert the Galilean transformation given in Eqs. (5.17)–(5.20). That is, determine four equations that give the value of  $(x', y', z', t')$  in terms of  $(x, y, z, t)$ .

**95.** Derive Eqs. (5.37) and (5.38) using the method suggested in the text. That is, calculate the values of  $A$  and  $C$ .

**96.** Start with the Lorentz transformation as given in Eqs. (5.37) and (5.38), and of course (5.18) and (5.19). Invert that transformation to obtain  $x'$ ,  $y'$ ,  $z'$ , and  $t'$  in terms of  $x$ ,  $y$ ,  $z$ , and  $t$ .

**97.** Derive the velocity addition formulas for the directions perpendicular to the relative motion of the two reference frames. That is, we have calculated the velocity addition formula by taking time derivatives of the Lorentz transformation equations, but we have only compared  $v_x$  and  $v'_x$ . In this problem, you are to obtain expressions for  $v_y$  and  $v_z$  in terms of the velocity components as measured in the primed frame,  $\vec{v}'$ .

**98.** Calculate the time difference measured by a clock carried by an astronaut on the International Space Station for one year compared with a clock on the Earth's surface at the equator. Assume that the ISS is in an equatorial orbit at 300 km altitude. You'll need to include the fact that the clock on the Earth's surface is also moving. HINT: Because the speeds are so small, the binomial expansion of  $\gamma$  will be useful.

**99.** The pole-and-barn paradox. A 20-m long pole (proper length) is traveling at  $0.9c$  in the direction along its axis. It encounters a barn whose front door and back door are open, with a proper distance of 10 m between the doors. From the point of view of the farmer in the barn, the pole is length contracted to 8.73 m. He plans to close the back door, wait until the pole enters the front door and is entirely within the barn, and then shut the front door. He will then have "captured" the pole. However, from the point of view of the pole, the *barn* is moving at  $0.9c$  in the other direction, and the barn is length contracted to 4.37 m; therefore the pole will never fit.

Resolve this paradox in the following way. Assume that the pole is in the primed frame with the origin at the front of the pole, and that the barn is in the unprimed frame with the front door of the barn at the origin. At the instant that the two origins coincide (the front of the pole is at the front door of the barn) is when  $t = t' = 0$ . Draw two spacetime diagrams (very carefully, on graph paper), one from the point of view of each frame, and draw the world lines of the following four objects: the front and back of the pole and the barn. Obtain the spacetime coordinates  $(x, t)$  in both frames of the following four events: A) the front of the pole passes through the front door ( $x = x' = 0, t = t' = 0$ ), B) the front of the pole passes through the back door, C) the back of the pole passes through the front door, and D) the back of the pole passes through the back door. Show that events that are simultaneous in one frame are not in another; however, the spacetime interval between events is invariant.

**100.** (a) Starting with the nonrelativistic expression for a particle's kinetic energy  $K$  in terms of its mass  $m$  and velocity  $v$  ( $K = mv^2/2$ ), along with the nonrelativistic expression for a particle's momentum  $p$  ( $p = mv$ ), obtain a relation between  $K$  and  $p$ . (b) Starting with the relativistic expression for a particle's total energy  $E$  in terms of its mass  $m$  and velocity  $v$  ( $E = \gamma mc^2$ ), along with the relativistic expression for a particle's momentum  $p$  ( $p = \gamma mv$ ), obtain a relation between  $E$  and  $p$ . (answer:  $E^2 = p^2 c^2 + (mc^2)^2$ ) NOTE: this problem is one-dimensional.

**101.** Derive the work-energy theorem (in one dimension) in the relativistic case. That is,  $W = \int F dx$  should be equal to the change in kinetic energy  $\Delta K$ . HINT: use Newton's second law, of course,  $F = dp/dt$ , and the fact that  $p = \gamma mv$ , then integrate by parts.

Finally, if you use the fact that

$$\int \frac{x dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + C,$$

and evaluate the integral from  $0 \rightarrow v$  so that  $\Delta K = K$ , you should obtain what we already know,  $K = E - E_0 = (\gamma - 1)mc^2$ .

**102.** The correct expression for the kinetic energy of a particle is  $K = \gamma mc^2 - E_0$ , where  $E_0$  is its rest energy. What is the maximum speed for which the classical kinetic energy of a particle,  $\frac{1}{2}mv^2$ , is correct to within 1%?

**103.** Calculate the same speed as you did in Problem 102, but this time in the ultrarelativistic limit. That is, the energy of a particle that is moving close to the speed of light is  $E \approx pc$  (for a photon that's the exact relation between energy and momentum). What is the minimum speed for which the ultrarelativistic energy approximation is correct to within 1%?

**104.** Show that the quantity  $\gamma_r(1 - \beta_r)$  is less than unity when  $u > 0$  and equal to unity when  $u = 0$ .

**105.** Derive Eq. (5.70) from Eqs. (5.68b) and (5.69).

**106.** Show that the velocities in Eq. (5.75) agree with the momenta in Eq. (5.71).

## Solutions

**85.** Dividing the length by  $c$  (a conversion factor) gives distance in units of time. (a) My height is  $6' 0.5'' = 183$  cm, which is  $6.10 \times 10^{-9}$  s. (b) This is about 750 m =  $2.50 \times 10^{-6}$  s. (c) The equatorial radius of the Earth is 6378 km, so that the circumference is  $C = 2\pi R_{\oplus} = 4.01 \times 10^7$  m, which is  $0.134$  s  $\approx \frac{1}{7.5}$  s. (d) 1 A.U. is about 150 million km, or  $500$  s  $\approx 8.3$  minutes.

**86.** Using the binomial expansion,  $\beta^2 = (1 - \epsilon)^2 \approx 1 - 2\epsilon$ , and therefore

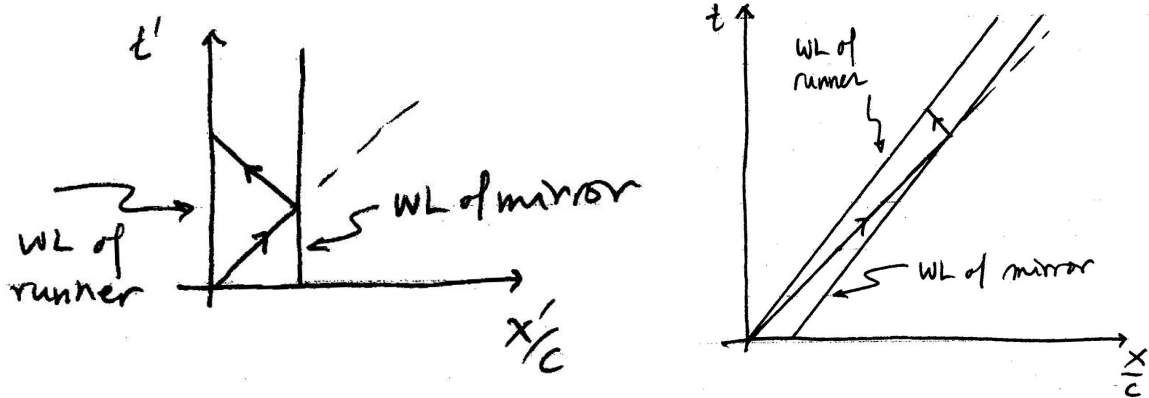
$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} \approx \frac{1}{\sqrt{2\epsilon}}.$$

**87.** In the ultra-relativistic regime, which this is, it is simplest to express the speed as  $\beta = 1 - \epsilon$ . In this case  $\epsilon = 9 \times 10^{-9}$ . From the previous problem, I can solve for  $\gamma \approx 7.45 \times 10^3$ . The total energy of a proton, therefore, is  $E = \gamma mc^2 = 6.99 \times 10^6$  MeV = 6.99 TeV = 1.13  $\mu$ J. Therefore, there must be about  $9 \times 10^5$  protons in the beam for it to have 1 J. Each bunch (of which there are many) in the LHC consists of about  $10^{11}$  protons, which means that each bunch has about 0.1 MJ of energy.

**88.** The speed relative to the stationary coordinate system is  $\beta = v/c = \Delta x/c\Delta t = 0.6$ . But the moving clock measures  $\Delta t_0 = \sqrt{(10 \text{ s})^2 - (6 \text{ s})^2} = 8$  s. Since this is less than 10 s, we are led to the conclusion that moving clocks run slow. (If you want to express the distance in SI units, note that 6 light-seconds is 1.8 million km.)

**89.** Yes, he will be able to see himself. Why? From his perspective, he's not moving. Therefore, all the usual laws of physics apply, including the law of reflection as well as Einstein's postulate of the constancy of the speed of light. Below on the left is the space-time diagram in the moving frame, in which the boy and the mirror are stationary, and

on the right is the spacetime diagram in the “lab” frame where the world lines of the boy and the mirror are diagonal straight lines.



90. First, since  $u = 4 \times 10^7$  m/s, let's calculate  $\beta_r = 0.13343$  and  $\gamma_r = 1.0090$ . (a) Since the runway is at rest in the Earth's frame of reference, 3600 m is its proper length,  $L_0 = 3600$  m. The pilot sees it contracted to  $L = L_0/\gamma = 3567.8$  m. (b) From the point of view of one observer, there is no confusion, elapsed time is just the distance traveled divided by the velocity, an elementary physics result:  $\Delta t = L_0/u = 90 \mu\text{s}$ . (c) Method 1: From the pilot's point of view, the runway is length contracted, but it is moving with speed  $u$ , so the elapsed time is again the elementary result of distance divided by velocity  $\Delta t' = L/u = (3567.8 \text{ m})/(4 \times 10^7 \text{ m/s}) = 89.2 \mu\text{s}$ . Method 2: The pilot is measuring proper time, so his clock runs slow (compared with the clock on the ground), and therefore  $\Delta t' = \Delta t/\gamma = 89.2 \mu\text{s}$ .

Final check: The spacetime interval between the two events (spacecraft at start of runway, and spacecraft at end of runway) must be invariant

$$\begin{aligned} (\Delta t')^2 - \frac{(\Delta x')^2}{c^2} &= (\Delta t)^2 - \frac{(\Delta x)^2}{c^2} \\ (89.2 \mu\text{s})^2 - \frac{(0 \text{ m})^2}{c^2} &= (90.0 \mu\text{s})^2 - \frac{(3600 \text{ m})^2}{c^2} \\ &= (89.2 \mu\text{s})^2. \end{aligned}$$

91. Since length contraction is given by  $L = L_0/\gamma$ , we want  $\gamma = 2$ , and since

$$\beta^2 = 1 - \frac{1}{\gamma^2} = \frac{3}{4},$$

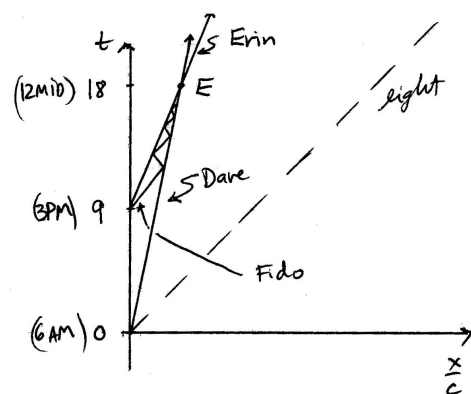
or  $\beta = \sqrt{0.75} = 0.866$ , which means  $v = 2.596 \times 10^8$  m/s.

92. First, let's describe the situation in the “lab” frame, where physics is normal.



Dave's world line has a slope of  $c\Delta t/\Delta x = c/v = 5$ , and he has  $\gamma = \sqrt{25/24}$ . Erin's world line (when she is moving) has a slope of 2.5, and her relativistic factor is  $\gamma = \sqrt{25/21}$ . They meet at Event  $E$ , at 12 midnight in coordinate time. This is all elementary physics - no relativity. Where do they meet? Well, the spatial position is 3.6 hours, because Dave walked for 18 hours at 1 mile per hour, so he covered 18 miles, but dividing by  $c = 5$  mi/hr gives 3.6 hr. The coordinates of  $E$  are thus

$$E = (3.6 \text{ hr}, 18 \text{ hr}).$$



Fido travels at 4 mi/hr, the slope of his world line is  $\pm 1.25$ , depending on which direction he is traveling. He has  $\gamma = 5/3$ . He travels for 9 hours (coordinate time), so he covered a distance of 36 miles. In time units that is 7.2 hours.

To obtain the proper time displayed by the moving clocks, we can use either the spacetime interval, or equivalently, the time dilation relation. Dave is moving the entire time, so his elapsed time is  $\Delta t_0 = (18 \text{ h})/\gamma = 17.64 \text{ h}$ , and at event  $E$  his clock reads 11:38 PM. Erin remains stationary for 9 h, and then her clock starts “dilating” for an additional time  $\Delta t_0 = (9 \text{ h})/\gamma = 8.25 \text{ h}$ . So her total elapsed time is 17.25 h, and her clock reads 11:15 PM. Fido also is stationary for 9 h, and then  $\Delta t_0 = (9 \text{ h})/\gamma = 5.4 \text{ h}$ , so his clock reads 8:24 PM. When they all meet at  $E$ , their clocks are all different!

Note that Erin and Dave are just like the twins Peter and Paul. In Dave's frame, he is at rest and it's Erin who travels away and then returns. Therefore she's the younger “twin,” and she ages 23 minutes less than Dave.

**93.** Approximate answer: The best you can do is to travel at the speed of light, in which case it will take you 400 years (coordinate time) to travel 400 ly, because  $c = 1$  ly/y. But if you wish to age only 10 years, then you need  $\gamma = 40$  to result in the proper time dilation, which gives  $\beta = \sqrt{1599/1600} = 0.999\ 687\ 451$ .

Exact answer: Your speed is  $\beta$ , which means that your total travel time is  $\Delta t = 400$  ly/ $\beta c$ , and the time dilation is given by  $\Delta t = \gamma\Delta t_0$ , where  $\Delta t_0 = 10$  y. This gives  $\gamma\beta = 40$ . Since  $\beta$  is very close to 1, the approximation above is good. An exact answer can be found by squaring  $\gamma\beta = 40$  to give

$$\gamma^2\beta^2 = \frac{\beta^2}{1 - \beta^2} = 40^2,$$

and solving for  $\beta$  gives  $\beta = \sqrt{\frac{1600}{1601}} = 0.999\ 687\ 646$ .

**94.** The  $y'$ ,  $z'$ , and  $t'$  equations can be simply flipped, so that the primed variables are on the left-hand-side. Equation (5.20) can be solved for  $x'$ , and then invoking absolute time gives

$$x' = x - ut' = x - ut.$$

**96.** The most straightforward method is to treat the Lorentz transformation as two equations and two unknowns, with the unknowns being  $x'$  and  $t'$ . This, however, involves

some algebra, so the simplest way is to write the Lorentz transformation as a matrix equation

$$\begin{pmatrix} x \\ t \end{pmatrix} = \gamma_r \begin{pmatrix} 1 & \beta_r \\ \beta_r & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix},$$

where I've set  $c = 1$  for notational clarity. Inverting a  $2 \times 2$  matrix is straightforward, and acting on the above equation — from the left — with the inverse gives

$$\frac{1}{\gamma_r} \frac{1}{(1 - \beta_r^2)} \begin{pmatrix} 1 & -\beta_r \\ -\beta_r & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} x' \\ t' \end{pmatrix},$$

where  $(1 - \beta_r^2)$  is the determinant of the first matrix. This, of course, is the identical transformation, but with the sign of  $\beta_r$  switched. This is because the primed frame and the unprimed frame are identical, except for the relative direction of motion of the other frame. Of course, the coordinates in the directions perpendicular to the relative velocity are still identical

$$y' = y \quad z' = z.$$

**97.** For the  $y$  velocity, we have

$$v_y = \frac{dy}{dt} = \frac{dy'}{dt} = \frac{dy'}{dt'} \frac{dt'}{dt} = v'_y \gamma_r \left(1 - \beta_r \frac{v_x}{c}\right).$$

But we want a transformation equation, which means that we need only primed quantities on the right hand side, so we need to use our previously obtained transformation for  $v_x$

$$\beta_x = \frac{\beta_r + \beta'_x}{1 + \beta_r \beta'_x}$$

which results, after some algebraic rearrangement

$$\beta_y = \frac{1}{\gamma_r} \frac{\beta'_y}{1 + \beta_r \beta'_x}.$$

**98.** The time dilation formula gives  $\Delta t' = \Delta t / \gamma \approx \Delta t \left(1 - \frac{1}{2}\beta^2\right)$ , and the time difference is

$$\Delta t' - \Delta t = -\Delta t \left(\frac{1}{2}\beta^2\right),$$

where for LEO at  $h = 300$  km,  $v = \sqrt{GM_\oplus / (R_\oplus + h)} = 7.732$  km/s. Since  $\Delta t = 3.156 \times 10^7$  s, the difference is  $\boxed{-10.5 \text{ ms}}$ .

However, as Hafele and Keating state in their 1971 report on a clock comparison between commercial airliners and the ground

Special relativity predicts that a moving standard clock will record less time compared with (real or hypothetical) coordinate clocks distributed at rest in an inertial reference space.... Because the earth rotates, standard clocks distributed at rest on the surface are not suitable in this case as candidates for coordinate clocks of an inertial space. Nevertheless, the *relative* [my emphasis] timekeeping behavior of terrestrial clocks can be evaluated by reference to

hypothetical coordinate clocks of an underlying nonrotating (inertial) space.... General relativity predicts another effect that (for weak gravitational fields) is proportional to the difference in the gravitational potential for the flying and ground reference clocks.<sup>25</sup>

We therefore need to compare two different “proper” time intervals

$$\Delta t_{ISS} - \Delta t_{Eq} = \Delta t \left( -\frac{1}{2}\beta_{ISS}^2 + \frac{1}{2}\beta_{Eq}^2 \right),$$

where  $\Delta t \left( \frac{1}{2}\beta_{Eq}^2 \right)$  is a correction term. Given that the equatorial speed of the Earth’s rotation is  $v = 463.3$  m/s, this correction is only +0.038 ms. In addition, the general relativistic correction is  $\Delta t(gh/c^2)$ , where  $g = 9.81$  m/s<sup>2</sup> and  $h = 300$  km, which gives +1.03 ms. Including general relativity, the astronaut’s clock will have lost 9.5 ms. It is probably impossible to tell (without a clock) that a person has aged not one year, but one year minus 9.5 ms.

**100.** (a) Since  $K = mv^2/2$  and  $\vec{p} = m\vec{v}$ , if we square the momentum  $p^2 = \vec{p} \cdot \vec{p} = m^2v^2$ , and eliminate  $v^2$  from both equations, we obtain

$$K = \frac{p^2}{2m},$$

where what we mean by  $p^2$  is  $\vec{p} \cdot \vec{p}$ . (b) The relativistic case is just as straightforward conceptually, but with a little more algebra. In this case, it makes sense to calculate  $p^2c^2$  so that all terms are in “energy units.” Using the same definition for  $p^2$ , I get

$$p^2c^2 = (\gamma^2m^2v^2)c^2 = (mc^2)^2\gamma^2\beta^2 = (mc^2)^2(\gamma^2 - 1),$$

where, in the last step, I’ve used the fact that  $\beta^2 = 1 - (1/\gamma^2)$ . Now, noting that  $E = \gamma mc^2$  so that  $E^2 = \gamma^2(mc^2)^2$ , the momentum can be written  $p^2c^2 = E^2 - (mc^2)^2$ , or

$$E^2 = p^2c^2 + (mc^2)^2.$$

**101.** The fundamental definition of work (in one dimension) is

$$W \equiv \int F dx = \int \frac{dp}{dt} dx = \int dp \frac{dx}{dt} = \int v dp,$$

where I’ve used Newton’s second law in the form  $F = dp/dt$ , as well as a fundamental property of differentials that allows me to let the  $dt$  in the denominator be associated with  $dx$  rather than  $dp$ . Integrating this by parts gives

$$\int v dp = vp - \int p dv.$$

The reason we do this is because we know  $p$  as a function of  $v$ , but not the other way around. The integral becomes

$$\int p dv = \int \gamma mv dv = m \int \frac{v dv}{\sqrt{1 - v^2/c^2}} = mc^2 \int \frac{\beta d\beta}{\sqrt{1 - \beta^2}},$$

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<sup>25</sup>Hafele and Keating, “Around-the-world atomic clocks: Predicted relativistic time gains,” *Science* **177** (4044), 166-168, (1971).

where I've changed variables to obtain the integral given. From the work-kinetic energy theorem,  $W = \Delta K$ , so that we have

$$\Delta K = \left( vp + mc^2 \sqrt{1 - \beta^2} \right)_i^f = \gamma mc^2 \Big|_i^f = (\gamma_{final} - \gamma_{initial}) mc^2.$$

If we take the initial value of the kinetic energy to be zero, then  $K = (\gamma - 1)mc^2$ .

Note that only the change in kinetic energy is determined. The actual value of  $K$  is undetermined up to an additive constant, just as in nonrelativistic Newtonian theory.

**102.** Approximate solution: Since  $\gamma$  can be expanded for small velocities to give

$$\gamma \approx 1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4,$$

the approximate kinetic energy is  $K = \gamma mc^2 - mc^2 \approx \left( \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 \right) mc^2 \approx \frac{1}{2}mv^2$ . If we assume that these two terms give the exact kinetic energy, then when the second term becomes 0.01 of the first term we have  $\frac{3}{8}\beta^4 = (0.01)\frac{1}{2}\beta^2$ , or  $\beta^2 = \frac{4}{3} \times 10^{-2}$ , or  $\beta = 0.11547$ .

Exact solution: We can do this problem very generally, by letting the fractional error be  $f = 0.01$ . The fractional error is defined as

$$\frac{K_r - K_c}{K_r} = f,$$

where  $K_r = (\gamma - 1)mc^2$  and  $K_c = \frac{1}{2}mv^2$ . From here we wish to solve for  $\beta(f)$ , which is essentially algebraic. Simplifying and grouping, I get  $\gamma(1 - f) = (1 - f) + \frac{1}{2}\beta^2$ . Squaring and writing  $\gamma$  in terms of  $\beta$ , I get a “bi-cubic” equation in  $\beta$

$$-\frac{1}{4}\beta^6 + \left[ \frac{1}{4} - (1 - f) \right] \beta^4 + \left[ (1 - f) - (1 - f)^2 \right] \beta^2 = 0.$$

Since there is no constant term, we can divide by  $\beta^2$  to get a “bi-quadratic”,

$$\beta^4 + (3 - 4f)\beta^2 - 4f(1 - f) = 0,$$

and solving for the two roots (when  $f = 0.01$ ) gives

$$\beta^2 = 1.33 \times 10^{-2}, \quad -2.97$$

Only the positive root makes sense ( $\beta$  must be real), I get  $\beta < 0.11541$ , or, when  $v$  is 11% of  $c$ , then the true kinetic energy is 1% larger than  $\frac{1}{2}mv^2$ . The solution to the biquadratic, for arbitrary  $f$ , is

$$\beta^2 = \frac{1}{2} \left[ \pm \sqrt{9 - 8f} - (3 - 4f) \right],$$

where we are interested in the positive root. If  $f$  is small, then we can make the approximation  $\sqrt{9 - 8f} = 3\sqrt{1 - 8f/9} \approx 3(1 - 4f/9)$ , and we get  $\beta^2 \approx 4f/3$ , so if  $f = 0.01$  then  $\beta = 0.115$ , as in the approximate solution.

Approximate solution revisited: Starting from the first equation  $(\gamma - 1)(1 - f) = \frac{1}{2}\beta^2$ , we can expand  $\gamma - 1$  for small velocities,

$$\gamma - 1 = (1 - \beta^2)^{-1/2} - 1 \approx \left( 1 - \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \dots \right) - 1,$$

where we only need to retain the  $\beta^4$  term. Plugging in, and retaining only the lowest order terms, gives  $\beta^2 = 4f/3$ , as above. Hence, both methods gives the same result when terms of order  $\beta^4$  are retained.

**103.** In the ultra-relativistic limit,  $E_u \approx pc$ . Since  $E > E_u$  is always true, we are looking for the velocity such that

$$\frac{E - E_u}{E} = f,$$

where  $f = 0.01$ . That is, what speed results in a 1% error for  $E_u$ ? The condition is

$$\sqrt{p^2c^2 + (mc^2)^2} - pc = f\sqrt{p^2c^2 + (mc^2)^2},$$

and rearranging and squaring gives

$$p^2c^2 = (mc^2)^2 \frac{(1-f)^2}{f(2-f)}.$$

On the other hand,  $p = \gamma mv$ , so that  $p^2c^2 = \gamma^2\beta^2(mc^2)^2$ , so that we really have to solve an equation for  $\gamma\beta$ , just like in Problem 93. That equation is

$$\gamma^2\beta^2 = \frac{(1-f)^2}{f(2-f)} = 49.25,$$

where the numerical value is for  $f = 0.01$ . As before, if  $\gamma^2\beta^2 = x$ , then  $\beta^2 = x/(x+1)$ , or

$$\beta = 1 - f = 0.99.$$

So you can't be moving slower than 99% of the speed of light.

**105.** Setting  $c = 1$  changes Eq. (5.68b) to

$$p_+^2 = (p - p_-)^2,$$

and plugging this into Eq. (5.69) results in

$$E = \sqrt{(p - p_-)^2 + m^2} + \sqrt{p_-^2 + m^2}.$$

In this equation, all quantities are measured in energy units. The easiest way to solve this is to bring the second term on the right side of the equation over to the left side, and then square both sides

$$\left(E - \sqrt{p_-^2 + m^2}\right)^2 = (p - p_-)^2 + m^2.$$

Expanding the squared terms on both sides gives

$$E^2 - 2E\sqrt{p_-^2 + m^2} + p_-^2 + m^2 = p^2 - 2pp_- + p_-^2 + m^2,$$

and the last two terms on each side cancel. Again, isolating the square root on one side, and then squaring gives

$$\left(E^2 - p^2 + 2pp_-\right)^2 = \left(2E\sqrt{p_-^2 + m^2}\right)^2 = 4E^2(p_-^2 + m^2).$$

Here, we can make a simplification by noting that  $E^2 - p^2 = M^2$ , where  $M$  is the mass of the kaon. Grouping terms results in the quadratic equation

$$0 = 4M^2 p_-^2 - 4M^2 p p_- + (4E^2 m^2 - M^4),$$

or

$$0 = p_-^2 - p p_- + (E^2 m^2 / M^2 - M^2 / 4).$$

Restoring the factors of  $c$  results in Eq. (5.70)

$$(p_- c)^2 - (p c)(p_- c) + \left( \frac{E^2 (m c^2)^2}{(M c^2)^2} - \frac{(M c^2)^2}{4} \right) = 0.$$

**106.** The momentum  $pc$  is a function of the velocity  $\beta$  in the following way

$$pc = (\gamma m v)c = \gamma \beta m c^2 = \frac{\beta m c^2}{\sqrt{1 - \beta^2}}. \quad (5.76)$$

Using our knowledge of  $\beta_{\pm}$  these momenta are just 667.9 MeV/c and  $-12.7$  MeV/c, exactly as calculated previously. You can also confirm that the energies are identical.

# Chapter 6

## Introduction to Quantum Physics

*We have not assumed that the quantum theory, as opposed to the classical theory, is essentially a statistical theory, in the sense that only statistical conclusions can be drawn from the exact data . . . In the formulation of the causal law, namely, ‘If we know the present exactly, we can predict the future,’ it is not the conclusion, but rather the premise that is false. We cannot know, as a matter of principle, the present in all its details.*

— Werner Heisenberg

### 6.1 Wave-particle duality

The fundamental property of quantum physics, as opposed to classical physics (i.e., Newton’s laws regarding the motion of point particles), is the wave-particle duality of all objects: photons, electrons, etc. This duality can be expressed by the two fundamental equations of quantum physics<sup>1</sup>

$$E = h\nu, \tag{6.1}$$

$$p = \frac{h}{\lambda}. \tag{6.2}$$

These equations relate properties that are usually associated with infinite plane waves, frequency  $\nu$  and wavelength  $\lambda$ , with properties that are usually associated with discrete particles, energy  $E$  and momentum  $p$ . Of course, Maxwell’s equations of electrodynamics also predict that electromagnetic waves carry energy and momentum, but Eqs. (6.1) and (6.2) imply that these properties are discrete, not continuous.

Recall that Eq. (6.1) was an *ad hoc* hypothesis, put forward by Planck to understand the blackbody spectrum, and confirmed by Einstein and Bohr in their theoretical models of

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<sup>1</sup>If you prefer  $\hbar$  rather than  $h$ , these can be written

$$\begin{aligned} E &= \hbar\omega, \\ \vec{p} &= \hbar\vec{k}. \end{aligned}$$

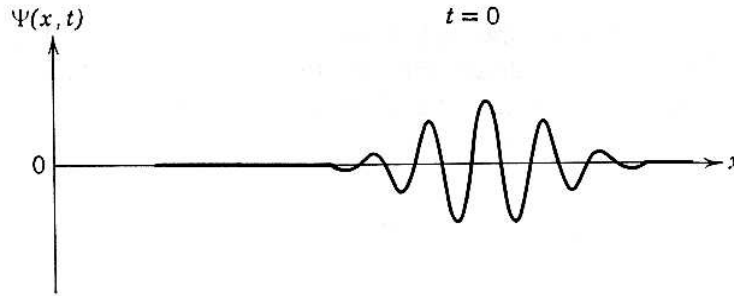


Figure 6.1: Snapshot of a one-dimensional wave function  $\Psi(x, t)$  at  $t = 0$ , meant to describe an electron. From Eisberg and Resnick, *Quantum Physics*, Figure 3-7.

the photoelectric effect and the hydrogen atom. On the other hand, Eq. (6.2) was proposed by de Broglie so that the symmetry between light and matter would be complete.

In any case, let's investigate some of the consequences of these two expressions of wave-particle duality. Consider the electron from de Broglie's point of view. He claimed that, in addition to its particle properties, it had wave properties as well. An interesting question to ask is, "What is the velocity of an electron?" Assuming that it is a wave, the velocity of an electron is presumably the velocity of the wave that describes that electron,  $v_w = \lambda\nu$ , where I've used  $v_w$  to denote the "wave velocity." On the other hand, assuming that it is a particle,<sup>2</sup> we can obtain  $v_p$ , the "particle velocity," from a knowledge of either the momentum,  $p = mv_p$ , or the kinetic energy,  $E = mv_p^2/2$ . How are these two velocity definitions related? Well, we can use our wave-particle duality equations to obtain the relation

$$v_w = \lambda\nu = \frac{h}{p} \frac{E}{h} = \frac{h}{mv_p} \frac{mv_p^2/2}{h} = \frac{v_p}{2}. \quad (6.3)$$

This seems to be saying that the wave velocity is one half the of the particle velocity! If these two pictures of the electron are to be consistent, then they should predict the same velocity.

From our extensive experience with baseballs and blocks on inclined planes we know what  $v_p$  means, but what exactly does  $v_w$  describe? The "wave velocity" in the equation  $v_w = \lambda\nu$  is actually the *phase velocity* of an infinite plane wave. That is, it is the velocity of the peaks and troughs of a sinusoidal oscillation that extends indefinitely in both directions. Even if you concede that an electron has a de Broglie wavelength, this *can't* be what is meant, because we know that electrons are localized objects. Detectors, Geiger counters for example, emit a discrete "click" when they observe an electron. They don't emit a continuous hum. Therefore, in order to describe an electron as a wave, what we need is a localized oscillation, a "wave packet," shown in Fig. 6.1. This figure depicts a *wave function*  $\Psi$  (we'll see what this means in Chapter 7) that is meant to describe an electron. It oscillates with a reasonably well-defined wavelength, but an amplitude that is only large in a small region of space. That is, it is localized: it is only nonzero where we might expect

<sup>2</sup>In this book I will consider only nonrelativistic quantum mechanics. For this reason, we can describe kinematics with nonrelativistic theory.





Figure 6.2: This is a plot of  $\Psi$  in Eq. (6.6) for  $t = 0$ , which is a “snapshot” of the beat phenomenon. The width of one “packet,” denoted by the horizontal line, is  $\Delta x$  as in Eq. (6.7), which can be expressed as  $2\pi/\Delta k$ .

to find the electron.

How can we create a wave packet mathematically? The answer is similar to the mathematics involved in the phenomenon of “beats.” There, two plane waves traveling in the same direction but with slightly different frequencies and wavelengths were added together (recall the linear superposition property of waves that allowed us to do that). Due to the interference between the two plane waves, the result was a function that could be expressed as a single wave with an amplitude that was modulated. In order to obtain the wave packet depicted in Fig. 6.1, however, we would need to add an *infinite* number of plane waves, each with a different wavelength and frequency (and possibly a different amplitude), so that constructive interference will occur near the location of the electron, and destructive interference will occur elsewhere. The addition of an infinite number of quantities means, of course, integration, and we’ll do that integration in Chapter 7, but for now we can get a feeling for the physical concept by reproducing beats, i.e., by adding just two waves. If the wave function of the electron  $\Psi$  is the sum of two plane waves with equal amplitudes  $\Psi_0$ , then we can write<sup>3</sup>

$$\Psi(x, t) = \Psi_0 \sin(k_1 x - \omega_1 t) + \Psi_0 \sin(k_2 x - \omega_2 t). \quad (6.4)$$

If we define the average wavenumber and average frequency as

$$\bar{k} \equiv \frac{k_1 + k_2}{2}, \quad \bar{\omega} \equiv \frac{\omega_1 + \omega_2}{2}, \quad (6.5)$$

then using a standard trigonometric identity<sup>4</sup> (as in the beat problem), I can express Eq. (6.4) as

$$\Psi(x, t) = \left[ 2\Psi_0 \cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right) \right] \sin(\bar{k}x - \bar{\omega}t), \quad (6.6)$$

where  $\Delta k \equiv (k_1 - k_2)$  and  $\Delta\omega \equiv (\omega_1 - \omega_2)$ . Figure 6.2 depicts  $\Psi$  at an instant of time. Just as in the physics of beat phenomena, the quantity in the square brackets is slowly varying in both space and time, and therefore can be interpreted as an amplitude, sometimes called an “envelope,” so that the result of adding two waves is a single wave

<sup>3</sup>We’ll see in Chapter 7 that what we mean by  $\Psi$  is the following: if  $\Psi$  is a function that represents the particle we are interested in, then  $|\Psi(\vec{r}, t)|^2 d^3r$  is the probability of finding that particle in a small volume  $d^3r = dx dy dz$  surrounding the position  $\vec{r}$ . In this interpretation,  $|\Psi(\vec{r}, t)|^2$  is called the probability density.

<sup>4</sup> $\sin a + \sin b = 2 \cos\left(\frac{a-b}{2}\right) \sin\left(\frac{a+b}{2}\right)$

(with a wavenumber and frequency equal to the *average* of the two original waves) whose amplitude is modulated at the *difference frequency*, or beat frequency,  $\omega_{beat} = |\omega_1 - \omega_2|$ . Or, if you were to take a snapshot of the wave function at a particular time, its amplitude would be modulated at the difference wavenumber,  $k_{beat} = |k_2 - k_1|$ . In this case, the modulation continues undiminished infinitely far away, which is why this mathematical model is not a good description of a *localized* electron. However, we can take just one period of the amplitude modulation, and consider that to be an approximation to the wave packet we are looking for.

This approximation to a localized wave packet — one modulation of Eq. (6.6) — is a good way to introduce two key physical concepts: group velocity and the Heisenberg uncertainty principle.

## 6.2 Dispersion, phase velocity, and group velocity

To determine how our wave packet moves, we first need to calculate the distance between neighboring minima of the envelope function, i.e., the quantity in square brackets in Eq. (6.6). Since the cosine function is zero when its argument is  $\pm\pi/2$ , and if we take the time of the snapshot to be  $t = 0$ , then  $x = \pm\pi/\Delta k$  must denote the positions of two neighboring minima of the envelope function. The distance between these minima is

$$\Delta x = \frac{2\pi}{\Delta k} = \frac{4\pi}{k_1 - k_2}. \quad (6.7)$$

This distance, then, is the spatial size of our wave packet. That is, the localization of the electron is determined by the spread in the wavelengths of the waves that we have used to construct our packet. If the wavelengths are widely separated (that is,  $[k_1 - k_2]$  is large) then  $\Delta x$  is small, and if the wavelengths are close ( $[k_1 - k_2]$  is small) then  $\Delta x$  is large.

What about the speed of the wave packet? The *phase* velocity of the wave is defined as

$$v_{ph} = \frac{\bar{\omega}}{\bar{k}}, \quad (6.8)$$

because this is the speed of the peaks and troughs of the oscillation. But how fast does the *envelope* move? This is the speed that any information carried by the wave can travel, and it is called the *group velocity*,  $v_g$ . The speed of the envelope function is just the ratio of the coefficients of  $x$  and  $t$ , which is

$$v_g \approx \frac{\Delta\omega}{\Delta k}. \quad (6.9)$$

Recall, though, that this is just an approximation. The correct method is to add many different waves (an infinite number) whose wavelengths and frequencies are very close together. In this limit, the  $\Delta$  becomes an infinitesimal  $d$ , so that the correct expression for the group velocity is a derivative

$$v_g \equiv \left. \frac{d\omega}{dk} \right|_{k_0}, \quad (6.10)$$

where the derivative is evaluated at the center of all the wavenumbers,  $k_0$ .

What does this derivative mean? In an operational sense, in order to evaluate the derivative, we first need to know  $\omega$  as a function of  $k$ . This function,  $\omega(k)$ , is called the “dispersion relation.” For light traveling in a vacuum, the relationship between  $\omega$  and  $k$  is just  $c = \omega/k$ , which can be rewritten as

$$\omega = ck, \quad (6.11)$$

which is a simple dispersion relation. Taking the required derivative we find that

$$v_g = \frac{d\omega}{dk} = c. \quad (6.12)$$

This just states that *all* wave packets travel at the same group velocity, regardless of the frequency or wavelength. Since dispersion describes the case where the phase velocity *does* depend on wavelength, this situation is said to have no dispersion, or to be “dispersionless.”

The term *dispersion* comes from Newton’s investigations into the color of sunlight. He showed that sunlight is composed of all colors by passing sunlight through a glass prism which spread out, or dispersed, the different colors of the rainbow. Why are different wavelengths of light refracted differently? Because they have different speeds in the glass. Or, as you have learned in your studies of optics, the index of refraction of glass is a function of wavelength. One question that we can ask, but Newton couldn’t (because he viewed light as made up of “corpuscles,” or particles, not waves), is “How are the phase velocity and group velocity related to the index of refraction?” The definition of the refractive index  $n$  is

$$n \equiv \frac{c}{v_p}, \quad (6.13)$$

where  $c$  is the speed of light in vacuum, and  $v_p$  is the speed of light in the material in question. As you might have guessed, it is the *phase* velocity,  $\omega/k$ , of electromagnetic waves in that medium. Solving for  $\omega$  to obtain the dispersion relation gives

$$\omega = \frac{ck}{n(k)}, \quad (6.14)$$

where, for example, the function  $n$  for borosilicate crown glass is shown in Fig. 6.3. If the function  $n(k)$  is known, then the group velocity of a wave packet can be calculated, assuming that the wavenumber content of the packet is known.

What about an electron matter wave? What is the correct dispersion relation? To determine this, an analysis similar to what was used in Eq. (6.3) will give us the function  $\omega(k)$ . That is, the dispersion relation is nothing but the relationship between energy and momentum, but written in the language of waves using Eqs. (6.1) and (6.2). That is

$$E = \frac{p^2}{2m} \quad \text{becomes} \quad \hbar\omega = \frac{(\hbar k)^2}{2m} \quad \text{or} \quad \omega = \frac{\hbar}{2m} k^2. \quad (6.15)$$

How can we interpret this electron dispersion relation? First, we see that the phase velocity of the electron wave is not constant (as with light in a vacuum) but depends on its wavelength (or its momentum, as de Broglie would say)

$$v_{ph} \equiv \frac{\omega}{k} = \frac{\hbar k}{2m}. \quad (6.16)$$

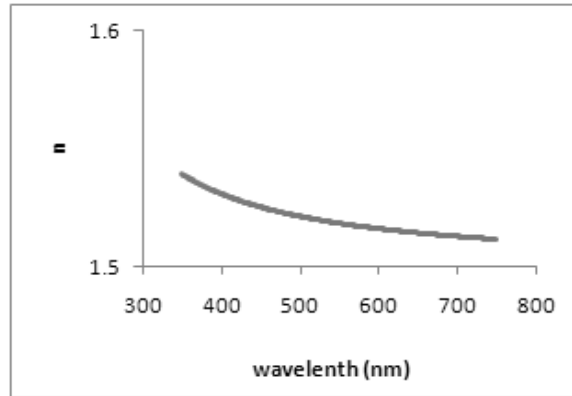


Figure 6.3: The index of refraction  $n$  as a function of wavelength  $\lambda$  for BK7, borosilicate crown glass, a mixture of boron, oxygen, sodium, aluminum, silicon, and potassium. Indices of refraction are determined empirically for transparent solids using the Sellmeier equation

$$n^2(\lambda) = 1 + \sum_i A_i \frac{\lambda^2}{\lambda^2 - \lambda_i^2}$$

where  $\lambda_i$  are the wavelengths where  $n$  becomes infinite and the phase velocity goes to zero. These points are called absorption “resonances.” They occur where the driving frequency of the light matches the natural frequency of oscillation of the atoms in the solid, and the Sellmeier equation is valid only when  $\lambda$  is not near any of these resonances. They typically are in the UV when they match a frequency that is resonant with the light electrons, and are in the IR when they resonate with the heavier ions. For BK7 the resonances are at  $\lambda_1 = 77.5$  nm,  $\lambda_2 = 141.5$  nm, and  $\lambda_3 = 10.2$   $\mu$ m. The first two are in the ultraviolet, and the last is in the infrared.

Second, the group velocity can be determined via differentiation

$$v_g \equiv \frac{d\omega}{dk} = \frac{\hbar k}{m} = \frac{p}{m} = \frac{mv}{m} = v_p, \quad (6.17)$$

where the last equalities were obtained using the de Broglie wavelength definition and the formula for the nonrelativistic momentum. Note that the group velocity is *equal* to the particle velocity! That is, the wave packet travels at the speed we would expect if the electron were a particle. So this is what we mean when we say that the electron is a wave:

An electron is a wave *packet* governed by a dispersion relation that results in a group velocity equal to the particle velocity.

### Dispersion relations

Equation (6.15),  $\omega(k) = \hbar k^2/2m$ , is the dispersion relation for nonrelativistic electron matter waves. From a knowledge of  $\omega(k)$ , both the phase velocity and group velocity can be obtained. In this case, the group velocity is *larger* than the phase velocity; specifically,

$v_g = 2v_{ph}$ . This means that the envelope travels faster than the peaks and troughs of the oscillation. Although Fig. 6.1 is only a snapshot, if we were to watch the packet as a function of time, the peaks and troughs would appear to drift backward (with respect to the envelope) as the entire packet traveled forward. This is not always the case. For example, you can show (Problem 113) for surface waves in water that the phase velocity is larger than the group velocity. This means that peaks and troughs are “born” near the back of the envelope, grow in size and move faster than the envelope, and then disappear in the front of the envelope.

Another interesting case of dispersion is that of electromagnetic waves in a plasma. This dispersion must be taken into account in order to understand, for example, radio waves moving through (or reflecting off) the Earth’s ionosphere. The dispersion relation of these waves is

$$\omega^2 = \omega_p^2 + k^2 c^2, \quad (6.18)$$

where  $\omega_p$  is the “plasma frequency,” the natural frequency of electric oscillations of the plasma medium. It is defined as

$$\omega_p^2 = \frac{ne^2}{\epsilon_0 m_e},$$

where  $n$  is the electron number density and  $m_e$  is the electron mass. Notice that if  $\omega \gg \omega_p$ , then the wave propagates as if it were in a vacuum,  $\omega \approx kc$ . The structure of Eq. (6.18) means that the phase velocity is *greater* than  $c$

$$\frac{\omega^2}{k^2} = c^2 + \frac{\omega_p^2}{k^2} > c^2. \quad (6.19)$$

This means that, in contrast to “normal” media, the index of refraction — see Eq. (6.14) — is *less* than unity

$$n = \frac{kc}{\omega} = \frac{kc}{\sqrt{k^2 c^2 + \omega_p^2}} < 1. \quad (6.20)$$

Of course, information does not travel at a speed greater than  $c$ , because, as you will show in Problem 111, the group velocity is *less* than  $c$ .

The interpretation of a dispersion equation such as Eq. (6.18) can be done in two different ways. First, we can treat it as an *initial value problem*, where we assume that there is some small disturbance with a particular wavelength (and hence wavenumber) superimposed on an otherwise uniform plasma. Then the dispersion relation tells us the frequency that the medium will oscillate at. For Eq. (6.18), if  $k = 0$ , which means that disturb the entire plasma uniformly, then it “rings” at the plasma frequency. Smaller wavelengths ring at higher frequencies. The second method is to treat it as a *boundary value problem*. Suppose we have an antenna at the edge of our plasma that is emitting waves at a frequency  $\omega$ . Then, solving Eq. (6.18) for  $k$  tells us the wavelength of the plasma response. For example, if our antenna oscillates with a frequency  $\omega > \omega_p$ , then a wave with wavenumber  $k$ , given by Eq. (6.18), will propagate away from the antenna into the plasma. However, if  $\omega < \omega_p$ , then the wave cannot propagate at all because  $k$  is imaginary

$$k = \frac{\sqrt{\omega^2 - \omega_p^2}}{c}. \quad (6.21)$$

Such a wave is called “evanescent,” and the amplitudes of the electric and magnetic fields oscillate at  $\omega$  but decay exponentially away from the antenna rather than propagating.

### 6.3 The Heisenberg uncertainty principle

We discussed several versions of the Heisenberg uncertainty principle in Chapter 2, but we have not yet been able to understand the reasons *why* it holds, until now. The reason for such an “indeterminacy” principle to hold in quantum physics is that objects are described with a language appropriate to waves, not particles. Remember that we do observe particle-like behavior, which means that any wave packet must be localized (i.e., it is only spread over a distance  $\Delta x$ ). But in order to mathematically describe a localized oscillation, a pure plane wave of only one wavelength will not suffice. We *must* add more than one plane wave, which means that we don’t know exactly what wavelength we have — there is a spread in wavelengths and wavenumbers ( $\Delta k$ ). But de Broglie says that if there is a spread in wavelengths, there must be a spread in momenta ( $\Delta p$ ). We can show with our simple model of two plane waves — Eq. (6.6) — that there is an inverse relationship between  $\Delta x$  and  $\Delta p$ . In fact, one way of looking at Heisenberg’s inequality is that it is simply a consequence of the wave description.

Looking at Fig. 6.2 and focusing on one modulation of the sinusoidal wave, Eq. (6.7) gives the length of that one modulation. Since we don’t know where the electron is within that modulation (all we know is that it *is* within it), that must be the uncertainty in position

$$\Delta x = \frac{2\pi}{\Delta k}. \quad (6.22)$$

On the other hand, since we don’t know the exact wavenumber (there is an uncertainty  $\Delta k$  in wavenumber), there must be an uncertainty in momentum, given by de Broglie

$$\Delta p = \hbar \Delta k. \quad (6.23)$$

Eliminating  $\Delta k$  from these two equations results in

$$\Delta p \Delta x = h. \quad (6.24)$$

For this particular wave packet, the more localized it is, the less spread out the momentum content of the packet will be, such that the product of  $\Delta p$  and  $\Delta x$  is always Planck’s constant. For wave packets with other shapes, the product of  $\Delta p$  and  $\Delta x$  will have a different value, although it will always be proportional to  $h$ . Of course, in this example, the uncertainties in  $x$  and  $k$  are just approximate.

Heisenberg developed a general rule that says for *any* wave packet the product must always be greater than  $\hbar/2$

$$\Delta p \Delta x \geq \frac{\hbar}{2}. \quad (6.25)$$

#### Rigorous mathematics

What exactly is meant by  $\Delta p$  and  $\Delta x$ ? The rigorous answer is found in analyzing the statistics of many measurements, formally similar to looking at the radioactive decay of

many nuclei, as considered in the calculation of average lifetime on page 68. The way to calculate probabilities is to envision a large number of identical copies of a system. On each of these copies a measurement of some quantity is made, and in general, a different outcome is obtained from each measurement.<sup>5</sup> Here, we are measuring the momentum  $p$  and the position  $x$ . If we wish to measure the quantity  $A$ , then the  $i$ th measurement is  $A_i$ , and the average, or mean, will be the sum of the measured values divided by the number of measurements:  $A_{ave} = \sum_{i=1}^N A_i/N$ . For example, if you were to throw a standard, six-sided die 10 times and obtain the following values: 2, 4, 5, 3, 4, 1, 2, 3, 3, and 6, then the mean value would be 3.3. What would your theoretical prediction of this mean value be? Since the probabilities of obtaining each of the six different outcomes are identical,  $\frac{1}{6}$ , the predicted mean,  $\langle A \rangle$ , would be

$$\langle A \rangle = \sum_{i=1}^N P_i A_i, \quad (6.26)$$

where  $P_i$  is the probability of obtaining the  $i$ th outcome.<sup>6</sup> In the case of the die, this sum would be 3.5.

What is the deviation of the  $i$ th measurement from the mean? This deviation is defined as

$$\Delta A_i = A_i - \langle A \rangle. \quad (6.27)$$

However, since each deviation is different, it is useful and interesting to calculate the mean deviation. Some measurements will be larger than the mean (a positive deviation) and some will be smaller (a negative deviation) so that the average deviation should be zero, if we have defined the mean properly. In fact, this is true, and it can be shown

$$\begin{aligned} \langle \Delta A \rangle &= \sum_{i=1}^N P_i (A_i - \langle A \rangle) \\ &= \langle A - \langle A \rangle \rangle \end{aligned} \quad (6.28)$$

$$\begin{aligned} &= \langle A \rangle - \langle A \rangle \\ &= 0, \end{aligned} \quad (6.29)$$

where the third line is obtained by noting that that

$$\langle \langle A \rangle \rangle = \sum_{i=1}^N P_i \langle A \rangle = \langle A \rangle \sum_{i=1}^N P_i = \langle A \rangle. \quad (6.30)$$

If the mean deviation is zero, how are we to determine the spread in the values of the measurement? The best way is to square the deviations  $(\Delta A_i)^2$ , which is positive definite, and then take its average.<sup>7</sup> This quantity leads to the “standard deviation,” or

<sup>5</sup>see Taylor, *An Introduction to Error Analysis*, Chapter 4, for more complete information.

<sup>6</sup>Technically, I should divide by  $\sum_i P_i$  to “normalize” the mean, but if this sum is unity, there is no need.

<sup>7</sup>Another method is to consider the absolute value of the deviation  $|\Delta A_i|$ , which is also positive definite, but the average of the square is easier to work with. In some texts, the quantity  $\langle |\Delta A| \rangle$  is called the “average deviation.”

the “root-mean-square deviation,” because of the order of operations. The square of the  $i$ th deviation is

$$(\Delta A_i)^2 = A_i^2 - 2A_i\langle A \rangle + \langle A \rangle^2, \quad (6.31)$$

which, when averaged becomes

$$\langle (\Delta A)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2, \quad (6.32)$$

where again I’ve used the fact that  $\langle \langle A \rangle \rangle = \langle A \rangle$ . Taking the square root gives the standard deviation

$$\Delta A_{rms} = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}. \quad (6.33)$$

This quantity, the standard (or *rms*) deviation, is the correct quantity for use in the Heisenberg uncertainty relation. For example, the well-known position-momentum uncertainty relation should read

$$\Delta x_{rms} \Delta p_{x,rms} \geq \frac{\hbar}{2}. \quad (6.34)$$

How can we translate these statistical concepts to the prediction of quantum mechanical measurements? In quantum mechanics, the average is just the expectation value given in Eq. (7.59). The expectation value of the square of an observable quantity is defined as

$$\langle A^2 \rangle = \int \Psi^* (\hat{A}\hat{A}\Psi) d^3r, \quad (6.35)$$

where the operator  $\hat{A}$  acts twice in succession on the wave function  $\Psi$  in the integrand. This is an extension of the definition of probability defined in the footnote on page 167.

## 6.4 Bohr’s Principle of Complementarity

In the early days of quantum mechanics, and still today, it is not completely clear what it all means. That is, how to calculate the predicted probability of an experimental outcome is straightforward, and these predictions agree with the experiments quite well. But what exactly does the wave function  $\Psi$  represent? Why is there a wave-particle duality? These questions have plagued scientists, and philosophers of science for over 100 years.

Niels Bohr struggled with these ideas, and in an attempt to reconcile the mathematics of quantum mechanics with our fundamental philosophical notions learned from a study of classical mechanics (causality, etc.), he developed his “principle of complementarity.” Essentially it states that the wave picture and the particle picture are complementary ways to view an electron, say, and they are both necessary for a complete picture, but can only be used one at a time. For example, when light is traveling in space it can be viewed as a wave, but when it interacts with matter, it must be viewed as a particle, a photon. The photoelectric effect provides a good example. You can measure the wavelength of the incoming ultraviolet light with a diffraction grating — hence, it must be a wave — but when it hits the metal plate you must interpret it as a particle, a quantum of energy.

Bohr’s 1947 version of his principle is



“The very nature of the quantum theory...forces us to regard the space-time coordination and the claim of causality, the union of which characterizes the classical theories, as complementary but exclusive features of the description, symbolizing the idealization of observation and definition, respectively.”<sup>8</sup>

If you are interested in this philosophical issue, probably Bohr’s most complete version is contained in his article in the book *Albert Einstein: Philosopher-Scientist*.

### Collateral Reading

- “The Duality in Matter and Light,” by Berthold-Georg Englert, Marlan O. Scully, and Herbert Walther, *Scientific American*, December 1994, pages 56-61.
- *Albert Einstein: Philosopher-Scientist*, Paul Schilpp, editor, MFJ Books, 1949. (ERAU: QC 16.E5 A43 2000)

### Problems

**107.** Calculate the de Broglie wavelength of (a) Usain Bolt when he is breaking the world record running the 100-m sprint in 9.69 s? (b) a typical nitrogen molecule in the air at room temperature? (c) a proton in the LHC, as in Problem 87.

**108.** The latter part of Eq. (6.3) is a relation between the de Broglie wavelength of an electron and its particle velocity  $v_p$ . (a) Obtain a formula that expresses the de Broglie wavelength  $\lambda$  as a function of the kinetic energy  $K$  (and the mass  $m$ ) of the electron. (b) Repeat part (a), but treat the electron as fully relativistic.

**109.** Make a plot of de Broglie wavelength versus kinetic energy for (a) electrons and (b) protons. Restrict your range of energy values to the nonrelativistic regime, i.e., restrict the kinetic energy to 5% of the rest energy,  $K \leq 0.05E_0$ . Be sure to properly label your axes.

**110.** Show that the dispersion relation for a fully relativistic particle can be written

$$\omega^2 = \omega_0^2 + k^2 c^2,$$

where  $\omega_0 = mc^2/\hbar$ .

**111.** The dispersion relation for a relativistic electron is formally identical to the dispersion relation that describes electromagnetic waves propagating in a plasma

$$\omega^2 = \omega_p^2 + k^2 c^2,$$

where  $\omega_p$  is the plasma frequency, and is constant, just like  $\omega_0$ . Obtain the phase velocity and group velocity of these waves, as well as the relationship between the two velocities.

**112.** Use the relativistic expressions for energy and momentum of an electron to prove that the group velocity  $v_g$  of a wave packet equals the particle velocity  $v$  of the electron. You can use the results of Problem 111.

<sup>8</sup>Bohr, *Nature*, **121** 580 (1928); quoted in Pais, *Inward Bound*, page 262.

**113.** For deep water waves (i.e., where the wavelength of the waves is much smaller than the depth of the water, typical of waves you generate by splashing in a pool), the dispersion relation is

$$\omega^2 = gk,$$

where  $g$  is, of course, the acceleration due to gravity. Show that the phase velocity is twice the group velocity, or  $v_{ph} = 2v_g$ . This means that crests and troughs move *faster* than the envelope of the wave packet.<sup>9</sup>

**114.** The dispersion relations in Problem 113 were both approximations. For surface water waves of any wavelength, the dispersion relation is

$$v_{ph} = \frac{\omega}{k} = \sqrt{\frac{g \tanh(kh)}{k}}.$$

Obtain the group velocity and the effective index of refraction for this dispersion relation.

**115.** Show that the smallest possible uncertainty in the position of a particle of mass  $m$  whose speed is  $\beta$  is

$$\Delta x_{min} = \frac{h}{4\pi mc} \sqrt{1 - \beta^2}.$$

Analyze this result in the limits  $\beta \rightarrow 0$  and  $\beta \rightarrow 1$ .

**116.** An electron moves between two walls (in one dimension) that are a distance  $L$  apart, and bounces elastically from the walls. Except for the walls, there are no forces on the electron. (a) If the electron can be represented by a matter wave  $\Psi$  with a de Broglie wavelength, and the matter wave has a node at each wall, show that the permitted de Broglie wavelengths are

$$\lambda = \frac{2L}{n} \quad n = 1, 2, 3, \dots$$

(b) If  $L = 1 \text{ \AA}$  (the size of an atom), calculate the allowed values of the kinetic energy of the electron.

**117.** If an electron is confined to a region of space the size of an atom,  $1 \text{ \AA}$ , (a) what is the uncertainty in the momentum of the electron,  $\Delta p$ ? (b) What is the kinetic energy of an electron with momentum  $\Delta p$ ? (c) Does this give a reasonable value for the kinetic energy of an electron in an atom? Why or why not?

**118.** Make the same calculations as in Problem 117 but for a proton confined in a uranium nucleus.

**119.** Measurements of an observable  $A$  will take on a range of values. The average value, or *expectation value*, is denoted  $\langle A \rangle$ , and each measurement can be expressed as the expectation value plus some deviation  $\Delta A$ ,

$$A = \langle A \rangle + \Delta A.$$

---

<sup>9</sup>For waves typical of tsunamis, the wavelength  $\lambda$  is much *larger* than the water depth  $h$ , and these waves are dispersionless

$$\frac{\omega}{k} = \sqrt{gh}.$$

This can be considered to be the definition of  $\Delta A$ . Show that

$$\langle(\Delta A)^2\rangle = \langle A^2\rangle - \langle A\rangle^2.$$

Note that  $\langle\Delta A\rangle = 0$ . Note also that  $\sqrt{\langle(\Delta A)^2\rangle}$  is just the “root-mean-square” of  $\Delta A$ .

## Solutions

**107.** (a) Usain Bolt is 86 kg, and is moving at 10.32 m/s, which means  $\lambda = h/mv = 7.5 \times 10^{-37}$  m. (b) The “root-mean-square” speed of a thermal particle is

$$v_{rms} = \sqrt{\frac{3kT}{m}} = c\sqrt{\frac{3kT}{mc^2}},$$

where  $kT \approx \frac{1}{40}$  eV, and, since the molar mass of  $N_2$  is 28 g,  $mc^2 = 28 \times 931$  MeV, I get  $\beta = 1.7 \times 10^{-6}$ . The nonrelativistic de Broglie wavelength can be written

$$\lambda = \frac{hc}{\beta E_0},$$

where  $E_0 = mc^2$ . I get  $\lambda = 0.28$  fm. (c) The previous particles were both nonrelativistic, but this one is relativistic, so that the relativistic expression for momentum must be used,  $p = \gamma mv$ , and therefore

$$\lambda = \frac{hc}{\gamma\beta E_0}.$$

Here, since  $\beta = 1 - 9 \times 10^9$ , it means that  $\gamma = 7.45 \times 10^3$ . I get  $\lambda = 1.77 \times 10^{-19}$  m.

**108.** (a) The nonrelativistic relation between  $K$  and  $p$  is  $K = \frac{p^2}{2m}$ , so that  $p = \sqrt{2mK}$  and the de Broglie wavelength is

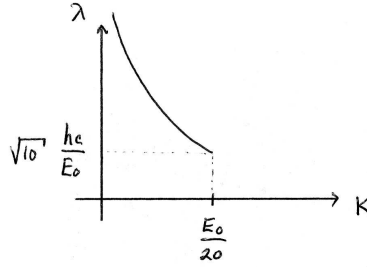
$$\lambda(m, K) = \frac{h}{\sqrt{2mK}}.$$

(b) Relativistically, we must use  $E^2 = p^2c^2 + E_0^2$ , where  $E = E_0 + K$ . Eliminating  $E$  between these two equations gives the relationship between  $K$  and  $p$ , which is  $pc = \sqrt{K^2 + 2E_0K}$ , so that the de Broglie wavelength can be written

$$\lambda(m, K) = \frac{hc}{\sqrt{K^2 + 2E_0K}} = \frac{h}{\sqrt{2mK + \frac{K^2}{c^2}}}.$$

The second form is useful in a comparison with the nonrelativistic case. You can easily see that if  $K \ll 2E_0$  then the relativistic expression reduces to the nonrelativistic version.

**109.** Nonrelativistically,  $\lambda = h/p = h/\sqrt{2mK} = hc/\sqrt{2E_0K}$ . If we restrict  $K < E_0/20$ , then the de Broglie wavelength will be restricted to  $\lambda > \sqrt{10}hc/E_0$  (see sketch). This, of course, holds true for both electrons and protons, with the only difference being the value of  $E_0$ . The maximum value for the kinetic energy is 25.5 keV in the case of electrons, and 46.9 MeV for protons.



What is the full relativistic relation? From Problem 108, we have that

$$\lambda = \frac{h}{\sqrt{2mK}} \frac{1}{\sqrt{1 + \frac{K}{2E_0}}} \approx \lambda_{NR} \left(1 - \frac{K}{2E_0}\right),$$

where I used the binomial approximation in the last step. If the largest value of  $K$  is  $E_0/20$ , then the correction term is  $K/2E_0 = 1/40$ , which means that our nonrelativistic approximation is 2.5% larger than the true value. A fairly good approximation.

**110.** The simplest method is to start with the relativistic energy formula

$$E^2 = p^2c^2 + (mc^2)^2,$$

and replace  $E$  and  $p$  with their wave equivalents:  $E = \hbar\omega$  and  $p = \hbar k$ . I obtain

$$(\hbar\omega)^2 = (\hbar k)^2c^2 + (mc^2)^2 \quad \text{or} \quad \omega^2 = k^2c^2 + \left(\frac{mc^2}{\hbar}\right)^2.$$

**111.** The phase velocity is given in Eq. (6.19)

$$v_{ph} = \frac{\omega}{k} = \sqrt{c^2 + \frac{\omega_p^2}{k^2}}.$$

Taking the derivative leads in a straightforward manner to

$$v_g = \frac{d\omega}{dk} = \frac{1}{2} (k^2c^2 + \omega_p^2)^{-1/2} 2kc^2 = \frac{kc^2}{\omega} = \frac{c^2}{v_{ph}},$$

This relation between the group and phase velocities,  $v_g = c^2/v_{ph}$ , says that if  $v_g < c$ , which it must be, then  $v_{ph} > c$ . The fact that the phase velocity is greater than  $c$  does not violate Einstein's axiom of relativity, because no information travels at that speed. Information is carried by the wave packet, which travels at the group velocity.

**112.** The first step is to obtain the dispersion relation for a relativistic electron matter wave, i.e., we need  $\omega$  as a function of  $k$ . The only step in Eq. (6.15) that must be changed is the fact that the energy-momentum relationship is not  $E = p^2/2m$ , but  $E^2 = p^2c^2 + m^2c^4$ . Following the same chain of reasoning gives

$$\omega = \frac{E}{\hbar} = \frac{\sqrt{p^2c^2 + E_0^2}}{\hbar} = \sqrt{k^2c^2 + \omega_0^2},$$

where  $\omega_0 \equiv E_0/\hbar$ . This can be written in an analogous form to the plasma dispersion relation

$$\omega^2 = \omega_0^2 + k^2c^2.$$

Using the fact that the phase velocity can be written

$$v_{ph} = \frac{\omega}{k} = \frac{E}{p}$$

along with the solution to Problem 111, we have

$$v_g = \frac{c^2}{v_{ph}} = \frac{c^2 p}{E} = c^2 \frac{\gamma m v_p}{\gamma m c^2} = v,$$

where I used the relativistic expressions for energy and momentum.

**113.** The phase velocity can be written in a number of ways

$$v_{ph} = \frac{\omega}{k} = \frac{\sqrt{gk}}{k} = \sqrt{\frac{g}{k}}.$$

The group velocity can most easily be found with implicit differentiation, which gives  $2\omega d\omega = g dk$ , which leads to

$$v_g = \frac{d\omega}{dk} = \frac{g}{2\omega} = \frac{g}{2\sqrt{gk}} = \frac{1}{2}v_{ph}.$$

**115.** Again using the uncertainty principle

$$\Delta x \sim \frac{\hbar}{\Delta p},$$

but we need to arrive at a reasonable approximation for the *maximum* value of  $\Delta p$ . One possibility is that we know the particle's momentum, but we don't know what direction it is traveling, so that  $\Delta p_{max} \approx 2p = 2\gamma mv$ . Also, if we are looking for the maximum value of  $\Delta p$ , the fastest the particle can travel is at  $v = c$ , so the conclusion is

$$\Delta x_{min} \sim \frac{\hbar}{2\gamma mc} = \frac{h}{4\pi mc} \sqrt{1 - \beta^2},$$

as desired. Why are we allowed to set  $v = c$  in one spot but not inside  $\beta$ ? Certainly as  $\beta \rightarrow 1$ ,  $v \rightarrow c$ , and  $\Delta x_{min} \rightarrow 0$ , as expected because the momentum tends toward infinity and hence  $\Delta p$  does also. What about in the other limit? When  $\beta \rightarrow 0$  we have

$$\Delta x_{min} \rightarrow \frac{h}{4\pi mc} = \frac{\lambda_C}{4\pi},$$

where  $\lambda_C$  is the Compton wavelength. The Compton effect experiment shows that this is a fundamental limit on the measurement of the position of a particle (an electron in the case of Compton scattering), so this makes sense. If we had not let  $v \rightarrow c$  before taking the  $\beta \rightarrow 0$  limit, the result would have been an infinitely large value of  $\Delta x_{min}$ , which doesn't make sense.

**116.** (a) This is formally identical to the analysis of standing waves on a string that is fixed at both ends—here, the wave function  $\Psi$  replaces the string displacement  $y$ . As a function of position,  $\Psi = \Psi_0 \sin(kx)$ , where, if I choose one of the walls to be at  $x = 0$ , then

I've already satisfied one boundary condition. For the other,  $\Psi(x = L) = \Psi_0 \sin(kL) = 0$ , which means that  $kL = n\pi$ ,  $n = 1, 2, 3, \dots$ . Rearranging this condition, we have our allowed wavelengths

$$\lambda = \frac{2L}{n}.$$

(b) Nonrelativistically, the kinetic energy is a function of the momentum,  $K = E = p^2/2m = \hbar^2 k^2/2m$ , and since the allowed wavenumbers are  $n\pi/L$ , the allowed values of the kinetic energy are

$$E_n = \frac{\hbar^2}{8mL} n^2 \quad n = 1, 2, 3, \dots$$

Another way to write this expression is to include factors of  $c$ . This allows an easy numerical calculation

$$E_n = \frac{(hc)^2}{8L(mc^2)} n^2 = \frac{(1240 \text{ MeV fm})^2}{8(0.1 \text{ nm})^2(0.511 \text{ MeV})} n^2 = E_1 n^2,$$

where  $E_1 = 37.6 \text{ eV}$ . The allowed energies are therefore  $E_1, 4E_1, 9E_1$ , etc. These energies are of the same “order of magnitude” of the electronic energies in the hydrogen atom, which are  $\sim 13.6 \text{ eV}$ .

**117.** (a) Using the Heisenberg relation, I get  $\Delta p \sim \hbar/2\Delta x = 5.28 \times 10^{-28} \text{ kg m/s}$ . (b) If this is the actual momentum  $p$ , then  $K = p^2/2m_e = 1.53 \times 10^{-19} \text{ J} = 0.953 \text{ eV}$ . (c) This is reasonable, because the Bohr energies for hydrogen are on the order of  $10 \text{ eV}$ . To be more precise, I note that the total energy is a sum of the kinetic and potential energies,  $E = K + U$ , but for the stable circular orbits I find that  $U = -2K$ , so that  $E = -K$ , or  $K = -E = 13.6 \text{ eV}$  for the ground state. This is within a factor of 10, so it is reasonable. Also, I assumed a *minimum* uncertainty relation, when in reality there should be an inequality, or  $K \geq 0.953 \text{ eV}$ , which shows that the electron in the ground state of a hydrogen atom is *not* in a minimum uncertainty state.

**118.** The calculation is the same, except for  $m_p$  and the diameter of the “box,”

$$D \sim 2R = 2R_0 A^{1/3} = 2(1.2 \text{ fm})(235)^{1/3} = 14.8 \text{ fm},$$

so the inequality is  $K \geq 7.58 \times 10^{-15} \text{ J} = 47.3 \text{ keV}$ . Of course,  $\alpha$  particles are ejected from nuclei with several MeV of energy, so the inequality certainly holds, but that would imply that we are essentially completely uncertain about where the protons are inside the nucleus!

**119.** See Section 6.3.

# Chapter 7

## Introduction to Quantum Mechanics

*... one should particularly remember that the human language permits the construction of sentences which do not involve any consequences and which therefore have no content at all—in spite of the fact that these sentences produce some kind of picture in our imagination; e.g., the statement that besides our world there exists another world, with which any connection is impossible in principle, does not lead to any experimental consequence, but does produce a kind of picture in the mind. Obviously such a statement can neither be proved nor disproved. One should be especially careful in using the words “reality,” “actually,” etc., since these words very often lead to statements of the type just mentioned. — Werner Heisenberg<sup>1</sup>*

This chapter signals a shift in our focus. Previously, the coverage has been less mathematical and more conceptual. The reason for this was that I had to quote results from mathematics that we had not covered yet. The nuts and bolts of quantum mechanics are highly mathematical, and our main focus will be to solve the Schrodinger equation (a partial differential equation) for several simple situations. Rather than it being just a mathematical exercise, however, I want to use it as a tool to explain some of the principles and rules that we have taken for granted. The first four sections cover the Schrodinger equation and how to solve it, the fundamental axioms of quantum mechanics, and a simple introduction to wave packets. These constitute the basics necessary to begin to understand the weirdness of quantum mechanics and enable you to continue on to more advanced studies. The final two sections — on symmetry and the Schrodinger equation in three dimensions — give just a taste of what you will encounter in the future.

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<sup>1</sup>Heisenberg, *The Physical Principles of the Quantum Theory*, page 15.

## 7.1 The Schrodinger Equation

In a broad sense, the development of the Schrodinger equation in 1926 by Erwin Schrodinger [Nobel Prize, Physics, 1933] was similar to the development of the classical laws of dynamics by Isaac Newton. That is, each was searching for a mathematical law that would describe observations of the physical world. In addition, each did not create something out of nothing — they both built on the work of previous physicists.



Newton had the incredibly good fortune to have at his disposal the experiments of Galileo and the observations of Kepler. Galileo had developed an understanding of constant acceleration (i.e., kinematics), and had proven his “length-time” theorem, which stated that under constant acceleration the total distance traveled was proportional to the time squared ( $d \propto t^2$ ).<sup>2</sup> Kepler had formulated his three Laws which described the orbital motion of the planets. From these, Newton was able to complete his dynamical laws and construct his law of gravitation. This unified the dynamical theories of motion on the Earth’s surface with motion in the heavens.

Two centuries later, Schrodinger had at his disposal the so-called “old quantum theory” (Chapter 6), which was comprised of seemingly inconsistent theories—most notably the notion of wave-particle duality. Were elementary particles (such as electrons and photons) particles or waves? It was hard to say, as different experiments revealed different properties. Schrodinger was able to develop a *wave* equation that gave a prescription for prediction the probability of detecting a *particle*, thereby combining the two viewpoints. (It would be helpful to reread Pauling’s viewpoint on page 10.)

### How to “derive” the Schrodinger equation

How did Schrodinger develop his eponymous equation? Just like Niels Bohr developed his atomic model, Schrodinger started with the classical wave equation and modified it with a set of *ad hoc* (and sometimes inconsistent) assumptions. *How* he obtained it, however, is less important *that* he obtained it. For the Schrodinger equation occupies the same logical place in the structure of physics that Newton’s second law occupies. That is, no one can state *why* it is correct, they can only state that it correctly predicts the outcomes of experiments. For this reason, I cannot derive it, but I will argue instead that it is plausible. Of course, the ultimate test is experiment, which it has passed perfectly since its discovery.

We would like to obtain an equation of motion for a particle, an electron say, that describes its wave character and its particle character at the same time. We still believe that energy is conserved, so that for this particle

$$E = K + U, \tag{7.1}$$

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<sup>2</sup>Galileo even made some crude measurements of the value of  $g \approx 4.7 \text{ m/s}^2$ . Others at around the same time had made better estimates,  $\sim 9.1 \text{ m/s}^2$ , but the primary difficulty was the lack of an accurate timekeeper.



where  $E$ ,  $K$ , and  $U$  are the total energy, kinetic energy and potential energy, respectively. This, of course, is a particle description. To merge this with the wave description, we must use the fundamental equations of quantum mechanics as listed on page 165

$$E = \hbar\omega, \quad \vec{p} = \hbar\vec{k}. \quad (7.2)$$

We know that  $\omega$  is the wave property analogous to the particle property  $E$ , and similarly for  $\vec{k}$  and  $\vec{p}$ , so that using the nonrelativistic kinetic energy  $p^2/2m$  means that our energy conservation equation becomes

$$\hbar\omega = \frac{\hbar^2 k^2}{2m} + U. \quad (7.3)$$

Note that for  $U = 0$  this is just the dispersion relation for a nonrelativistic electron that we derived in Eq. (6.15). But what does it mean? How do we interpret this equation?

Schrodinger realized that since he wanted to describe a wave, he needed a wave *function*. Classically, the description of a plane wave always included some physical property (surface height for water waves, pressure for sound waves, electric and magnetic fields for electromagnetic waves) that was a traveling oscillation, which takes the mathematical form

$$\exp\left(i[\vec{k} \cdot \vec{r} - \omega t]\right). \quad (7.4)$$

This describes a plane wave with angular frequency  $\omega$  and wavenumber  $k$  traveling in the direction of the vector  $\vec{k}$ . Using Euler's formula<sup>3</sup> it can be written in terms of a cosine and a sine, and in one dimension the dot product becomes  $\vec{k} \cdot \vec{r} = k_x x$ . Recall that in elementary wave analysis, this function is usually written as a pure sinusoid

$$\sin\left(\frac{2\pi}{\lambda} x - 2\pi\nu t\right) = \sin\left(\frac{2\pi}{\lambda}[x - vt]\right), \quad (7.5)$$

where the wave speed  $v$  (this is the *phase* velocity) is given by  $v = \omega/k = \nu\lambda$ . While adequate in elementary wave mechanics, the sinusoidal notation is not sufficient in quantum mechanics for two reasons. First, it is much simpler to manipulate exponential functions than trigonometric functions, and it is useful to retain the ability to discuss sine and cosine functions simultaneously. Second, the Schrodinger equation *requires* the use of complex variables, and this is not possible using only trigonometric functions. Schrodinger called the wave function  $\Psi$  and assumed it had the form of a plane traveling wave

$$\Psi = \Psi_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}, \quad (7.6)$$

where  $\Psi_0$  is a complex constant, so that  $\Psi$  is a complex function of  $x$ ,  $y$ ,  $z$ , and  $t$ .

The next step that Schrodinger took had no justification, but it is usually these kinds of insights that lead to a radically new way of thinking, and this one certainly did. As Born put it, it ushered in "... a new mode of thought in regard to natural phenomena."<sup>4</sup> Schrodinger multiplied Eq. (7.3) by  $\Psi$

$$\hbar\omega\Psi = \frac{\hbar^2 k^2}{2m}\Psi + U\Psi, \quad (7.7)$$

<sup>3</sup> $e^{i\theta} = \cos\theta + i\sin\theta$ .

<sup>4</sup>Born, "The statistical interpretation of quantum mechanics," Nobel Prize lecture, 1954.

and realized that partial derivatives of  $\Psi$  would give the proper multiplicative factors that are in Eq. (7.7), as long as  $\Psi$  took the form in Eq. (7.6). That is, the classical wave equation was a partial differential equation, so to obtain a *quantum* wave equation Schrodinger utilized the fact that the derivatives of the three dimensional plane wave in Eq. (7.6) have the form

$$\begin{aligned}\frac{\partial}{\partial t}\Psi &= -i\omega\Psi \\ \frac{\partial}{\partial x}\Psi &= ik_x\Psi \\ \frac{\partial}{\partial y}\Psi &= ik_y\Psi \\ \frac{\partial}{\partial z}\Psi &= ik_z\Psi.\end{aligned}\tag{7.8}$$

In fact, the spatial derivatives can all be written in a more compact manner

$$\nabla\Psi = i\vec{k}\Psi,\tag{7.9}$$

where  $\nabla$  is the gradient operator. With these partial derivatives, the energy conservation equation that was modified to include wave properties, Eq. (7.7), can be written as a partial differential equation

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = -\frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r},t) + U(\vec{r})\Psi(\vec{r},t).\tag{7.10}$$

This is the Schrodinger equation, where I've written explicitly the dependence of the function  $\Psi$  on position and time, and I've used  $\Psi(\vec{r},t)$  as a shorthand for  $\Psi(x,y,z,t)$ . Equation (7.10) is sometimes called the "time-dependent Schrodinger equation."

## The Born probability interpretation

If this function  $\Psi$  somehow describes an electron, how is it related to the electron that we measure in the laboratory? Almost immediately after Schrodinger proposed his equation, Max Born showed that  $\Psi(\vec{r},t)$  gave information regarding the probability of finding the electron at position  $\vec{r}$  at time  $t$ . That is,  $\Psi$  is called the PROBABILITY AMPLITUDE, and  $|\Psi|^2$  is the PROBABILITY DENSITY. Formally,  $|\Psi(\vec{r},t)|^2 dV$  is proportional to the probability of finding the electron in a volume  $dV$  near position  $\vec{r}$  at time  $t$ , where  $|\Psi|^2 \equiv \Psi^*\Psi$ , and  $\Psi^*$  is the complex conjugate of  $\Psi$ . Integrating over all space

$$\int |\Psi(\vec{r},t)|^2 dV\tag{7.11}$$

gives the probability of finding the electron anywhere in the universe at time  $t$ , and this should be equal to unity

$$\int |\Psi(\vec{r},t)|^2 dV = 1.\tag{7.12}$$

This last requirement is known as *normalization*. That is, the wave function  $\Psi$  may need to be multiplied by a constant so that it satisfies Eq. (7.12), i.e., so that it is normalized.

A word of caution: since it is a solution to the Schrodinger equation (7.10), the probability amplitude  $\Psi$  is, in general, complex, but the probability density — and hence the probability — is real. This makes sense because probability is a physically measurable quantity, and therefore it must be real.

## 7.2 Solving the Schrodinger Equation

Since the Schrodinger equation is a linear partial differential equation, the general solution can be found by means of a technique called “separation of variables.” This technique is different from that of the same name in ordinary differential equation theory. Here, we make the assumption that the function of several variables  $\Psi(x, y, z, t)$  is a product of functions of only one variable

$$\Psi(x, y, z, t) = X(x)Y(y)Z(z)T(t). \quad (7.13)$$

It is simplest to first separate out only the time dependence, and make the following assumption for the form of the solution

$$\Psi(\vec{r}, t) = \psi(\vec{r})T(t), \quad (7.14)$$

where again,  $\vec{r}$  stands for all three spatial variables,  $(x, y, z)$ . Note that the full wave function is denoted by the upper case Greek letter  $\Psi$ , while the time-independent wave function is denoted by the lower case Greek letter  $\psi$ . When Eq. (7.14) is substituted into Eq. (7.10) the Schrodinger equation takes the form

$$i\hbar \frac{1}{T(t)} \frac{\partial}{\partial t} T(t) = -\frac{1}{\psi(\vec{r})} \frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}) + U(\vec{r}), \quad (7.15)$$

where I have divided the entire equation by  $\Psi$ .

The next step is the key conceptual step involved in this technique. Note that the left-hand-side of Eq. (7.15) is a function of only one variable,  $t$ , and the right-hand-side is a function of only  $\vec{r}$ . If this equation is to hold for *all* values of  $t$  and  $\vec{r}$ , then each side of the equation must be equal to a constant; in fact they must be equal to the same constant. In this case, following the logic from Eqs (7.1) and (7.7), the constant *must* be the total energy  $E$ . Setting each side equal to  $E$ , we have obtained an *ordinary* differential equation for  $T(t)$ , and a *reduced* partial differential equation (only three independent variables) for  $\psi(\vec{r})$ . The equation for  $T(t)$  is

$$i\hbar \frac{dT}{dt} = ET, \quad (7.16)$$

whose solution is — up to a multiplicative constant —

$$T(t) = \exp(-iEt/\hbar), \quad (7.17)$$

The full solution can therefore be written

$$\Psi(\vec{r}, t) = \exp(-iEt/\hbar)\psi(\vec{r}). \quad (7.18)$$

I have not included the integration constant in  $T(t)$  since I will eventually absorb it into my final solution for  $\psi$ . Note that since  $E/\hbar = \omega$  the argument of the exponential is just  $-i\omega t$ , which means that the time dependence of  $\Psi$  is just that of a plane wave, exactly our initial guess for its form.

The equation for  $\psi(\vec{r})$  is

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}) + U(\vec{r})\psi(\vec{r}) = E\psi(\vec{r}). \quad (7.19)$$

This is called the time-independent Schrodinger equation, and is sometimes written

$$H\psi = E\psi, \quad (7.20)$$

where  $H$  is the Hamiltonian operator (in this case it's a differential operator that acts on the function  $\psi$ ) and is given by

$$H = -\frac{\hbar^2}{2m}\nabla^2 + U(\vec{r}). \quad (7.21)$$

Note that Eq. (7.20) has the form of an *eigenvalue* equation, in which an operator acting on a function results in a constant times that same function. In certain situations, this means that there is a solution only when  $E$  is one of a set of discrete values. In other words, the energy is quantized, hence the origin of the term QUANTUM MECHANICS.

### 7.2.1 One-dimension

We'll now solve Eq. (7.20) in one dimension. Even though  $\psi$  and Eq. (7.20) are three dimensional, all of the essential physics can be understood by restricting our focus to only one spatial dimension. That is, we'll assume that both  $U$  and  $\psi$  are functions only of  $x$ , not of  $y$  nor  $z$ . In this restricted regime the Hamiltonian operator is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + U(x), \quad (7.22)$$

and the time-independent Schrodinger equation simplifies to

$$\frac{d^2\psi}{dx^2} = \left[ \frac{2m(U - E)}{\hbar^2} \right] \psi. \quad (7.23)$$

Even though  $U$  is in general an arbitrary function of  $x$ , the *form* of the solution to Eq. (7.23) is clear. If  $U > E$ , then  $\psi$  must be an exponential function of  $x$ ,  $\psi \sim e^{\pm\kappa x}$ , but if  $U < E$ , then  $\psi$  must be a complex exponential function of  $x$ ,  $\psi \sim e^{\pm ikx}$ , where

$$\kappa^2 = \frac{2m(U - E)}{\hbar^2} \quad \text{and} \quad k^2 = \frac{2m(E - U)}{\hbar^2}. \quad (7.24)$$

What do these solutions mean physically? They mean that in the regions of space where the particle has a positive kinetic energy  $K$ , i.e.,  $U < E$  — recall the energy conservation Equation (7.1) — then the wave function is oscillatory in space *and* time, and our particle

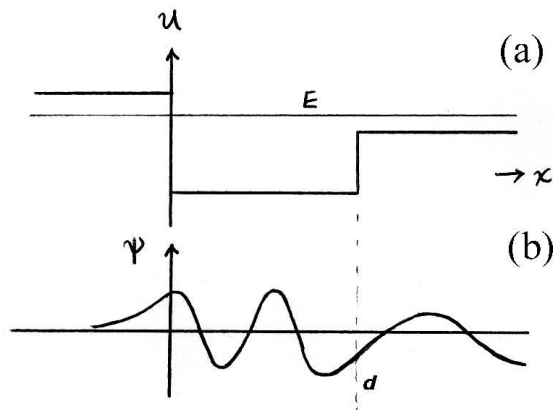


Figure 7.1: (a) Plot of potential energy  $U(x)$  and total energy  $E$ . Note that the kinetic energy is positive for  $x > 0$ , but negative otherwise. (b) Possible wave function  $\psi$  consistent with  $U$  in (a). Note both the exponential decay for  $x < 0$  and the abrupt change of wavelength at  $x = d$ .

is described by a traveling wave. This is the usual wave-particle duality. On the other hand, in the regions of space where energy conservation requires that the kinetic energy is *negative*, i.e.,  $U > E$ , then the spatial behavior of  $\psi$  is “evanescent.” This region of space, where the total energy of the particle is less than its potential energy, is called the “classically forbidden region.” Elementary classical mechanics would not allow, say, a ball to roll to the top of a hill unless the ball possessed a minimum amount of kinetic energy at the bottom of the hill. Elementary quantum mechanics, on the other hand, requires a nonzero wave function in these “forbidden” regions. Since the wave function is related to the probability of the particle being observed in this region, it means that quantum mechanics allows particles to be where they energetically cannot be. As we shall see, this is the fundamental notion behind “quantum tunneling.”

Figure 7.1(a) depicts a potential energy function  $U$  that is a function of position  $x$ , and, given a particular total energy  $E$  (that is constant) for the particle under description, Fig. 7.1(b) shows a possible solution for  $\psi(x)$ . Note that for  $x < 0$ , where  $U > E$ , the wave function is evanescent, i.e., it exponentially decays. For positive  $x$ , there are two regions, each with a different value of  $U$ , hence each with a different value of  $k$ , and hence each with a different value of wavelength.

### General requirements for $\psi$

Regardless of the shape of  $U(x)$ , there are several general requirements that  $\psi(x)$  must satisfy. They are determined by the properties of the governing differential equation (7.23), as well as physical properties that  $\psi$  must have in order to be useful as a probability amplitude.

First, the total energy  $E$  must be greater than the global minimum of  $U$ ,  $U_{min}$ , for  $\psi$  to be normalizable. If the entire domain,  $-\infty < x < +\infty$ , is classically forbidden, then  $\psi$  everywhere takes the form  $e^{+\kappa x}$  or  $e^{-\kappa x}$ , and these diverge as  $x \rightarrow -\infty$  or  $x \rightarrow +\infty$ ,

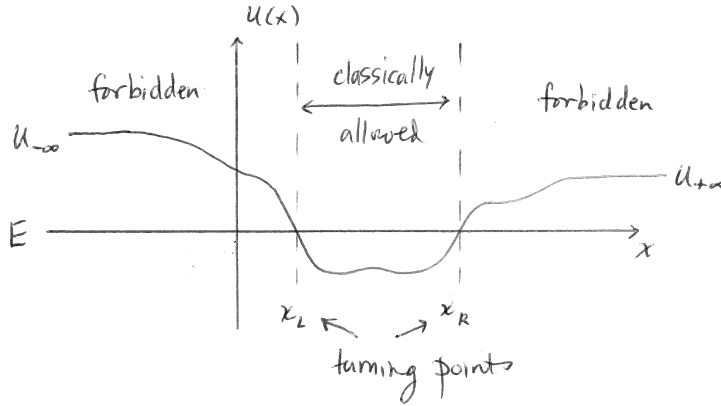


Figure 7.2: Generic potential energy function  $U(x)$  which tends to constant values  $U_{+\infty}$  and  $U_{-\infty}$  as  $x \rightarrow \pm\infty$ . Here, the total energy is  $E = 0$ , and the classical turning points, as well as the forbidden and allowed regions are labeled.

respectively. Thus, there is no way to satisfy Eq. (7.12).

Second, in order for a particle represented by  $\Psi$  to be “bound,” i.e., restricted to a localized region of space, the total energy  $E$  must be less than the potential energy at  $\pm\infty$ , i.e.,

$$E < U_{+\infty} \quad \text{and} \quad E < U_{-\infty} \quad (7.25)$$

where  $U_{\pm\infty} = U(x \rightarrow \pm\infty)$ . Classically, this would mean that the particle is trapped between two classical “turning points,”  $x_{\text{turn}}$ , defined to be where  $E = U(x_{\text{turn}})$ . See Fig. (7.2). Because  $\psi$  must evanescently decay when  $x > x_R$  and when  $x < x_L$ , the wave function — and hence the probability of finding the particle — is confined mostly to the classically allowed region. If, on the other hand, either  $E > U_{+\infty}$  or  $E > U_{-\infty}$ , as is true in Fig. (7.1)(a), then the particle is bounded on one side but can escape on the other. This is called a “free particle,” to distinguish it from a bound particle.<sup>5</sup>

In addition to the requirement that the wave function must be normalizable, Eq. (7.12), it must also be continuous and its slope must be continuous.<sup>6</sup> When  $U$  changes continu-

<sup>5</sup>Sometimes the word “free” is meant to imply that there are no forces acting on the particle, i.e., the potential energy is constant over all space. You will have to infer which is meant by the context.

<sup>6</sup>The function  $\psi$  must be continuous because there are no sources or sinks of probability — that is, in this formalism, particles are not created nor are they destroyed. The fact that the derivative of  $\psi$  must be continuous comes from the Schrodinger equation itself. Integrating Eq. (7.23) over an infinitesimal domain,

$$\int_{x_0-\epsilon}^{x_0+\epsilon} \frac{d}{dx} (\psi') dx = \int_{x_0-\epsilon}^{x_0+\epsilon} k^2 \psi dx.$$

In the limit that  $\epsilon \rightarrow 0$ , the right-hand-side goes to zero, because neither  $k$  (and therefore  $U$ ) nor  $\psi$  diverge. This integral becomes a condition on  $\psi'$

$$\psi' \Big|_{x_0-\epsilon}^{x_0+\epsilon} = 0,$$

which means that  $\psi'$  is continuous.

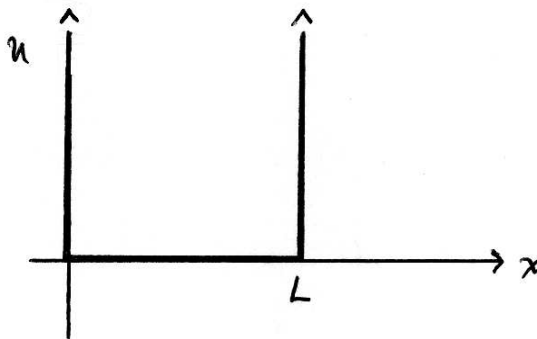


Figure 7.3: Potential energy  $U(x)$  for a one-dimensional infinite square well, also known as a “particle-in-a-box.” The potential energy is zero within the well, but is infinitely large outside the well, restricting the particle unambiguously to be located in the region  $0 < x < L$ .

ously with  $x$ , then these requirements are automatically taken care of in the solution of the second-order differential equation Eq. (7.23). However, when  $U$  changes discontinuously, as in Fig. 7.1(a), it is simplest to solve Eq. (7.23) separately in each region where  $U$  is constant, and then patch the solutions together at the boundaries. As you can see, these requirements have been satisfied in Fig. 7.1(b). Such a potential energy function is called “piecewise constant,” and is the form of  $U$  that we now tackle explicitly.

### Particle in a box

Our first potential energy function is one of the simplest situations possible, known as the 1D infinite square well, or the 1D “particle-in-a-box,” and is depicted in Fig. 7.3 and specified by

$$U(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{otherwise} \end{cases} \quad (7.26)$$

How do we go about solving Eq. (7.23) for such a function  $U$ ? There are four steps. **First**, for any finite energy  $E$  the value of  $\kappa$  is infinitely large outside the potential well, when  $x < 0$  and  $x > L$ . This means that our boundary conditions for  $\psi$  require that  $\psi = 0$  in those regions. I want to stress that although this potential energy function is not physical (in realistic situations there would be an upper bound on  $U$ ), it is mathematically the simplest. **Second**, it is straightforward to solve Eq. (7.23) in the region  $0 < x < L$ . In this region,  $U = 0$ , so that  $k = \sqrt{2mE}/\hbar$ , and the two linearly independent solutions to

$$\frac{d^2\psi}{dx^2} = -k^2\psi \quad (7.27)$$

are either  $e^{+ikx}$  and  $e^{-ikx}$ , or  $\sin(kx)$  and  $\cos(kx)$ .

If we were looking for traveling wave solutions — if our particle was moving freely through space — the exponential form would be preferable. However, we are now in the

position that Niels Bohr was in 1913 when he was trying to understand the hydrogen atom. That is, we are looking for *stationary state* solutions to this problem. Any particle in this potential well must be bound (and not traveling), and therefore cannot escape. For this reason, the trigonometric form is simpler. In addition, the trigonometric functions simplify the application of the boundary conditions. Our most general solution is therefore

$$\psi(x) = A \sin(kx) + B \cos(kx), \quad (7.28)$$

where  $A$  and  $B$  are the two arbitrary complex constants. **Third**, we now apply the boundary conditions at  $x = 0$  and  $x = L$ . The only way for  $\psi(x = 0)$  to be zero is for the constant  $B$  to be zero. The boundary condition at  $x = L$  is slightly trickier. For the other term in Eq. (7.28) to vanish at the right side of the well requires that

$$\sin(kL) = 0, \quad (7.29)$$

or

$$kL = n\pi, \quad n = 1, 2, 3, \dots \quad (7.30)$$

This requires  $k$  to take on discrete values, and given the definition of  $k$ , it requires the energy  $E$  also to take on discrete values

$$E_n = \frac{\pi^2 \hbar^2}{2mL^2} n^2, \quad n = 1, 2, 3, \dots \quad (7.31)$$

Here,  $n$  is the quantum number that denotes the energy level of the particle in the box.

This mathematics should appear familiar to you. In your introductory study of resonance phenomena in waves on strings you solved the classical wave equation subject to the boundary conditions. Formally the two problems are identical, and if you calculate the allowed wavelengths of  $\psi$ , you'll find  $\lambda = 2L/n$ , exactly the result for standing waves on a string. **Fourth**, and last, we take our solution that satisfies the time-independent Schrodinger equation along with the proper boundary conditions, and normalize it. That is, we don't yet know the correct value of  $A$ , but integrating over all space (the probability of finding our particle *somewhere* is unity) gives

$$\int_0^L |\psi(x)|^2 dx = |A|^2 \int_0^L \sin^2(kx) dx = \frac{L}{2} |A|^2. \quad (7.32)$$

Since this integral must be unity,  $A$  must satisfy

$$|A|^2 = \frac{2}{L}. \quad (7.33)$$

Since  $A$  is complex, this equation only sets a condition on its amplitude, and the most common choice is  $A = \sqrt{2/L}$ . Of course, any arbitrary phase is possible, which means that  $A = \sqrt{2/L} e^{i\phi}$  is also correct, for any value of  $\phi$ . For most cases, the simple choice of  $\phi = 0$  is usually made.

Finally, we have the complete solution to the time-independent Schrodinger equation: an infinite set of energies and their associated wave functions

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad E_n = \frac{\pi^2 \hbar^2}{2mL^2} n^2 \quad n = 1, 2, 3, \dots \quad (7.34)$$



These are what Bohr would call “stationary states.” If a particle is placed into the ground state,  $n = 1$ , it would remain there forever (i.e., it would be stationary) if there were no external influence. However, just like the hydrogen atom, if it is placed into an excited state,  $n > 1$ , then after a finite amount of time it would emit a photon and make a transition to a state with a lower energy, a lower “energy level.”

### 7.2.2 Two dimensions

A slightly more complicated potential energy function can now be considered, and with very little extra effort. It is called a “particle in a 2D box.” The potential energy  $U$ , and by extension  $\psi$ , is now a function of both  $x$  and  $y$  and takes a form similar to Eq. (7.26)

$$U(x, y) = \begin{cases} 0 & 0 < x < L, \quad 0 < y < L \\ \infty & \text{otherwise} \end{cases} \quad (7.35)$$

This is a symmetric box, whose width in the  $x$  direction is the same as its width in the  $y$  direction. Of course, we could choose an asymmetric box, with different widths, say  $L_x$  and  $L_y$ , but that extra level of complexity does not add any more physical understanding. In fact, in the next section I will extend this analysis to encompass identical particles, and for the analogy to hold, symmetry is required.

The time-independent Schrödinger equation, Eq. (7.20), is still a separable partial differential equation because the Hamiltonian operator is

$$H = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + U(x, y). \quad (7.36)$$

In order to be *separable*, i.e., solvable by the technique of separation of variables, the potential energy function  $U$  must be able to be expressed as a sum of two terms

$$U(x, y) = U_x(x) + U_y(y), \quad (7.37)$$

each of which is a function of only one of the variables. Since  $U = 0$  in the present problem (in the region of interest), we can use this technique.

Our guess, then, for the wave function  $\psi$  is a product of two functions

$$\psi(x, y) = X(x)Y(y). \quad (7.38)$$

Inserting this into Eq. (7.20) with  $H$  given by Eq. (7.36) and  $U$  given by Eq. (7.37) results in

$$-\frac{\hbar^2}{2m} \left( Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} \right) + (U_x + U_y)XY = EXY. \quad (7.39)$$

Dividing by  $\psi$  and rearranging

$$\left( -\frac{\hbar^2}{2m} \frac{1}{X} \frac{\partial^2 X}{\partial x^2} + U_x \right) + \left( -\frac{\hbar^2}{2m} \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} + U_y \right) = E. \quad (7.40)$$

Just like Eq. (7.15), I have grouped all the functional dependence on  $x$  into the first term in parentheses, and all the functional dependence on  $y$  into the second term. Recall

that the energy  $E$  is a constant and not a function of either variable. By the same logic that followed Eq. (7.15), each term in parentheses must be a constant. Since we can use whatever symbol we wish, it makes sense to use  $E_x$  and  $E_y$  for the two constants, which means that the *ordinary* differential equation for  $X(x)$  is

$$\frac{d^2 X}{dx^2} = \left[ \frac{2m(U_x - E_x)}{\hbar^2} \right] X, \quad (7.41)$$

which is identical (in form) to Eq. (7.23). Because of this identical form, all conclusions that we made about the solution  $\psi$  to Eq. (7.23) will also hold for the solution  $X$  to Eq. (7.41). In particular, the classification of the solution's character into either oscillatory or evanescent depending on the sign of  $U - E$  is still valid, and therefore a division of the domain into "classically allowed" and "classically forbidden" is still useful. For the present case of a particle in a 2D box,  $\psi$  must be oscillatory in the region where  $U = 0$ , and  $\psi = 0$  elsewhere.

The form of the differential equation for  $Y(y)$  has the same form as Eq. (7.41), and the constants must satisfy

$$E_x + E_y = E. \quad (7.42)$$

This means that the total energy depends on two quantum numbers, one for the  $x$  dependence and one for the  $y$  dependence. As we saw in Chapter 4, each dimension allows one "degree of freedom," and therefore requires one quantum number to describe the particle's state. For this two-dimensional problem, the particle is governed by two quantum numbers.

Since the solutions for both  $X(x)$  and  $Y(y)$  are given by Eq. (7.34), we can immediately write down the full, normalized solution to Eq. (7.40) as

$$\psi_{nm}(x) = \frac{2}{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right) \quad (7.43)$$

with an associated energy

$$E_{nm} = E_x + E_y = \frac{\pi^2 \hbar^2}{2mL^2} (n^2 + m^2) \quad \text{where} \quad n, m = 1, 2, 3, \dots \quad (7.44)$$

Each quantum state is characterized by two quantum numbers,  $n$  and  $m$ , and both the wave function  $\psi$  and the energy are symmetric functions of these two quantum numbers, which means that nothing changes under the interchange  $n \leftrightarrow m$ .

## Degeneracy

Degeneracy, in the quantum sense, refers to two (or more) linearly independent states that have the same energy. In the case of a particle in a symmetric 2D box, the states  $(n, m) = (1, 2)$  and  $(2, 1)$  are degenerate. That is, they have the linearly independent eigenfunctions

$$\psi_{12}(x) = \frac{2}{L} \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi y}{L}\right) \quad (7.45)$$

and

$$\psi_{21}(x) = \frac{2}{L} \sin\left(\frac{2\pi x}{L}\right) \sin\left(\frac{\pi y}{L}\right), \quad (7.46)$$

but they both have an energy of

$$E_{12} = E_{21} = 5 \frac{\pi^2 \hbar^2}{2mL^2} = 5E_G, \quad (7.47)$$

where  $E_G \equiv \pi^2 \hbar^2 / 2mL^2$  is the energy of the ground state in the 1D case. This property, degeneracy, has the same meaning as that in linear algebra, where linearly dependent eigenvectors of a given matrix have identical eigenvalues. In fact, in the Heisenberg representation of quantum mechanics, the operators are matrices and the states are vectors. The time-independent Schrödinger equation is thus a matrix equation, whose solutions are eigenvectors and eigenvalues.

A more familiar example of degeneracy is the ground state of an electron in a hydrogen atom, where two different quantum states have the same energy. These two states are both written as  $1s^1$ , which means they have the quantum numbers  $n = 1$ ,  $\ell = 0$ , and  $m_\ell = 0$ . However, they have different values for the  $z$ -component of the electron's spin,  $m_s = +\frac{1}{2}$  and  $m_s = -\frac{1}{2}$ . In reality, these two states do not have exactly the same energy because of the magnetic interaction between the electron's spin and orbital motion, and between the spin of the electron and the spin of the proton. However, these energy differences are small and result in fine structure in the hydrogen spectrum. Therefore, the degeneracy of these states is only approximate.

### Accidental degeneracy

A useful thing to do is to list the energies of the stationary states in numerical order. In this way, it is straightforward to predict the spectrum (i.e., the frequencies of any emitted photons) by calculating the energy differences between various states.<sup>7</sup> For the particle in a 2D box, some of the energy levels are

$n$	$m$	$E/E_G$
1	1	2
1	2	5
2	1	5
2	2	8
⋮	⋮	⋮
5	5	50
1	7	50
7	1	50
⋮	⋮	⋮

---

<sup>7</sup>Recall that, by Bohr's postulate, the frequency of an emitted photon when the particle makes a jump from state  $n_1$  to state  $n_2$  is given by

$$h\nu = E_{n_1} - E_{n_2}.$$

As we saw above, the (1,2) and (2,1) states are degenerate, but their degeneracy is a consequence of the symmetry of the problem — it's an expected degeneracy. Another expected degeneracy is the states (1,3) and (3,1), and you can list many others. However, there is another degeneracy in the list above that is *not* due to the symmetry of the problem. The (5,5) and (1,7) states have the same energy, and this is called an *accidental degeneracy*. The reason for this name is that there is no obvious underlying symmetry that is responsible for this degeneracy.

### Superposition states

Even though Eq. (7.34) lists all possible solutions to the time-independent Schrodinger equation for this potential energy function, the most general solution to the full, time-dependent, Schrodinger equation must include the time dependence, Eq. (7.17). Each of the solutions in Eq. (7.34), however, has a different energy, so the theory of partial differential equations states that the general solution to Eq. (7.10) is a linear combination of all the possible solutions

$$\Psi(x, t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar} = \sum_n c_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \exp\left(-i\frac{\pi^2 n^2 \hbar}{2mL^2} t\right), \quad (7.48)$$

where  $c_n$  are the (complex) amplitudes of each state. This form for  $\Psi$  is not a product, like Eq. (7.14). However, the product solution is not the most general solution. It does result in one of Bohr's stationary states, but solutions are not required to be stationary (see below for the simplest nontrivial example), and the linear combination in Eq. (7.48) is required to obtain all possible solutions.

The general solution, Eq. (7.48), tells us two new facts. First, particles do not have to be in a single energy level, i.e., a single state. If more than one  $c_n$  is nonzero, then the particle is in what is called a "superposition" state. Of course, if you were to measure the energy of the particle, you would only obtain one result, with the *probability* of obtaining  $E_n$  proportional to  $|c_n|^2$  (see postulate 5 in Sec. 7.3). This is one of the strange things about quantum mechanics — the wave function gives information about the probability of measuring a certain quantity rather than a specific value for that quantity. Second, when the particle is in a superposition state, it is not stationary. That is, when only one  $c_n$  is nonzero, then the probability density is just

$$|\Psi|^2 = \left| c_n \psi_n e^{-iE_n t/\hbar} \right|^2 = |c_n \psi_n|^2, \quad (7.49)$$

which is not a function of time because the complex exponentials have a modulus of 1. That is, the particle is in a stationary state, which means that whenever you measure the energy you will obtain the same value. However, when the particle is in a superposition state, for example, if it's a mixture of two quantum states,  $n$  and  $m$ ,

$$\Psi = c_n \psi_n e^{-iE_n t/\hbar} + c_m \psi_m e^{-iE_m t/\hbar}, \quad (7.50)$$

where  $n \neq m$ , then the probability density *is* time dependent, because of the cross terms in the absolute value,

$$|\Psi|^2 = \left| c_n \psi_n e^{-iE_n t/\hbar} + c_m \psi_m e^{-iE_m t/\hbar} \right|^2 \quad (7.51a)$$

$$\begin{aligned}
&= |c_n\psi_n|^2 + |c_m\psi_m|^2 \\
&\quad + c_n^*c_m\psi_n\psi_m e^{-i(E_m-E_n)t/\hbar} + c_m^*c_n\psi_m\psi_n e^{-i(E_n-E_m)t/\hbar}.
\end{aligned} \tag{7.51b}$$

If the amplitudes are real, then we can write

$$|\Psi|^2 = |c_n\psi_n|^2 + |c_m\psi_m|^2 + 2c_n c_m \psi_n \psi_m \cos([E_m - E_n]t/\hbar). \tag{7.52}$$

This time dependence means that the probability of finding the particle at a particular position  $x$  *changes with time*, which complicates the interpretation of measurements further.

## 7.3 The Postulates of Quantum Mechanics

Just like learning how to use Newton's laws of dynamics, there are several rules for solving quantum mechanical problems. They are called "postulates" or "axioms" because they can't be proven, but are justified *post facto*. These "rules" of quantum mechanics can be boiled down to just a few. As Rolf Winter says in his quantum mechanics textbook, "If you are formally inclined and mathematically skillful, you might consider clipping [this] paragraph and throwing the rest of the book away."<sup>8</sup>

1. The wave function (or state function)  $\Psi$  contains all the information that can be known about the system.
2. The time development of  $\Psi$  is determined by the Schrodinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi. \tag{7.53}$$

3. Observable quantities are represented by Hermitian operators  $A$ . These operators have a complete set of eigenfunctions and eigenvalues

$$A\Psi_n = a_n\Psi_n. \tag{7.54}$$

4. The wave function can be represented by the expansion

$$\Psi = \sum_n c_n \Psi_n, \quad \text{where} \quad c_n = \int \Psi_n^* \Psi d^3r. \tag{7.55}$$

5. The possible results of a measurement of  $A$  are the eigenvalues  $a_n$ , and the probability of obtaining  $a_n$  in any given measurement is  $|c_n|^2$ .

Axiom 1 means that, just like the quantity  $\vec{r}(t)$  describing the trajectory of a particle in classical mechanics, the wave function  $\Psi(\vec{r}, t)$  has within it complete information about the system, although it cannot be directly measured.

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<sup>8</sup>Winter, *Quantum Physics*, page 131.

Axiom 2 says that the Schrodinger equation is the quantum counterpart to Newton's second law

$$\frac{d^2}{dt^2} (m\vec{r}) = \vec{F}(\vec{r}, t), \quad (7.56)$$

which describes how  $\vec{r}(t)$  evolves in time. Neither of these equations can be “derived” but must be chosen so that they agree with experiment.

Axioms 3-5 are at the heart of the quantum weirdness. Classically, given  $\vec{r}(t)$ , a solution to Newton's second law, it is straightforward to predict where the particle will be, as well as its velocity and kinetic energy, at a particular time  $t$ . They are just  $\vec{r}(t)$ ,  $\vec{v} = d\vec{r}(t)/dt$ , and  $mv^2/2$ . The corresponding predictions in quantum theory require more involved calculations. For example, if we wish to measure the energy  $E$ , which is an observable quantity, we first form the corresponding operator  $H$ , and express  $\Psi$  as a linear combination of the eigenfunctions of  $H$ , as shown in Eq. (7.55) in general, or Eq. (7.48) for the specific case of the 1D square well. The only possible results of any measurement are the eigenvalues  $E_n$  of  $H$ . For example, if the particle is in the 1D infinite square well, only  $\pi^2\hbar^2n^2/2ma^2$  are possible energy measurements.

Other Hermitian operators that we have seen include the position  $x$ , which is just a multiplicative operator (i.e., the operation needed is simple multiplication), and momentum  $p_x$ , which is a differential operator. “Hermitian” simply means that the eigenvalues are real, and not complex. Of course, since the result of a measurement is one of the eigenvalues, and in the laboratory we only measure real quantities, the eigenvalues of any allowed operator (that represents a quantity that can be measured — i.e., an observable) must be real.

Of course, since  $\Psi$  must be normalized, you can show (Problem 120) that this normalization requirement demands that

$$\sum_n |c_n|^2 = 1. \quad (7.57)$$

This simple result arises from the fact that the sine functions are “orthonormal” over the interval  $0 < x < L$ , which means that

$$\int_0^L \psi_n \psi_m dx = \delta_{nm}, \quad (7.58)$$

where  $\delta_{nm}$  is the Kronecker delta symbol, and is equal to 1 if  $n = m$ , but is equal to zero if  $n \neq m$ . Of course, the mathematics in this section is just the mathematics involved in the Fourier expansion of a function over the interval  $0 < x < L$ .

## Measurement

There has been quite a bit written about the measurement process in quantum mechanics and what it means. At this point, we can add a two axioms to our list that define what it means to make a measurement.

6. The measurement of an observable  $A$  that yields the value  $a_n$  will leave the system in state  $\Psi_n$ . This is what is meant by the “collapse” of the wave function.

7. If the system is in state  $\Psi$ , then the average of many measurements of  $A$  is called the “expectation value” and is

$$\langle A \rangle = \int \Psi^* A \Psi dV. \quad (7.59)$$

Axiom 6, along with axiom 2, points out another aspect of quantum weirdness, which has to do with how the wave function  $\Psi$  evolves in time. There are two ways in which it can change: the first is in a continuous, causal, manner as described by the Schrodinger equation (Axiom 2), and the second is discontinuous and probabilistic which occurs when an observation is made (Axiom 6). In the first case the system is isolated and is not disturbed by any measurement apparatus. This structure is called the “Copenhagen” interpretation of quantum mechanics, because it was hammered out by Bohr and his colleagues.

The integral in axiom 7 is interpreted in the following way. Just as with Eq. (3.25) describing the average value of the roll of a loaded die, or with Eq. (3.27) describing the average lifetime of a radioactive nucleus, you might expect that the expectation value of  $A$  would be an integral of  $A$  times the probability, which is  $\Psi^*\Psi$ . This is almost correct. However, it turns out that to agree with experiment (always our goal) the operator  $A$  must act to the right on the wavefunction  $\Psi$ , and only then is the result multiplied by  $\Psi^*$ . Sometimes the integrand is written as  $\Psi^*(A\Psi)$  to show this explicitly. It turns out that the order doesn’t really matter for multiplicative operators like the position  $x$ , but for differential operators, like momentum  $p_x = -i\hbar\partial/\partial x$ , it must be clear that the derivative is acting only on  $\Psi$  and not on  $\Psi^*$ .

## 7.4 Free particles and wave packets

While you might think that a free particle, with no forces acting on it, would be the simplest case to solve, it turns out that there are subtle issues with normalization that makes the 1D particle-in-a-box the simpler case mathematically, which is why we solved that case first. In general, while the time-independent Schrodinger equation ( $H\psi = E\psi$ ) *looks* like an eigenvalue equation which is solvable only for certain, discrete, values of  $E$  (and indeed that is the way it behaves in many cases), for a free particle *any* value of  $E$  is allowed (see page 188 for the distinction between bound and free).

Here, we consider a particle moving in one dimension, just like in Sec. 7.2.1, but now  $U = 0$  for all  $x$ . Since the particle feels no force, this requires that  $dU/dx = 0$ , which means that  $U$  can take on any constant value, and we can choose  $U = 0$  without loss of generality. With these caveats, the time-independent Schrodinger equation can be rewritten as

$$\nabla^2\psi + k^2\psi = 0, \quad (7.60)$$

where  $k^2 = 2mE/\hbar^2$ . This equation is called the Helmholtz equation.<sup>9</sup> For our purposes, a simple solution is

$$\psi(x, y, z) = \psi_0 e^{i\vec{k}\cdot\vec{r}}, \quad (7.61)$$

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<sup>9</sup>For those of you with a mathematical bent, it is an elliptic partial differential equation, and shows up in electrostatic phenomena, diffusion and heat flow, waves in solids, and even nuclear reactors.

where  $\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$ . Remember that the full solution to the time-dependent Schrodinger equation requires us to multiply  $\psi$  by the exponential factor  $\exp(-i\omega t)$  so that

$$\Psi(x, y, z, t) = \psi_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}. \quad (7.62)$$

This, of course, is just a traveling plane wave, traveling in the direction of  $\vec{k}$  with a phase speed of  $\omega/k$ . We have just described a particle with a matter wave of energy  $E$  (remember that  $\omega = E/\hbar$ ).

While this is a perfectly good description of a wave, it is a *horrible* description of a particle, because this wave has a nonzero amplitude everywhere! The prescription that  $|\Psi|^2$  is the probability density means that the particle could be anywhere in the universe with equal probability,  $|\Psi|^2 = |\psi_0|^2$ , which is a constant. This unphysical behavior is “explained” by Heisenberg: if we know the kinetic energy exactly, then we know the momentum exactly, and then we have no knowledge of the location. To describe a particle, then, we need to do what we did at the end of Chapter 6, and that is to construct a wave packet, which is a traveling wave with a fairly well-defined frequency and wave number, but a nonzero amplitude only in a localized region of space. In three dimensions, the math can get quite tricky, so I’ll restrict my analysis to one dimension.

### One-dimensional wave packet

In one dimension Eq. (7.62) becomes

$$\Psi(x, t) = \psi_0 e^{i(k_x x - \omega t)}. \quad (7.63)$$

To construct a wave packet we will extend our development in Chapter 6 where we added two waves with slightly different wavenumbers. In that case, the superposition resulted in what looked like beats, with each beat resembling a localized wave packet. However, the entire wave function still extended to infinity, and therefore was not localized. To construct a *truly* localized wave packet, we must add together, in superposition, an *infinite* number of different waves, all with different wavelengths and possibly different amplitudes. The mathematical representation is not a sum, but an integral<sup>10</sup>

$$\Psi(x, t) = \int_{-\infty}^{\infty} \psi_0(k_x) e^{i(k_x x - \omega t)} dk_x, \quad (7.64)$$

where  $\omega$  is a function of  $k_x$

$$\omega(k_x) = \frac{E}{\hbar} = \frac{\hbar k_x^2}{2m}. \quad (7.65)$$

This is just the dispersion relation that we obtained in Chapter 6, and it is needed here because each wave with a particular wavelength  $k_x$  is associated with particular energy  $E$ , and hence a particular angular frequency  $\omega$ .

Depending on the exact functional form of  $\psi_0(k_x)$  the integral in Eq. (7.64) ranges from merely difficult to impossible to evaluate analytically. However, there are methods of approximating the integral that are simple and at the same time can result in fairly deep

<sup>10</sup>This integral is called a Fourier transform. It is a logical extension of a Fourier series, in the limit where the allowed values of the wavelength are continuous, not discrete.



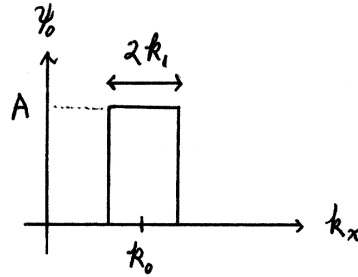


Figure 7.4: A simple amplitude function that allows the evaluation of  $\Psi(x, t)$ . This amplitude is given by  $\psi_0 = A$  when  $k_0 - k_1 < k_x < k_0 + k_1$ , and  $\psi_0 = 0$  for all other values of  $k_x$ .

physical insight. One shape of  $\psi_0(k_x)$  for which an approximation is fairly easy is shown in Fig. 7.4 and is defined by

$$\psi_0(k_x) = \begin{cases} A & \text{for } k_0 - k_1 < k_x < k_0 + k_1 \\ 0 & \text{otherwise} \end{cases} \quad (7.66)$$

This is called a “box” function where  $2k_1$  is the width of the box. It means that the waves we are adding all have wavelengths *near*  $2\pi/k_0$ . If  $k_1$  is small enough, then we can expand all the functions of  $k_x$  inside the integrand in a Taylor series around  $k_0$ . In particular, the dispersion relation can be approximated by

$$\omega(k_x) \approx \omega_0 + \omega'_0(k_x - k_0), \quad (7.67)$$

where I have ignored the higher order terms in the Taylor series, and I have defined  $\omega_0 \equiv \omega(k_0)$  and

$$\omega'_0 \equiv \left. \frac{d\omega}{dk_x} \right|_{k_0}. \quad (7.68)$$

If this approximation is made, then the integral in Eq. (7.64) becomes straightforward, and you can show (see Problem 135) that evaluation gives

$$\Psi(x, t) = (2Ak_1) \left[ \frac{\sin [k_1(x - \omega'_0 t)]}{k_1(x - \omega'_0 t)} \right] e^{i(k_0 x - \omega_0 t)}. \quad (7.69)$$

I’ve written this in a suggestive manner. The first factor,  $(2Ak_1)$ , is just a constant that must be determined through normalization (see Problem 136). The second factor in the square brackets is the envelope (amplitude) of the wave packet. Remember that any function that can be written as  $f(x - vt)$  retains its shape but “travels” to the right. The factor inside the square brackets is just such a function, and therefore moves at speed  $\omega'_0$ , as given by Eq. (7.68). This is what we really mean when we say “group velocity,” and as long as the distribution of wavenumbers is narrow, it is equal to the derivative of  $\omega$  with respect to  $k_x$ , evaluated *at the center* of the amplitude distribution. The last factor, the exponential, is a traveling plane wave with a frequency and wavelength corresponding to the *center* of the amplitude function  $\psi(k_x)$ . In total, then,  $\Psi$  is an amplitude-modulated traveling plane wave, whose phase velocity is  $\omega_0/k_0$  and whose group (or envelope) velocity is  $\omega'_0$ .

## 7.5 Symmetry in the 1D Schrodinger Equation

*[Q]uantum symmetries ... simply have no place in the classical context, a first example being the ... permutation symmetry of states of  $n$  identical particles. These symmetries hold the key to the exclusion principle and to quantum statistics.*

— Abraham Pais

Symmetry plays a large role in the modern understanding of physics. For example, the fact that space is translationally invariant (*here* exhibits the same physical laws as *there*) automatically implies that momentum is a conserved quantity. And the fact that space is isotropic (physics is the same in all directions) implies that angular momentum is a conserved quantity. These are just two examples of Noether's Theorem, proved in 1915 by Emmy Noether. At its most simplistic level, this theorem states that "to every symmetry there corresponds a conserved quantity."

In this section I want to explore some of the ramifications that follow from the symmetry properties of a simple system: the one-dimensional, time-independent Schrodinger equation.

### Theorem

Consider the one-dimensional, time-independent Schrodinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi + U(x)\psi = E\psi. \quad (7.70)$$

If  $U(x)$  is symmetric (even) in  $x$ , i.e.,  $U(-x) = U(x)$ , then the solutions  $\psi(x)$  can always be taken to be purely even or purely odd in  $x$ , i.e.,  $\psi(-x) = \pm\psi(x)$ .

### Proof

First, assume that  $\psi(x)$  is a solution to Eq. (7.70). Second, let's investigate whether or not  $\psi(-x)$  is also a solution. That is, we wish to determine if

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(-x) + U(x)\psi(-x) = E\psi(-x). \quad (7.71)$$

is satisfied or not. To do this, let's make the transformation  $x \rightarrow -x$ , which is called reflection, or the parity operation.<sup>11</sup> We need to know how each of the terms in Eq. (7.71) transform. It is straightforward to see that

$$\frac{d^2}{dx^2} \rightarrow \frac{d^2}{d(-x)^2} = \frac{d^2}{dx^2},$$

<sup>11</sup>Rigorously, the parity operator, commonly written as  $\hat{P}$ , is defined by its effect on a function

$$\hat{P}f(x) = f(-x).$$

If the function  $f$  is even, then  $\hat{P}f(x) = (+1)f(x)$ , and if the function  $f$  is odd, then  $\hat{P}f(x) = (-1)f(x)$ . This means that the eigenvalues of  $\hat{P}$  are  $\pm 1$ . Liboff, in his book *Introductory Quantum Mechanics*, has a good discussion of parity on pages 182-186.

$$U(x) \rightarrow U(-x) = U(x),$$

and, of course,

$$\psi(-x) \rightarrow \psi(-(-x)) = \psi(x).$$

Therefore, Eq. (7.71) transforms to Eq. (7.70), which means that  $\psi(-x)$  is also a solution to Eq. (7.71).

Since both  $\psi(x)$  and  $\psi(-x)$  are solutions of Eq. (7.70), then it is also true that the even linear combination

$$\psi_+ \equiv \psi(x) + \psi(-x) \tag{7.72}$$

is a solution. In addition, the odd combination  $\psi_- \equiv \psi(x) - \psi(-x)$  is a solution. Finally, since  $\psi(x) = \frac{1}{2}(\psi_+ + \psi_-)$ , this means that we can always express  $\psi$  as a linear combination of even and odd functions. **Q.E.D.**

### Implications

There are several important ramifications of this theorem that are far reaching. I will discuss two of them here, the second of which is an explanation of the Pauli exclusion principle.

**1. Operators.** Equation (7.70) can be rewritten in the form

$$\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + U(x) - E \right] \psi(x) = 0, \tag{7.73}$$

or

$$\mathcal{L}(x)\psi(x) = 0, \tag{7.74}$$

where  $\mathcal{L}$  is a differential operator that happens to be symmetric. That is, it has “even parity,” or  $\mathcal{L}(-x) = (+1)\mathcal{L}(x)$ . Clearly, if  $\psi(x)$  is a solution to Eq. (7.74), then  $\psi(-x)$  is also a solution. There are several ordinary differential equations of mathematical physics that have this property. For example, the simple harmonic oscillator equation

$$\left( \frac{d^2}{dt^2} + \omega^2 \right) x = 0 \tag{7.75}$$

has even parity. Similarly, Legendre’s equation

$$\left( (1 - z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} + \ell(\ell + 1) \right) y = 0, \tag{7.76}$$

which we will encounter in Sec. 7.6 when solving the angular part of the three-dimensional, time-independent Schrödinger equation in spherical coordinates, exhibits even parity. On the other hand, Laguerre’s equation,

$$xy'' + (1 - x)y' + ay = 0, \tag{7.77}$$

obtained from the radial part of the 3D Schrödinger equation, does not exhibit any parity (neither even nor odd). Therefore, solutions to Laguerre’s equation cannot be written as a sum of even and odd terms.

**2. The Pauli exclusion principle.** Symmetry arguments similar to the theorem I have just proven can be used to explain the Pauli exclusion principle. I'll use as my example the solutions to the Schrodinger equation for one particle in a one-dimensional infinite square well

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad E_n = \frac{\pi^2 \hbar^2}{2mL^2} n^2 \quad n = 1, 2, 3, \dots \quad (7.78)$$

Back in Sec. 7.2 we interpreted this to mean that the particle can occupy any of the states, or energy levels.

What happens if we try to put *two* particles in the same box? We need to develop and solve the Schrodinger equation for two particles, which means that we need to be careful about distinguishing the position of particle 1,  $x_1$ , from the position of particle 2,  $x_2$ . Of course, the wave function  $\psi$  must depend on the positions of both particles, and since the Schrodinger equation is just a statement of energy conservation, it can be written as

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx_1^2} - \frac{\hbar^2}{2m} \frac{d^2}{dx_2^2}\right) \psi(x_1, x_2) + U(x_1, x_2) \psi(x_1, x_2) = E \psi(x_1, x_2). \quad (7.79)$$

The first term is the sum of the kinetic energies and the second term is the sum of the potential energies of the two particles. I wrote the potential energy as  $U(x_1, x_2)$  because it is in general determined by any external forces as well as any interaction forces between the two particles. If they are electrons, say, then there will be a repulsive potential energy that depends on  $|x_1 - x_2|$  in addition to the external force that holds them bound in the 1D box.

The simplest case is to assume that the two particles are *not* interacting. Then  $U = 0$  in the box, there is no potential energy due to any interaction between the two particles, and we can use our technique of separation of variables to solve the Schrodinger equation. Let

$$\psi(x_1, x_2) = \phi(x_1) \xi(x_2), \quad (7.80)$$

where  $\phi$  is the wave function describing particle 1 and  $\xi$  is the wave function describing particle 2. That is, we are guessing that the two-particle wave function can be expressed as a product of one-particle wave functions. It is pretty clear that with this guess the Schrodinger equation will separate into two equations, one for  $x_1$  and the other for  $x_2$  (just like the result for a two-dimensional box in Sec. 7.2.2), and the boundary conditions on each separate wave function will be the same as before. Hence, the full normalized solutions are

$$\phi_{n_1}(x_1) = \sqrt{\frac{2}{L}} \sin\left(\frac{n_1 \pi x_1}{L}\right) \quad n_1 = 1, 2, 3, \dots \quad (7.81)$$

$$\xi_{n_2}(x_2) = \sqrt{\frac{2}{L}} \sin\left(\frac{n_2 \pi x_2}{L}\right) \quad n_2 = 1, 2, 3, \dots \quad (7.82)$$

where the total energy is

$$E = \frac{\pi^2 \hbar^2}{2mL^2} (n_1^2 + n_2^2). \quad (7.83)$$

Notice that this is mathematically identical to the case of one particle in a two-dimensional square well.

Let's say we place particle 1 in state 5 and particle 2 in state 6. The wave function would be

$$\psi(x_1, x_2) = \phi_5(x_1) \xi_6(x_2). \quad (7.84)$$

If the two particles are different, an electron and a proton, say, then we have no problem distinguishing them, and Eq. (7.84) is the correct wave function. But, what if they are both electrons and are therefore indistinguishable?<sup>12</sup> That is, we cannot tell them apart? In this case, how do we know that we haven't put particle 1 in level 6 and particle 2 in level 5? Our wave function must indicate the fact that the particles are indistinguishable. That is, it must encompass both possibilities at the same time. One way is to construct the following symmetric wave function

$$\psi_e(x_1, x_2) = \frac{1}{\sqrt{2}} \left[ \phi_5(x_1) \xi_6(x_2) + \phi_6(x_1) \xi_5(x_2) \right], \quad (7.85)$$

which turns out to be “even” under the interchange of the two particles

$$\psi_e(x_1, x_2) = \psi_e(x_2, x_1). \quad (7.86)$$

This wave function explicitly indicates that we don't know which particle is in which state. In fact, there's an equal probability of each of the two possibilities. A second way is to construct the *anti-symmetric* wave function

$$\psi_o(x_1, x_2) = \frac{1}{\sqrt{2}} \left[ \phi_5(x_1) \xi_6(x_2) - \phi_6(x_1) \xi_5(x_2) \right], \quad (7.87)$$

which is odd under the interchange of the two particles

$$\psi_o(x_1, x_2) = -\psi_o(x_2, x_1). \quad (7.88)$$

How do we decide which wave function to choose? The even or the odd? It turns out that *bosons*, particles with integer spin, must have symmetric wave functions, like  $\psi_e$ . On the other hand, *fermions*, particles with half-integer spin, must have anti-symmetric wave functions, like  $\psi_o$ . If you try to put two identical fermions in the same quantum state, say the ground state of the 1D box, the (antisymmetric) wave function is zero

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}} \left[ \phi_1(x_1) \xi_1(x_2) - \phi_1(x_1) \xi_1(x_2) \right] = 0, \quad (7.89)$$

and the wave function vanishes. Therefore, *two fermions cannot be in the same quantum state*. This is the precise formulation of the Pauli exclusion principle.

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<sup>12</sup>Recall that one of the features of quantum mechanics is that all electrons are identical. As David Griffiths (page 204 of *Introduction to Quantum Mechanics*) puts it

The fact is, all electrons are *utterly identical*, in a way that no two classical objects can ever be. It's not just that *we* don't happen to know which electron is which; *God* doesn't know which is which, because there is *no such thing* as “this” electron, or “that” electron; all we can legitimately speak about is “an” electron.

## 7.6 Three dimensions - central potential

An entire book can be written (and has) on solving the Schrodinger equation in three dimensions, especially for physically relevant cases, such as the hydrogen atom. In this section, I just want to give you a taste of the extra complexity, and so we'll look at just one special case — the 3D infinite spherical well.

The separation of variables in three dimensions is formally identical to that in two dimensions, which we have already carried out in the case of the particle in a 2D box. The time independent Schrodinger equation,  $H\psi = E\psi$ , becomes

$$-\frac{\hbar^2}{2m}\nabla^2\psi + U\psi = E\psi, \quad (7.90)$$

which is actually the form in which Schrodinger first obtained. If the potential energy can be written in the form

$$U(x, y, z) = f(x) + g(y) + h(z), \quad (7.91)$$

a sum of three functions, each of only one variable, then it is possible to choose Cartesian coordinates for separation, and write the wave function  $\psi$  as a *product* of three functions,  $\psi(x, y, z) = F(x)G(y)H(z)$ , each of only one variable.

However, a central potential is one in which the potential energy function depends only on  $r$ , the radial distance from the origin, and cannot be written in the form of Eq. (7.91). It can, however, be separated in spherical polar coordinates,  $r$ ,  $\theta$ , and  $\phi$ . Recall that in spherical polar coordinates the gradient operator can be written

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}, \quad (7.92)$$

and the Laplacian operator can be written

$$\nabla^2 = \nabla \cdot \nabla = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right), \quad (7.93)$$

so that our technique is to separate variables by making the guess

$$\begin{aligned} \psi(r, \theta, \phi) &= R(r)\Theta(\theta)\Phi(\phi) \\ &= R(r)Y(\theta, \phi), \end{aligned} \quad (7.94)$$

where the second line lumps the angular coordinates together into  $Y(\theta, \phi)$ , which is called a “spherical harmonic.” I first want to separate out the radial equation, so this combination makes sense. As usual, plugging this guess into the Schrodinger equation, dividing by  $\psi$ , and rearranging, results in

$$\begin{aligned} \frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\left(U(r) - E\right) &= -\frac{1}{Y}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right\} \\ &= \mathcal{L}. \end{aligned} \quad (7.95)$$

In the last line I've set both sides to a constant that I'll call  $\mathcal{L}$ .

It is customary to normalize the radial part of the wave function separately from the angular part. That is, even though all that is required is

$$\int |\psi(\vec{r})|^2 d^3r = 1, \quad (7.96)$$

where  $d^3r = r^2 dr \sin \theta d\theta d\phi$ , the integral can be split up into two factors, each of which we require to be unity

$$\int |R(r)|^2 r^2 dr = 1 \quad \int |Y(\theta, \phi)|^2 \sin \theta d\theta d\phi = 1. \quad (7.97)$$

Solving the angular equation for  $Y(\theta, \phi)$  is beyond the scope of this book, but it is fairly straightforward to solve the radial equation for  $R(r)$ .

### The radial equation

The differential equation for  $R(r)$  turns out to be easily solvable if we make a change of variable from  $R$  to  $u$

$$u(r) \equiv rR(r), \quad (7.98)$$

which leads to the ordinary differential equation<sup>13</sup>

$$-\frac{\hbar^2}{2m} \frac{d^2 u(r)}{dr^2} + \left\{ U(r) + \frac{\hbar^2 \mathcal{L}}{2m r^2} \right\} u(r) = Eu(r). \quad (7.99)$$

Notice that this is exactly the same form as the one dimensional Schrodinger equation, but with an “effective” potential energy

$$U_{eff}(r) = U(r) + \frac{\hbar^2 \mathcal{L}}{2mr^2}, \quad (7.100)$$

which is just the true potential energy  $U$  plus a term that describes the centrifugal potential energy. You are familiar with the fictitious centrifugal force, but what is the centrifugal potential energy? Recall that any conservative force can be written in terms of a potential energy, so the centrifugal force, i.e., that which feels like you are being pushed outwards, has only a radial component

$$F_r = m\omega^2 r, \quad (7.101)$$

and any potential energy is related to the force by  $F_r = -dU/dr$ , which gives  $U = m\omega^2 r^2/2$ . The angular momentum of a particle of mass  $m$  traveling in a circle of radius  $r$  with velocity  $v$  is  $L = mvr = m\omega r^2$ , so that the centrifugal potential energy can be written

$$U = \frac{m\omega^2 r^2}{2} = \frac{L^2}{2mr^2}. \quad (7.102)$$

<sup>13</sup>To obtain this form, I have rewritten the first term in Eq. (7.95) in the following way

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial R}{\partial r} \right) = \frac{1}{r} \frac{\partial^2}{\partial r^2} (rR).$$

It is clear, then, that  $\hbar^2\mathcal{L}$  is the square of the angular momentum. From Chapter 1, recall that the angular momentum is quantized, and takes on the possible values  $L = \sqrt{\ell(\ell+1)}\hbar$ , which implies that when we solve the differential equation for  $Y(\theta, \phi)$  we should find out that the allowed values of the separation constant are  $\mathcal{L} = \ell(\ell+1)$ . Indeed, this is the case. You can show that when solving the angular part of the Schrodinger equation for  $\Theta(\theta)$ , normalizable solutions can only be found for integer values of  $\ell$ .

### The infinite spherical well

One of the simplest cases, just as in one dimension, is that of an infinite well. In this case, however, the well is spherical in shape, and has a potential energy function that takes the form

$$U(r) = \begin{cases} 0 & r < a \\ \infty & r > a \end{cases} \quad (7.103)$$

For simplicity, let's look at the case where  $\mathcal{L} = 0$ . Of course, this means that  $\ell = 0$ , which is a spherically symmetric solution with zero total angular momentum. Just like in one dimension (because the Schrodinger equation is formally identical to that case) the potential energy is piecewise constant. Hence, we can break the solution up into two regions. Inside the well (for  $r < a$ ) the equation for  $u$  is

$$\frac{d^2u}{dr^2} = -k^2u, \quad (7.104)$$

where  $k^2 = 2mE/\hbar^2$ , and outside the well (for  $r > a$ ) the wave function must be zero, i.e.,  $u = 0$ . Since we are looking for stationary states, it's natural to look for solutions of the standing wave type—rather than the traveling wave type—so we can express the general solution as

$$u(r) = A \sin(kr) + B \cos(kr), \quad (7.105)$$

which means that  $R$  is

$$R(r) = \frac{u}{r} = A \frac{\sin(kr)}{r} + B \frac{\cos(kr)}{r}. \quad (7.106)$$

The wave function must be finite at the origin (i.e., as  $r \rightarrow 0$ ), which means that  $R$  must be finite also. This requires, therefore, that  $B = 0$ , because otherwise  $R \sim 1/r$ , and  $|\Psi|^2 \sim 1/r^2$ , which is not normalizable. The other boundary, at  $r = a$ , again requires that  $R = 0$ , just as in the one dimensional case. As in the one dimensional infinite square well, this second boundary limits the allowed values of the wavelength

$$ka = n\pi \quad n = 1, 2, 3, \dots \quad (7.107)$$

which limits the total energy to the allowed values

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\pi^2 \hbar^2}{2ma} n^2. \quad (7.108)$$

Interestingly, this has exactly the same *form* as the one dimensional infinite square well, although the meaning of  $a$  here and  $L$  in the one dimensional case are different. The



normalized wave functions are slightly different from the 1D case

$$R_{n0}(r) = \frac{1}{r} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi r}{a}\right), \quad (7.109)$$

where the radial wave function carries two quantum numbers, both  $n$  and  $\ell$ . It turns out that  $R_{n0}$  is related to a spherical Bessel function of order 0,  $j_0(x) = \sin x/x$ . The case where  $\ell = 1$  is covered in Problem 139.

## Problems

**120.** Show that in order to normalize  $\Psi$  in the one-dimensional particle-in-a-box, Eq. (7.57) must be satisfied. That is, evaluate the integral in Eq (7.12) using the wave-function in Eq. (7.48).

**121.** (a) Estimate the ground state energy of an electron in an atom by treating it as if it were in an infinite square well (1D) of width equal to an atomic diameter of  $10^{-10}$  m. (b) Estimate the ground state energy of an  $\alpha$  particle in a nucleus by treating it as if it were in an infinite square well (1D) of width equal to a nuclear diameter of  $10^{-14}$  m. (c) Estimate the ground state energy of an electron in a nucleus by treating it as if it were in an infinite square well (1D) of width equal to a nuclear diameter of  $10^{-14}$  m.

**122.** The fact that the ground state energy (or “zero-point” energy) is not zero is a consequence of Heisenberg’s indeterminacy principle. To show this, note that if a particle is bound in a one-dimensional box of size  $L$ , then  $\Delta x \approx L$ . Now carry out the following steps: (a) Using Heisenberg’s inequality, what is the required indeterminacy of the momentum,  $\Delta p$ ? (Use the correct principle  $\Delta x \Delta p \geq \hbar/2$ .) (b) What is the magnitude of the momentum  $p$  of a particle in the ground state of an infinite square well? Since the particle can be moving either to the left or to the right, the uncertainty of the momentum of a particle in the ground state is  $\Delta p \approx 2p$ . (c) Compare the two values obtained for  $\Delta p$ . Do they agree?

**123.** Obtain a Balmer-like formula for the spectrum of a particle of mass  $m$  in a 1D box. That is, obtain the counterpart to Eq. (4.18).

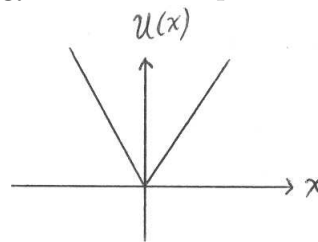
**124.** For a 1-kg ball in a 1-m<sup>2</sup> square two-dimensional box (i.e.,  $L = 1$  m), what is the lowest quantum energy allowed? If the ball had this kinetic energy, how long would it take to travel from one side of the box to the other? Is this time something that you think you could measure?

**125.** For a 2D infinite square well, what is the next highest energy level (above  $50E_G$ ) that exhibits an “accidental degeneracy”? List all the possible sets of quantum numbers for this state.

**126.** Enumerate the lowest 15 energy levels in a three-dimensional symmetric box (along with their quantum numbers). There should be one level that is accidentally degenerate. Which one is it?

**127.** What are the quantum numbers for the 3D symmetric box with energy  $E = 363 E_G = 363 (\pi^2 \hbar^2 / 2ma^2)$ . Hint: the degeneracy (some of which is accidental) is 13.

**128.** Sketch the wavefunctions  $\psi(x)$  for the lowest four energy levels of the potential well on the right. Make sure to utilize your knowledge of symmetry, continuity, and normalizability. Indicate the energies on a plot of  $U(x)$ , and draw four separate plots of the four wavefunctions. Label the locations of the classical turning points.



**129.** In three dimensions, the parity operator is defined as

$$\hat{P}f(x, y, z) = f(-x, -y, -z).$$

(a) What is the parity of the following function?

$$g = A(x + y + z)e^{(x^2+y^2+z^2)}.$$

(b) Can you express the effect of the parity operator if  $f$  is written as a function of spherical coordinates  $f = f(r, \theta, \phi)$ ?

**130.** (a) Given a wavefunction

$$\psi(x) = \begin{cases} A \sin(kx) & 0 < x < \pi/k \\ 0 & \text{otherwise} \end{cases}$$

find  $A$ . That is, normalize the wavefunction. (b) Find  $\langle x \rangle$ , the average position of the particle.

**131.** Show that the expectation value of  $x^2$  for a particle in state  $n$  of a 1D infinite square well is

$$\langle x^2 \rangle = L^2 \left( \frac{1}{3} - \frac{1}{2\pi^2 n^2} \right).$$

**132.** Show that the root-mean-square deviation for the position of a particle in state  $n$  of a 1D infinite square well is

$$(\Delta x)_{rms} \equiv \sqrt{\langle (\Delta x)^2 \rangle} = L \sqrt{\frac{1}{12} - \frac{1}{2\pi^2 n^2}}.$$

**133.** Continue the line of investigation that you started with Problem 132 by calculating  $\langle p_x \rangle$ ,  $\langle (p_x)^2 \rangle$ , and  $(\Delta p_x)_{rms}$  for the  $n$ th state in the 1D infinite square well. Recall the definition of expectation value in Eq. (7.59) and the fact that the momentum operator is

$$p_x \equiv \frac{\hbar}{i} \frac{\partial}{\partial x}.$$

To obtain the expectation value of  $p_x^2$  you will need to differentiate  $\psi$  twice

$$p_x^2 \psi = -\hbar^2 \frac{\partial^2 \psi}{\partial x^2}.$$

**134.** Using the results of Problems 132 and 133, show that all states in the 1D infinite square well satisfy Heisenberg's uncertainty relation.

**135.** Evaluate the integral in Eq. (7.64) for the amplitude function in Eq. (7.66) using the Taylor series approximation for  $\omega(k_x)$  in Eq. (7.67). You should obtain as your answer Eq. (7.69).

**136.** Normalize the wave function in Eq. (7.69). That is, determine  $A$ .

**137.** A more realistic wave packet can be obtained by using a more realistic amplitude function than in Eq. (7.66). In this problem we'll just worry about the spatial dependence, so that at some time  $t_0$  the wave function is

$$y(x) = \int A(k) \cos(kx) dk.$$

If you use a Gaussian for the amplitude distribution of wavenumbers

$$A(k) = \exp\left[-\frac{(k - k_0)^2}{2\bar{k}^2}\right],$$

where  $k_0$  is the “central” wavenumber (i.e., the wave with the largest amplitude) and  $\bar{k}$  gives a measure of the width of the amplitude distribution, then you can show that the wave packet can be written

$$y(x) = \sqrt{2\pi\bar{k}} e^{-\bar{k}^2 x^2 / 2} \cos(k_0 x),$$

which is a sinusoidal oscillation with a Gaussian envelope function. (a) Sketch (by hand, not on a computer)  $A(k)$ , making sure to label the axes and the locations of  $k = k_0$  and  $k = k_0 \pm \bar{k}$ . (b) Show that the integral over  $k$  results in the wave packet  $y(x)$  above. HINT: The technique for integrals of this type is to rewrite the cosine function as a sum of complex exponentials:  $2 \cos \phi = e^{i\phi} + e^{-i\phi}$ . Then you are left with an integral of exponentials, and “completing the square” in the exponent allows you to write the integral in the form of the error function  $\text{erf}(x)$

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz,$$

where  $\text{erf}(0) = 0$  and  $\text{erf}(\infty) = 1$ .

**138.** Consider a particle of mass  $m$  in a 1D infinite well of width  $L$  that is *not* in a stationary state. That is, assume that you place the particle (at  $t = 0$ ) in a superposition state (NOT a stationary state) that is half ground state and half first excited state:

$$f(x) = \sin\left(\frac{\pi x}{L}\right) + \sin\left(\frac{2\pi x}{L}\right).$$

This is not, of course, normalized. Express this initial wave function in the form

$$\Psi(x, t = 0) = c_1 \psi_1(x) + c_2 \psi_2(x),$$

where

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

are the *normalized* stationary eigenfunctions. (a) Normalize  $\Psi(x, t = 0)$ . That is, evaluate  $c_1$  and  $c_2$ . (b) Write the full time dependent wave function  $\Psi(x, t)$ . That is, recall that the general solution to the partial differential equation is

$$\Psi(x, t) = \sum_n c_n \psi_n(x) \exp\left(-i \frac{E_n}{\hbar} t\right),$$

where in this case all  $c_n$  are zero except for  $c_1$  and  $c_2$ . (c) Calculate the expectation value of the position

$$\langle x \rangle = \int_0^L \Psi^* x \Psi dx.$$

Careful. Not all of the complex exponentials will cancel because there are cross terms that survive. This is not a stationary state so this expectation value will be a function of time! (d) Calculate the expectation value of the kinetic energy. Predict your answer before you calculate!

**139.** Show that

$$u(r) = \frac{\sin(kr)}{kr} - \cos(kr)$$

is a solution to the radial part of the Schrodinger equation for the infinite spherical well when  $\ell = 1$ . That is, let  $\ell = 1$ , or  $\mathcal{L} = 2$ , so that in the region  $r < a$  Eq. (7.99) simplifies to

$$\frac{d^2 u(r)}{dr^2} + \left(k^2 - \frac{2}{r^2}\right) u(r) = 0,$$

where, as before,  $k^2 = 2mE/\hbar^2$ . This function  $u$  is the spherical Bessel function of order 1,  $j_1(kr)$ .

**140.** Form explicitly the differential operators for (a) the simple harmonic oscillator, (b) Legendre's equation, (c) Bessel's equation, (d) Hermite's equation, and (e) Laguerre's equation. Determine explicitly the parity of each operator.

## Solutions

**120.** Inserting the most general wave function into the normalization condition and exchanging the order of the sums and integral gives

$$\begin{aligned} 1 &= \int_0^L \left( \sum_{n=1}^{\infty} c_n^* \psi_n^*(x) e^{+iE_n t/\hbar} \right) \left( \sum_{m=1}^{\infty} c_m \psi_m(x) e^{-iE_m t/\hbar} \right) dx \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_n^* c_m e^{+i(E_n - E_m)t/\hbar} \int_0^L \psi_n^*(x) \psi_m(x) dx. \end{aligned}$$

Note that for the 1D square well  $\psi$  is real, so the complex conjugate symbol is not needed. However, I retain it to keep our result general. Since the wave functions  $\psi_n(x)$  are orthonormal, the integral becomes

$$\int_0^L \psi_n^*(x) \psi_m(x) dx = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \delta_{nm},$$

where  $\delta_{nm}$  is the KRONECKER DELTA, which is unity if  $n = m$  and zero otherwise, this integral “collapses” one of the infinite sums, and also reduces the time exponential to unity. Thus we are left with

$$1 = \sum_{m=1}^{\infty} c_m^* c_m.$$

**Q.E.D.**

**121.** (a) From the results of Problem 116, we have

$$E_1 = \frac{(hc)^2}{8L(mc^2)} = \frac{(1240 \text{ MeV fm})^2}{8(0.1 \text{ nm})^2(0.511 \text{ MeV})} = 37.5 \text{ eV}.$$

(b) Inserting  $mc^2 = 3727 \text{ MeV}$  and  $L = 10^{-5} \text{ nm}$  into the above formula,  $E_1 = 0.514 \text{ MeV}$ .

(c) Inserting  $mc^2 = 0.511 \text{ MeV}$  and  $L = 10^{-5} \text{ nm}$  into the above formula,  $E_1 = 3.75 \text{ GeV}$ .

The first two energies are reasonably close to the experimental values. The last, however, implies that electrons cannot be confined in the nucleus, so this is another argument against the pe model.

**122.** (a) Heisenberg gives

$$\Delta p \geq \frac{\hbar}{2\Delta x} \approx \frac{\hbar}{2L}.$$

(b) The momentum of the particle in the ground state, assuming that it is nonrelativistic, is

$$p = \sqrt{2mE_1} = \sqrt{2m \left( \frac{\pi^2 \hbar^2}{2mL^2} \right)} = \frac{\pi \hbar}{L},$$

and, of course,  $\Delta p \approx 2p = 2\pi\hbar/L$ . (c) The inequality becomes

$$\frac{2\pi\hbar}{L} \geq \frac{\hbar}{2L},$$

which is obviously satisfied. If the ground state energy were zero, then the ground state momentum would be zero, and Heisenberg’s inequality would read  $0 \geq \hbar/2L$ , which is not satisfied.

**124.** For a 2D infinite square well, the quantum numbers of the ground state are  $n = m = 1$ , so that the energy is

$$E_{11} = 2E_G = \frac{\pi^2 \hbar^2}{mL^2} = \frac{\pi^2 (1.05 \times 10^{-34} \text{ J s})^2}{(1 \text{ kg})(1 \text{ m})^2} = 1.09 \times 10^{-67} \text{ J}.$$

If  $K = E_{11}$  then, noting that it’s nonrelativistic, the speed would be  $v = \sqrt{2K/m}$  and the time it would take to travel across the box is  $\Delta t = L/v$ , or

$$\Delta t = L \sqrt{\frac{m}{2E_{11}}} = (1 \text{ m}) \sqrt{\frac{(1 \text{ kg})}{2(1.09 \times 10^{-67} \text{ J})}} = 2.14 \times 10^{33} \text{ s} \approx 7 \times 10^{25} \text{ years}.$$

Since this is much longer than the age of the universe, no, it is not technically feasible to measure it.

**125.** For a 2D symmetric well, it is useful to determine the energy for each set of quantum numbers as shown in the following table. The expected (symmetric) degeneracies are not listed, but the accidentally degenerate states are boxed. The energies are given as multiples of  $E_G = \pi^2 \hbar^2 / 2mL^2$ , the one-particle ground state energy.

$n_y \backslash n_x$	1	2	3	4	5	6	7	8	9
1	2	5	10	17	26	37	50	65	82
2		8	13	20	29	40	53	68	85
3			18	25	34	45	58	73	90
4				32	41	52	65	80	97
5					50	61	74	89	106
6						72	85	100	117
7							98	113	130

In order to determine the energy levels and possible transitions between them, it is helpful to list the states in increasing order of energy, as below. All states where  $n_x = n_y$  are not degenerate, but if  $n_x \neq n_y$  then the state is doubly degenerate (but an expected degeneracy).

level	$E/E_G$	$n_x, n_y$	level	$E/E_G$	$n_x, n_y$
1	2	1,1	13	34	3,5
2	5	1,2	14	37	1,6
3	8	2,2	15	40	2,6
4	10	1,3	16	41	4,5
5	13	2,3	17	45	3,6
6	17	1,4	18	50	1,7 - 5,5
7	18	3,3	19	52	4,6
8	20	2,4	20	53	2,7
9	25	3,4	21	58	3,7
10	26	1,5	22	61	5,6
11	29	2,5	23	65	1,8 - 4,7
12	32	4,4			

For those who are interested, the next accidentally degenerate state is the 30th energy level, with  $E = 85E_G$ , and  $n_x, n_y = 2, 9$  and  $6, 7$ . In addition, it's easy to show that  $2^2 + 11^2 = 5^2 + 10^2 = 125$  as well as  $10^2 + 10^2 = 2^2 + 14^2 = 200$ . In fact, there are an infinite number of accidental degeneracies in this system.

**126.** As in the solution to Problem 125, the same technique of listing all the states can be utilized. The states in increasing order of energy are listed. all are expected to be degenerate due to their symmetry, but  $E_{14} = 27E_G$  is quadruply degenerate by accident. Extra: show that the 17th and 21st energy levels are also accidentally degenerate.

level	$E/E_G$	$n_x, n_y, n_z$
1	3	1,1,1
2	6	1,1,2
3	9	1,2,2
4	11	1,1,3
5	12	2,2,2
6	14	1,2,3
7	17	2,2,3
8	18	1,1,4
9	19	1,3,3
10	21	1,2,4
11	22	2,3,3
12	24	2,2,4
13	26	1,3,4
14	27	1,1,5 - 3,3,3
15	29	2,3,4

**129.** (a) Since  $\hat{P}g = -g$ , the parity of  $g$  is odd. (b) The radial distance  $r$  remains the same, but the polar angle  $\theta$  is flipped around  $\pi/2$  and the azimuthal angle  $\phi$  is rotated one half a revolution. These all combine to give

$$\hat{P}f(r, \theta, \phi) = f(r, \pi - \theta, \pi + \phi).$$

**130.** (a) Normalization requires  $\int |\psi|^2 dx = 1$  or, in this case

$$A^2 \int_0^{\pi/k} \sin^2(kx) dx = \frac{A^2}{k} \int_0^{\pi} \sin^2 z dz = 1.$$

Since the last integral is equal to  $\pi/2$ , I get

$$A = \sqrt{\frac{2k}{\pi}}.$$

(b) Again, changing variables to let  $z = kx$ , we have

$$\langle x \rangle = A^2 \int_0^{\pi/k} x \sin^2(kx) dx = \frac{A^2}{k^2} \int_0^{\pi} z \sin^2 z dz.$$

Substituting in the value for  $A$  found before, and noting that the last integral is equal to  $\pi^2/4$ , we get

$$\langle x \rangle = \frac{\pi}{2k},$$

or  $\langle kx \rangle = \pi/2$ , as expected.

**131.** In this problem, a knowledge of integration techniques is very useful.

$$\langle x^2 \rangle = \frac{2}{L} \int_0^L x^2 \sin^2\left(\frac{n\pi x}{L}\right) dx \quad - \text{ let } y = \frac{n\pi x}{L}$$

$$\begin{aligned}
&= \frac{2}{L} \left( \frac{L}{n\pi} \right)^3 \int_0^{n\pi} y^2 \sin^2 y \, dy \quad - \text{let } \sin^2 y = \frac{1}{2} (1 - \cos 2y) \\
&= \frac{2L^2}{n^3\pi^3} \left( \frac{1}{2} \right) \left[ \frac{(n\pi)^3}{3} - \int_0^{n\pi} y^2 \cos 2y \, dy \right] \\
&= L^2 \left( \frac{1}{3} - \frac{1}{2\pi^2 n^2} \right) \qquad \qquad \qquad \mathbf{Q.E.D.}
\end{aligned}$$

The last integral was evaluated by integrating by parts twice to obtain  $n\pi/2$ .

**132.** Since  $\langle x \rangle = L/2$  and we've just calculated  $\langle x^2 \rangle$  then we have

$$(\Delta x)_{rms} = \langle x^2 \rangle - \langle x \rangle^2 = L^2 \left( \frac{1}{12} - \frac{1}{2\pi^2 n^2} \right).$$

**133.** First, the particle is bound, so therefore  $\langle p_x \rangle = 0$ . You can calculate  $\langle p_x^2 \rangle$  using the brute force method, i.e., by evaluating the expectation value integral, or you can be clever. First, note that  $H = p_x^2/2m = E$  (recall that  $U = 0$ ). Therefore

$$\langle p_x^2 \rangle = \langle 2mE \rangle = 2m \left( \frac{n^2 \pi^2 \hbar^2}{2mL^2} \right)$$

because the expectation value of  $E$  in state  $n$  is just the energy of that state (i.e., it's a stationary state). Thus we have  $\langle p_x^2 \rangle = n^2 \pi^2 \hbar^2 / L^2$ . Finally, the rot-mean-square value of the momentum is

$$(\Delta p_x)_{rms} = \sqrt{\langle p_x^2 \rangle - \langle p_x \rangle^2} = \frac{n\pi\hbar}{L}.$$

**134.** Heisenberg's inequality is

$$(\Delta x)_{rms} (\Delta p_x)_{rms} = L \sqrt{\frac{1}{12} - \frac{1}{2\pi^2 n^2}} \times \frac{n\pi\hbar}{L} = \hbar \sqrt{\frac{n^2 \pi^2}{12} - \frac{1}{2}} \geq \frac{\hbar}{2}.$$

If  $n = 1$  then the value of the square root is 0.568 which is greater than  $\frac{1}{2}$ . As  $n$  increases, the value of the square root increases, so that all states satisfy the inequality. In fact, the ground state is close to being a "minimum uncertainty" state.

**135.** With the Taylor series approximation of Eq. (7.67), the integral in Eq. (7.64) becomes straightforward

$$\Psi(x, t) \approx A \int_{k_0 - k_1}^{k_0 + k_1} e^{ik_x x} e^{-i[\omega_0 + \omega'_0(k_x - k_0)]t} dk_x.$$

Pulling out those terms that are constant (i.e., that don't depend on  $k_x$ ) and rearranging the integrand gives

$$\Psi(x, t) = A e^{-it(\omega_0 - \omega'_0 k_0)} \int_{k_0 - k_1}^{k_0 + k_1} e^{ik_x(x - \omega'_0 t)} dk_x.$$

The integral is now elementary and can be evaluated, and the arguments of the exponentials can be manipulated to obtain

$$\int_{k_0 - k_1}^{k_0 + k_1} e^{ik_x(x - \omega'_0 t)} dk_x = \frac{e^{i(k_0 + k_1)(x - \omega'_0 t)} - e^{i(k_0 - k_1)(x - \omega'_0 t)}}{i(x - \omega'_0 t)}$$



$$\begin{aligned}
&= 2 \frac{e^{ik_0(x-\omega'_0 t)}}{(x-\omega'_0 t)} \left[ \frac{e^{ik_1(x-\omega'_0 t)} - e^{-ik_1(x-\omega'_0 t)}}{2i} \right] \\
&= 2 \frac{e^{ik_0(x-\omega'_0 t)}}{(x-\omega'_0 t)} \sin [k_1(x-\omega'_0 t)].
\end{aligned}$$

Plugging this back into the expression for  $\Psi$  and manipulating the exponentials again results in Eq. (7.69)

$$\Psi(x, t) = (2Ak_1) \left[ \frac{\sin [k_1(x-\omega'_0 t)]}{k_1(x-\omega'_0 t)} \right] e^{i(k_0 x - \omega_0 t)}.$$

**136.** From a table of integrals I get

$$\int_0^\infty \frac{\sin^2(px)}{x^2} dx = \frac{\pi p}{2}.$$

The normalization integral thus becomes

$$1 = \int_{-\infty}^\infty |\Psi|^2 dx = 4|A|^2 k_1^2 \frac{1}{k_1^2} \int_{-\infty}^\infty \frac{\sin^2(k_1 z)}{z^2} dz$$

where  $z = x - \omega'_0 t$  is a change of variables. Since this integral must be unity

$$1 = 4|A|^2 \left( 2 \frac{\pi k_1}{2} \right)$$

or

$$A = \frac{1}{2\pi\sqrt{k_1}}.$$

**138.** (a)  $c_1 = c_2 = 1/\sqrt{2}$ . Check:  $c_1^2 + c_2^2 = 1$ . (b) This can be written down “by inspection,” because we know the energies of each of the states:

$$\Psi(x, t) = \frac{1}{\sqrt{2}} \left( \psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar} \right),$$

where  $\psi_n$  are the normalized eigenfunctions and  $E_n = (\pi^2 \hbar^2 / 2mL^2)n^2$  are the energies of the stationary states. (c) This is specific example of the superposition principle discussed on page 194. First we need

$$\Psi^* \Psi = \frac{1}{2} \left( \psi_1^2 + \psi_2^2 + 2 \cos \omega t \psi_1 \psi_2 \right),$$

where

$$\omega = \frac{(E_2 - E_1)}{\hbar} = \frac{3\pi^2 \hbar}{2mL^2}.$$

The expectation value of  $x$  is

$$\langle x \rangle = \int_0^L \Psi^* x \Psi dx = \frac{1}{2} \left[ \langle x \rangle_1 + \langle x \rangle_2 \right] + \cos \omega t \int_0^L x \psi_1 \psi_2 dx,$$

where  $\langle x \rangle_1 = \langle x \rangle_2 = L/2$  are the expectation values of  $x$  in states 1 and 2, respectively. the integral is

$$\int_0^L x \psi_1 \psi_2 dx = \left(\frac{L}{\pi}\right)^2 \frac{2}{L} \int_0^\pi y \sin y \sin 2y dy = -\frac{8}{9},$$

where I made the variable substitution  $y = \pi x/L$ . The final answer is

$$\langle x \rangle = \frac{L}{2} - \left(\frac{4}{3\pi}\right)^2 L \cos \omega t.$$

The average (expected) position oscillates about the center with a frequency equal to the *beat* frequency, and an amplitude equal to  $(4/3\pi)^2 \approx 0.18L$ .

**139.** This problem can be solved with a simple substitution. However, it is interesting to note that the spherical Bessel functions are defined as

$$j_0 = \sin z/z \quad y_0 = -\cos z/z$$

$$j_1 = \sin z/z^2 - \cos z/z \quad y_1 = -\cos z/z^2 - \sin z/z$$

The differential equation for  $R$  is (leaving  $\ell$  arbitrary)

$$r^2 R'' + 2rR' + [(kr)^2 - \ell(\ell + 1)] R = 0,$$

which can be written in dimensionless form, with  $z = kr$ , and  $w(z) = R(kr)$  as

$$z^2 w'' + 2zw' + [z^2 - \ell(\ell + 1)] w = 0,$$

which is nothing but the spherical Bessel function equation. The function  $R$  that satisfies the radial part of the Schrodinger equation for  $\ell = 1$  is

$$R(r) = A \left[ \frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr} \right] = A j_1(kr) \sim \frac{u(r)}{r}.$$

The allowed energies in this case are slightly more complicated than when  $\ell = 0$  because there is no simple formula for the zeros of this function. The zeros of  $j_1(z)$  occur when  $z = 4.493409, 7.725252, 10.904122$ , etc. So that the energies are given by  $ka = j_{n,s}$  where  $j_{n,s}$  is the  $s$ th zero of  $j_n$ .

# Bibliography

## Books

- Ernest S. Abers, *Quantum Mechanics*, Prentice-Hall, 2003. (ERAU: QC 174.12 .A24 2004)
- Hans A. Bethe and Roman Jackiw, *Intermediate Quantum Mechanics*, 2nd ed., W.A. Benjamin, New York, 1968. (ERAU: Reynolds' office)
- Henry A Boorse and Lloyd Motz, editors, *The World of the Atom*, Basic, 1966. (ERAU: QC 173 .B56)
- P. W. Bridgman, *Dimensional Analysis*, Yale University Press, 1922. (ERAU: Reynolds' office)
- Barbara Lovett Cline, *Men who made a new physics*, University of Chicago Press, 1987. (ERAU: QC 15 .C4 1987)
- Robert Eisberg and Robert Resnick, *Quantum Physics: of Atoms, Molecules, Solids, Nuclei, and Particles*, 2nd edition, Wiley, 1985. (ERAU: Reynolds' office)
- Michael W. Friedlander, *Cosmic Rays*, Harvard University Press, 1989. (ERAU: QC 485 .F75 1989)
- F. L. Friedman and L. Sartori, *The Classical Atom*, Addison-Wesley, 1965. (ERAU: QC 174.1 .07)
- George Gamow, *Thirty years that shook physics: The story of quantum theory*, Doubleday, 1966. (ERAU: QC 174.1 .G3)
- David J. Griffiths, *Introduction to Quantum Mechanics*, 2nd ed., Addison-Wesley, 2004.
- Werner Heisenberg, *The Physical Principles of the Quantum Theory*, Dover, 1930. (ERAU: QC 174.1 .H4 1949)
- Ernest M. Henley and Alejandro Garcia, *Subatomic Physics*, 3rd ed., World Scientific, 2007. (ERAU: QC 776 .H46 2007)
- J. D. Jackson, *Classical Electrodynamics*, 2nd ed., Wiley, 1975. (ERAU: 3rd ed. QC 631 .J3 1999)
- Richard L. Liboff, *Introductory Quantum Mechanics*, 3rd ed., Addison-Wesley, 1998. (ERAU: 2nd ed. QC 174.12 .L52 1992)
- B. R. Martin and G. Shaw, *Particle Physics*, Wiley, 1992. (ERAU: QC 793.2 .M38 1992)
- Walter E. Meyerhoff, *Elements of nuclear physics*, McGraw-Hill, 1967. (ERAU: Reynolds' office)
- Abraham Pais, *Subtle is the Lord*, Oxford, 1982. (ERAU: QC 16.E5 P35 2005)
- Abraham Pais, *Inward Bound: Of matter and forces in the physical world*, Oxford, 1986. (ERAU: Reynolds' office)

Linus Pauling, *General Chemistry*, Freeman, 1970, 3rd ed., Dover reprint, 1988. (ERAU: Reynolds' office)

Emilio Segre, *From X-rays to quarks: Modern physicists and their discoveries*, W. H. Freeman, 1980. (ERAU: QC 7 .S44 1980)

Eric R. Scerri, *The periodic table: Its story and significance*, Oxford University Press, 2007. (ERAU: QD 467 .S34 2007)

Paul Schilpp, editor, *Albert Einstein: Philosopher-Scientist*, MFJ Books, 1949. (ERAU: QC 16.E5 A43 2000)

J. J. Sakurai, *Modern Quantum Mechanics*, Benjamin-Cummings, 1985. (ERAU: Reynolds' office)

Peter Scott and William L. Burke, *Special Relativity Primer*, unpublished notes, 1991. (ERAU: Reynolds' office)

John Stachel, ed., *Einstein's Miraculous Year: Five Papers that Changed the Face of Physics*, Princeton University Press, 1998. (ERAU: QC7 .E52 1998)

Edwin F. Taylor and John Archibald Wheeler, *Spacetime Physics: Introduction to Special Relativity*, 2nd ed., Freeman, 1992. ERAU: QC 173.65 T37 1991)

John R. Taylor, *An Introduction to Error Analysis*, University Science Books, 1997. (ERAU: QC 39 .T4 1997)

Martinus Veltman, *Facts and mysteries in elementary particle physics*, World Scientific, 2003. (ERAU: QC 793.2 .V45 2003)

Steven Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, Wiley, 1972. (ERAU: Reynolds' office)

Rolf Winter, *Quantum Physics*, Wadsworth, 1979. (ERAU: Reynolds' office)

## Articles

G. Audi, O. Bersillon, J. Blachot, and A. H. Wapstra, "The NUBASE evaluation of nuclear and decay properties," *Nucl. Phys. A*, **729** 3-128 (2003).

R. W. Clay, A. G. K. Smith, and J. L. Reid, "Cosmic Ray Induced Noise in Gravitational Wave Detectors," *Pub. Astron. Soc. Australia* **14**(2) (1997).

Arthur H. Compton, "A quantum theory of the scattering of X-rays by light elements," *Phys. Rev.* **21** 483-502 (1923).

Hans Dehmelt, "A Single Atomic Particle Forever Floating at Rest in Free Space: New Value for Electron Radius," *Phys. Scr.* **T22**, 102-110 (1988). Measured an upper limit on the electron radius of  $10^{-22}$  m.

Albert Einstein, "On the electrodynamics of moving bodies," *Ann. Phys.* **17** 891 (1905). ([www.fourmilab.ch/etexts/einstein/specrel/www/](http://www.fourmilab.ch/etexts/einstein/specrel/www/))

G. F. FitzGerald, "The Ether and the Earth's Atmosphere," *Science* **13**, 390 (1889).

Stanley M. Flatte, "The Schrodinger equation in classical physics," *Am. J. Phys.*, **54**, 1088-1092 (1986).

Bretislav Friedrich and Dudley Herschbach, "Stern and Gerlach: How a Bad Cigar Helped Reorient Atomic Physics," *Physics Today*, **56**(12) 53-59 (December 2003).

Sheldon Goldstein, "Quantum Theory Without Observers - Part One", *Phys. Today*, March 1998, pp.42-46.

J. C. Hafele and Richard E. Keating, "Around-the-World Atomic Clocks: Predicted relativistic time gains," *Science* **177** (4044), 166-168, (1971).

Harkness and Wilson, *J. Amer. Chem. Soc.* **37** 1367 (1915).

Jun Luo, Liang-Cheng Tu, Zhong-Kun Hu, and En-Jie Luan, "New experimental limit on the photon rest mass with a rotating torsion balance," *Phys. Rev. Lett.* **90** 081801 (2003).

P. de Marcillac, N Coron, G Dambier, J Leblanc, and J-P Moalic, "Experimental detection of  $\alpha$ -particles from the radioactive decay of natural bismuth," *Nature* **422**(6934) 876-878 (2003).

Ernan McMullin, "The Origins of the Field Concept in Physics," *Physics in Perspective*, **4**, 13-39, 2002.

R. A. Millikan, "A direct determination of  $h$ ." *Phys. Rev.* **4**, 73-75 (1914).

H. G. J. Moseley, "The high frequency spectra of the elements," *Phil. Mag.*, **26** 1024 (1913).

E. Schrödinger, "An undulatory theory of the mechanics of atoms and molecules," *Phys. Rev.* **28**, 1049-1070 (1926).

Dennis W. Sciama, "Consistent neutrino masses from cosmology and solar physics," *Nature* **348**, 617-618 (13 December 1990).

Harold C. Urey, F. G. Brickwedde, and G. M. Murphy, "A hydrogen isotope of mass 2," *Phys. Rev.* **39**, 164-165 (1932).

## Web sites

HyperPhysics, Carl R. Nave, Department of Physics and Astronomy, Georgia State University, Atlanta, Georgia 30303-3088, RodNave@gsu.edu.

<http://hyperphysics.phy-astr.gsu.edu/hbase/hframe.html>

Nobel Laureates in Physics

[http://nobelprize.org/nobel\\_prizes/physics/laureates/index.html](http://nobelprize.org/nobel_prizes/physics/laureates/index.html)



# Appendix A

## Suggested Background Knowledge

At the beginning of our story ( $\sim 1900$ ) the laws of mechanics were quite well developed, and their dynamical consequences were understood. In the 200 years after Newton, his laws had been applied to systems as diverse as projectiles on the Earth's surface and binary stars in galaxies far, far away. The mathematical structure, though, had become much more rich and complex since Newton's day. In particular, Lagrange's and Hamilton's equations of motion had replaced Newton's second law as the most elegant and powerful formulations of classical mechanics. These formulations, however, required that energy (especially potential energy) play a more important role than force in the understanding of dynamical systems and in the intuition of physicists.

In the quest to understand the microscopic world, it was natural to use these powerful techniques — assuming, of course, that atoms behaved in a “mechanistic” way, analogous to macroscopic objects. This mechanistic requirement was already starting to fail in Maxwell's electromagnetic theory, since the concept of an “ether”<sup>1</sup> that fills all space was under increasing attack. Light, however, was the sole method by which experimental information about atoms was obtained. And for light to deliver reliable knowledge about matter, a deep understanding of the *interaction* between light and matter was required.

Since a coherent and accurate mixing of all these ideas would have to wait until the development of QED in the middle of the twentieth century, and since this book is about the first steps along that road, it is useful to review, at an elementary level, what was known about light, matter and their interaction at the turn of the century.

### Light

Prior to 1900, light was an electromagnetic wave. It had the usual wave properties: speed  $c$ , wavelength  $\lambda$ , frequency  $\nu$ , which are all related by

$$c = \lambda\nu. \tag{A.1}$$

All wave have properties that satisfy the wave equation, which had been studied since 1746 when Jean d'Alembert developed the governing (one-dimensional) partial differential

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<sup>1</sup>The ether, sometimes known as the “luminiferous aether” was thought to be the medium that allowed electromagnetic waves to propagate.

equation,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad (\text{A.2})$$

although it had been anticipated by John Bernoulli in 1727. This equation described the properties of taut strings, the surfaces of liquids, and eventually electric and magnetic fields. When Maxwell discovered that a combination of Faraday's law and Ampere's law resulted in a three-dimensional vector wave equation,

$$\frac{\partial^2 \vec{E}}{\partial t^2} + \frac{1}{\epsilon_0 \mu_0} \nabla \times \nabla \times \vec{E} = 0, \quad (\text{A.3})$$

where  $\epsilon_0$  and  $\mu_0$  are the permittivity and permeability of free space, respectively, and also that the speed of the predicted waves matched that of light,

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}, \quad (\text{A.4})$$

it put the nail in the coffin of Newton's claim that light was "corpuscular." Of course, when Planck, Einstein, and Compton explained their respective experiments in terms of photons, physicists had to wrestle with the conundrum of "wave-particle duality." An understanding of light, though, required an understanding of how it interacted with matter.

## Matter

Even though the atomic hypothesis was hinted at by Dalton (see Chapter 1), championed by Boltzmann, and confirmed by Rutherford, Einstein and Bohr, in Maxwell's day the theory of matter's interaction with light did not depend on its discreteness. In fact, for light traveling in a transparent medium, the governing wave equation is just Eq. (A.3) with  $\epsilon_0 \rightarrow \epsilon$  and  $\mu_0 \rightarrow \mu$ . That is, the correct permittivity and permeability of the medium must be used. Maxwell explicitly thought about  $\epsilon_0$  as describing the polarizability of the ether, especially when describing the displacement current in a vacuum capacitor.

For most transparent media,  $\mu \approx \mu_0$ , so that the speed of light is reduced by a factor equal to the index of refraction  $n$  of the material

$$v = \frac{c}{n}, \quad (\text{A.5})$$

where

$$n = \sqrt{\frac{\epsilon}{\epsilon_0}}, \quad (\text{A.6})$$

and therefore the speed of light in that material is

$$v = \frac{1}{\sqrt{\epsilon \mu}}. \quad (\text{A.7})$$

Note that for this description, which agreed with experimental observations, matter is assumed to be continuous. Of course, determining *why* a material had particular values for  $\epsilon$  and  $\mu$  would require not only an atomic description, but quantum mechanics as well.



## Light and Matter

Light is emitted by matter, and there are, in general, three types of spectra that can be observed. The first is a *continuous* spectrum, in which all colors (or wavelengths) are present, but with differing intensities. A hot piece of metal emits such a spectrum, which is usually close to a blackbody spectrum (see Appendix D). The second type is an *emission* spectrum. Here, only specific, discrete, wavelengths are present, all other colors being nonexistent. A low-density gas in a discharge tube, such as a neon light, has this type of spectrum. Each element has its own set of spectral lines, its own “spectral fingerprint,” which allows an unknown gas to be identified solely from the colors of the light it emits. This is how, for example, helium was discovered — looking at the solar spectrum — and it is also how  $\alpha$  particles were proven to be the nuclei of helium atoms. In fact, neon lights are not only made of neon. Different elements have their own characteristic color, so in fact different colored “neon” lights must actually be filled with different elements. For example, pure neon emits an orange glow, and argon is blue.

Finally, the third type is an *absorption* spectrum. This is characterized by a continuous spectrum with dark lines where certain wavelengths are absent. It was first independently discovered in a solar spectrum by Joseph von Fraunhofer in 1814.<sup>2</sup> The solar surface is a good approximation to a blackbody, so it emits a continuous spectrum. But as the light passes through the solar atmosphere (i.e., the corona), atoms there absorb specific wavelengths. In fact, for a given element, these are the *same* lines that are emitted by that gas when it is in a discharge tube! This, then, is how elements in the solar atmosphere are identified. Of course the solar spectrum contains many dark lines, and it is not always simple to pick out the few that pertain to any given element. This complexity is part of the reason that new elements have been thought to have been discovered in stellar spectra, only later to be found that they are due to unknown lines from a previously known element. These discrete spectra strongly imply that matter has some kind of discrete structure. In fact, since the wavelengths of these spectral lines were hundreds of nanometers, the structure of matter must be at least that small. Even if one assumed that matter were composed of atoms, the atoms themselves could not be point particles (i.e., structureless), but must have some kind of internal structure. Of course, it was Rutherford and Bohr who initially determined that structure.

What happens when light, when propagating in a vacuum, encounters matter? Like all waves that are described by a linear wave equation, like Eq. (A.2), light satisfies the principle of superposition, and is therefore subject to interference and diffraction. However, if light is confined to a cavity (or waves travel on a string that is held fixed at two ends), resonances<sup>3</sup> occur at frequencies corresponding to standing waves. The resonance condition is given by

$$L = n \left( \frac{\lambda}{2} \right) \quad n = 1, 2, 3, \dots \quad (\text{A.8})$$

which is a mathematical statement that an integer number of half wavelengths must fit

<sup>2</sup>These dark lines are sometimes called Fraunhofer lines.

<sup>3</sup>In this case, a resonance means that there is positive feedback. That is, waves with particular frequencies (and wavelengths) add constructively and therefore reinforce themselves. Otherwise the interference is destructive.

exactly in the confining space of the length  $L$ . This equation results from the fact that when light encounters the matter at the edges of the cavity, Maxwell's equations require it to satisfy certain "boundary conditions," which lead to Eq. (A.8). Along with the wave formula above, these allowed wavelengths imply that the resonant frequencies are

$$\nu_n = \frac{nc}{2L} \quad (\text{A.9})$$

Since the Schrodinger equation *is* a linear wave equation, interference is inherent in quantum phenomena. In fact, this viewpoint is one method that is used to understand Bohr's atomic model (see Chapter 4).

Another consequence of interference is the two slit experiment of Young. Not only is an "interference pattern" observed when light or sound waves passes through the two slits, de Broglie's assertion that electrons exhibit wave properties guarantees that their passage through the slits results in the same effect (see Chapter 6). Finally, the fact that X-rays are electromagnetic waves means that their diffraction from crystals allowed Moseley to measure their wavelength (see Chapter 4). It also allowed William and Lawrence Bragg to catalogue the geometry of the crystal structure of several solids, and it is still used today for this purpose. Indeed, in 1953 it was used by Rosalind Franklin to help deduce the structure of DNA.

# Appendix B

## Rutherford Scattering

*All science is either physics or stamp collecting.* — Ernest Rutherford

After studying radioactivity for several years, and winning a Nobel Prize (in Chemistry!) for his efforts (see Section 3.7), in 1911 Ernest Rutherford attacked the question of the composition of matter from a different perspective. Along with two students, Hans Geiger and Ernest Marsden, Rutherford directed the  $\alpha$  rays emitted by “radium emanation,”  $^{222}\text{Rn}$ , at several thin solid targets, primarily gold.<sup>1</sup> (He had by that time definitively determined that  $\alpha$  rays were nothing more than helium nuclei.) At this time, the prevailing model of the atom was J. J. Thomson’s “plum pudding” model, in which he envisioned a smeared out positive charge with electrons embedded like plums in a pudding. One way to test this model was to fire a charged particle at an atom and then measure its trajectory. This would give information about the location of the electric charges. In fact, this is the primary method that has been used over the past 100 years to investigate the structure of subatomic particles.

Rutherford’s results showed that the atom consisted of a small, massive “nucleus” that was positively charged, surrounded by several light, negatively charged electrons. The incoming  $\alpha$  particle was deflected only by the nucleus and not by the light electrons, so Rutherford developed a theory of scattering to analyze his results. This theory is sufficiently important that I will derive its general form, and then apply it to two specific situations.

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<sup>1</sup>Radon-222 is the daughter of  $^{226}\text{Ra}$ , and it  $\alpha$  decays to radium A ( $^{218}\text{Po}$ ) which then  $\alpha$  decays to  $^{214}\text{Pb}$  which then  $\beta$  decays to radium C ( $^{214}\text{Bi}$ ). This is part of the  $4n + 2$  natural decay series (see Section 3.8.1) starting with  $^{238}\text{U}$ . All of these isotopes were present and emitting  $\alpha$  particles, each with a characteristic energy. As Geiger and Marsden stated in “On a Diffuse Reflection of the  $\alpha$ -Particles,” *Proc. Roy. Soc.*, **82**, 495-500 (1909), “The tube contained an amount of emanation equivalent to about 20 milligrammes  $\text{RaBr}_2$  at a pressure of a few centimetres. The number of  $\alpha$ -particles expelled per second through the window was, therefore, very great, and, on account of the small pressure inside the tube, the different ranges of the  $\alpha$ -particles from the three products (i.e. emanation,  $\text{RaA}$ , and  $\text{RaC}$ ) were sharply defined.”

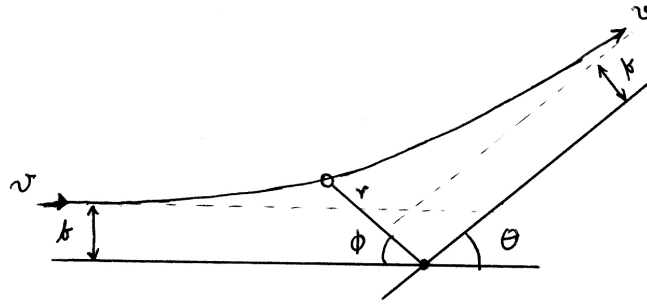


Figure B.1: Scattering geometry for a fixed, repulsive scattering center. The scattered particle initially has a speed  $v$ , and if no force were present, its straight line trajectory would take it to within a distance  $b$  — the impact parameter — of the scattering center. By symmetry, it will have a final speed  $v$ , but in a direction  $\theta$  relative to its initial direction.

## Scattering by a central force

The standard scattering problem is as follows: An object approaches a “scattering center” with speed  $v$ , and if it felt no force it would miss the scattering center by a distance  $b$ , known as the “impact parameter.” See Fig. B.1. If the force exerted on the object is in the radial direction, and depends only on the radial distance  $r$ , then the resulting trajectory will be symmetric, and the object will head away from the center asymptotically approaching a line that is also a distance  $b$  from the center. At any given instant, the object will be located at  $(r, \phi)$  relative to the center, and the scattering angle  $\theta$  is the direction that it is heading (relative to its initial direction) when it is far away from the center.

For any given force that the scattering center exerts on the object, the main theoretical prediction is the function  $\theta(b)$ . That is, how does the scattering angle  $\theta$  depend on the impact parameter  $b$ ? For repulsive forces, we can predict some general features of  $\theta(b)$ . First, if  $b = 0$ , then the object hits the (repulsive) center head on and simply “bounces” back, resulting in  $\theta = \pi$ . As  $b$  increases  $\theta$  must decrease, until for large  $b$ ,  $\theta$  must be small. In the limit that  $b \rightarrow \infty$ , it must be the case that  $\theta \rightarrow 0$ , as long as the scattering force gets weaker with distance. See Fig. B.2, which shows the general case of  $\theta(b)$ . It is correct for small  $b$  and for large  $b$  for all repulsive forces, but detailed shape depends on the actual force law.

## Hard sphere (billiard-ball) collisions

One of the simplest types of collisions to analyze is that of two solid spheres of radius  $R$ , and the only force that exists between them is a repulsive, elastic contact force when they touch. In this case,  $b = 0$  results in  $\theta = \pi$ , as we predicted above. However, if  $b$  is greater than  $2R$ , then the spheres miss each other completely, and there is no scattering, which means  $\theta = 0$ . Using the law of reflection (the angle of incidence equals the angle of

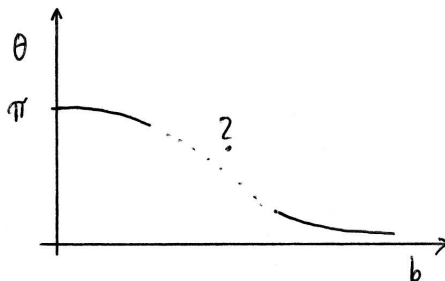


Figure B.2: Typical plot of scattering angle  $\theta$  versus impact parameter  $b$  for a repulsive force. A direct hit ( $b = 0$ ) must cause direct backscattering ( $\theta = \pi$ ), and as  $b$  increases,  $\theta$  must decrease toward zero. The exact form of the decrease depends on the form of the repulsive force, so that a measurement of  $\theta(b)$  can be inverted to infer  $F_r(r)$ .

reflection), you can show (see Problem 1) that  $b$  and  $\theta$  are related by

$$b = 2R \cos\left(\frac{\theta}{2}\right). \quad (\text{B.1})$$

To obtain  $\theta$  as a function of  $b$ , you simply need to invert the formula above

$$\theta = \begin{cases} 2 \arccos(b/2R) & b < 2R \\ 0 & b \geq 2R \end{cases} \quad (\text{B.2})$$

## Rutherford scattering

Ernest Rutherford, of course, was interested in the case where the force law was a repulsive Coulomb force, which was the case in his experiment of  $\alpha$  particles scattering off gold nuclei. The gold nucleus acted as the scattering center — it was very massive and did not move very much during the “collision” — and the  $\alpha$  particles were the objects being scattered. If the repulsive force is

$$F = \frac{1}{4\pi\epsilon_0} \frac{(ze)(Ze)}{r^2}, \quad (\text{B.3})$$

where  $Ze$  is the positive charge of the scattering center, and  $ze$  is the positive charge of the object being scattered, then you can show that

$$b = \frac{zZ}{2K} \frac{e^2}{4\pi\epsilon_0} \cot\left(\frac{\theta}{2}\right), \quad (\text{B.4})$$

where  $K$  is the initial kinetic energy (when it is very far away) of the  $\alpha$  particle.<sup>2</sup> Notice that the scattering angle depends on the speed of the object, which was not true in the

<sup>2</sup>For comparison, in the case of gravitational scattering, where a particle traverses a hyperbolic orbit due to a central mass  $M$ , the relation between  $b$  and  $\theta$  is

$$b = \left(\frac{GM}{v_\infty^2}\right) \cot\left(\frac{\theta}{2}\right),$$

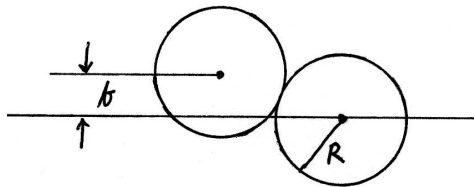


Figure B.3: Scattering geometry for two solid spheres of radius  $R$ .

billiard-ball case. There, since there was no force except when the spheres made contact, the speed did not matter at all, only the angle of the collision. Here, if the  $\alpha$  particle moves quickly, it spends less time in the region where the repulsive electric force is strong, and therefore the scattering angle is small.

**The Rutherford scattering formula** Unfortunately, Eq. (B.4) is not in the proper form for comparison with experimental results. Why not? Well, since the detector is typically located at an angle  $\theta$  from the initial projectile direction, or at least the number of particles that are deflected by an angle  $\theta$  is measured, we wish to predict the probability for an  $\alpha$  particle to be scattered into any angle between  $\theta$  and  $\theta + d\theta$ . In addition, there are many  $\alpha$  particles and many nuclei in the thin foil target, which means the density of the gold nuclei and the thickness of the foil must be taken into account. If all these factors are included, the probability above, that for scattering into any angle between  $\theta$  and  $\theta + d\theta$ , is given by  $N(\theta)d\theta$ . For the Coulomb force, Rutherford showed that

$$N(\theta) = \frac{nt}{4r^2} \left( \frac{zZ}{2K} \right)^2 \left( \frac{e^2}{4\pi\epsilon_0} \right)^2 \sin^{-4} \left( \frac{\theta}{2} \right), \quad (\text{B.5})$$

where  $n$  is the number density of the scatterers,  $t$  is the foil thickness, and  $r$  is the distance of the detector from the point where the beam hits the foil. Geiger and Marsden were able to reproduce the dependence of  $N$  on  $\theta$ ,  $Z$ ,  $t$ , and  $K$ . All of the measurements matched the predictions, which led to the acceptance of a “nuclear” atom.

## Problems

1. Derive the scattering formula, Eq. (B.1), for two solid spheres.

where  $v_\infty$  is the speed of the particle when it is infinitely far away. Note that, unlike the electrostatic case, there is no dependence on the mass of the scattered particle, and this formula was used by Newton to predict the deflection of light by the sun (remember that Newton believed light consisted of particles).

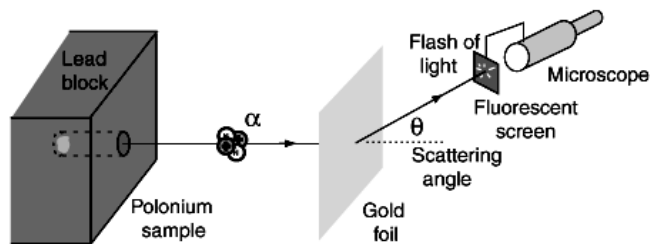
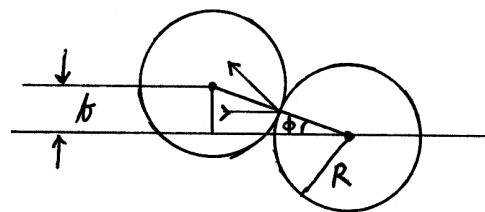


Figure B.4: Schematic of the Geiger-Marsden experiment. The scattering angle  $\theta$  was measured by noting a flash on the fluorescent screen in a darkened room. From Hyperphysics.

**Solution** The right triangle in the figure has a hypotenuse of  $2R$ , and hence  $\sin \phi = b/2R$ . The scattering angle  $\theta$  (see Fig. B.1) is given by  $\theta + 2\phi = \pi$ , where the factor of 2 comes from the fact that the angle of incidence equals the angle of reflection. Solving for  $b$  gives



$$b = 2R \sin \left( \frac{\pi}{2} - \frac{\theta}{2} \right) = 2R \cos \left( \frac{\theta}{2} \right).$$

2. Sketch the function  $\theta(b)$  for billiard-ball collisions. One method is to sketch  $b(\theta)$  from Eq. (B.1), and then invert the sketch (flip it mirror-like around the line  $b = \theta$ ).
3. Sketch the function  $\theta(b)$  for Coulomb collisions (i.e., Rutherford scattering).
4. In Geiger and Marsden's experiment,  $\alpha$  particles impinged on a gold foil. Consider one  $\alpha$  particle heading directly toward one gold nucleus ( $^{197}\text{Au}$  of course). How much initial kinetic energy must the  $\alpha$  particle have (when it is very far from the nucleus) in order to have its distance of closest approach (defined to be where its kinetic energy is zero) be equal to the radius of the nucleus?

**Solution** Gold has  $A \sim 197$  so that  $R = R_0 A^{1/3} = 1.2 \text{ fm} \times 5.8 \sim 7 \text{ fm}$ . The potential energy between the  $\alpha$  particle ( $z = 2$ ) and the gold nucleus ( $Z = 79$ ) when they are 7 fm apart is

$$\frac{1}{2}mv^2 = |U| = \frac{zZe^2}{4\pi\epsilon_0 D} = \left( \frac{e^2}{4\pi\epsilon_0} \right) \frac{(2)(79)}{(7 \text{ fm})} \approx 32 \text{ MeV}.$$

This is just what  $K$  must be when they are very far apart. However, this is much larger than the  $\sim 5 \text{ MeV}$   $\alpha$  particles that are emitted radioactively, so that Rutherford's nuclei did not get close enough to experience the strong force.

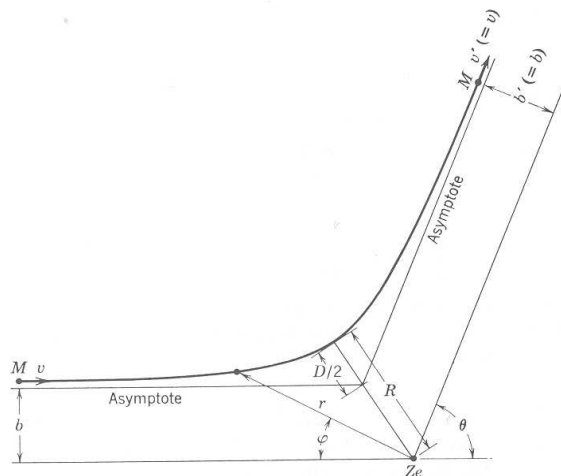
5. Rutherford scattering. In polar coordinates, with a scattering nucleus (of charge  $+Ze$ ) fixed at the origin, the equation of the trajectory of the  $\alpha$  particle (of charge  $+ze$ ) can be shown to be

$$\frac{1}{r} = \frac{1}{b} \sin \varphi + \frac{D}{2b^2} (\cos \varphi - 1),$$

where  $b$  is the “impact parameter,” and  $D$  is the “distance of closest approach” in a head on collision ( $b = 0$ ), which is given by

$$D = \frac{1}{4\pi\epsilon_0} \frac{zZe^2}{Mv^2/2}.$$

In a head-on collision, the  $\alpha$  particle will stop and turn around at this location distance from the nucleus. (a) Show that  $D$  is the distance at which the potential energy of the  $\alpha$  particle is equal to its initial kinetic energy ( $Mv^2/2$ ). (b) Show that the trajectory equation is a hyperbola. (In the figure,  $\theta$  is the scattering angle.)



**Solution** (a) The total energy of the  $\alpha$  particle is  $E = K + U$ . When it is very far from the gold nucleus, the potential energy is zero, so that  $E = K = \frac{1}{2}mv^2$ . At its distance of closest approach, the “turning point,” the kinetic energy is zero so that  $E = U = +zZe^2/4\pi\epsilon_0 r$ . The  $+$  sign indicates that the Coulomb force is repulsive. Since  $E$  is the same in both cases, setting  $K = U$  and  $r = D$  gives the formula for  $D$ . (b) This one is hard, and if anyone made a reasonable attempt, give them some credit.

6. In a collision between hard spheres, there is no scattering if  $b$  is larger than a maximum value. This means that you must “aim well” in order to see an effect. Not true for the  $1/r^2$  Coulomb force: *any* impact parameter will cause scattering. It is instructive to investigate the “Born approximation,” where we take the limit of large impact parameter (and thus a small scattering angle). Determine the relationship between  $b$  and  $\theta$  in this limit.

**Solution** Expanding the cotangent in Eq. (B.4) for small  $\theta$  gives

$$b \approx \frac{zZ}{2K} \frac{e^2}{4\pi\epsilon_0} \left( \frac{2}{\theta} \right),$$

which shows that  $b$  and  $\theta$  are inversely proportional. More interesting is that, in this limit, the quantity  $b\theta K$  is constant.



# Appendix C

## The Stern-Gerlach Experiment

*...quantum-mechanical states are to be represented by vectors in abstract complex vector space. —J.J. Sakurai<sup>1</sup>*

In 1922, Otto Stern and Walther Gerlach measured the magnetic moment of the silver atom using a technique that has come to be known as the “molecular beam method.” Due to the electron configuration of silver ( $Z = 47$ ), it was essentially a measurement of the magnetic moment of the electron (see Section 2.4 and page 94). It is also a demonstration of the simplest system that is inherently quantum mechanical, and it is instructive to realize just how inadequate our macroscopic intuition really is.

### Torques and Forces

Recall from elementary electromagnetism that electric dipoles and magnetic dipoles experience forces and torques due to electric fields and magnetic fields, respectively. For magnetic dipoles, if the magnetic field  $\vec{B}$  is uniform, then the force on the dipole is zero, but the torque on the dipole is equal to  $\vec{\tau} = \vec{\mu} \times \vec{B}$ , where  $\vec{\mu}$  is the magnetic dipole moment. This just says that field tries to align the moment with the field vector. More important for the Stern-Gerlach experiment is the fact that if the field is nonuniform, then the dipole feels a net force that is due to the gradient in the field. Specifically, a magnetic dipole feels a force

$$\vec{F} = \nabla (\vec{\mu} \cdot \vec{B}), \quad (\text{C.1})$$

which, if the field points primarily in the  $z$  direction, and its magnitude also varies in the  $z$  direction, becomes approximately

$$F_z \approx \mu_z \frac{\partial B_z}{\partial z}. \quad (\text{C.2})$$

This was exactly the case in the Stern-Gerlach experiment. Otto Stern had the idea for this experiment in 1921 in order to see if he could detect the “space quantization” of the atom. In Bohr’s atomic model, the angular momentum perpendicular to the plane of the electron orbit was quantized,  $L_z = n\hbar$ , which meant that the magnetic moment due to

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<sup>1</sup>Sakurai, *Modern Quantum Mechanics*, page 10.

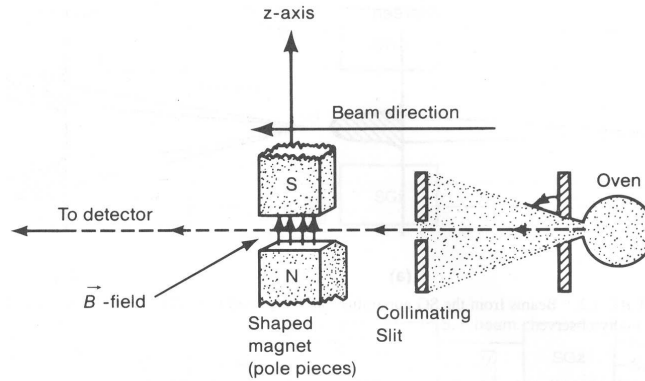


FIGURE 1.1. The Stern-Gerlach experiment.

Figure C.1: Schematic of the Stern-Gerlach experiment. Silver atoms were heated in an oven, allowed to escape and sent in a particular direction via a collimated slit, passed through a nonuniform magnetic field, and finally impinged on a photographic plate. Figure 1.1 from Sakurai, *Modern Quantum Mechanics*.

the orbital motion was also quantized. It can be expressed (see Problem 2 at the end of this appendix) in units of the Bohr magneton

$$\mu_z = -n\mu_B, \quad (\text{C.3})$$

exactly as in Eq. (2.9). From observations of the Zeeman effect, where the spectral lines of atoms that have been placed in a magnetic field are split into two, three, or more components, it was postulated that the magnetic moment vector of an atom was forced to be either parallel or anti-parallel to the external magnetic field. Since these two states have different energies (the torque equation above implies that there is a potential energy due to the interaction that is  $U = -\vec{\mu} \cdot \vec{B}$ ), this would explain the splitting of the spectral lines. Stern proposed to verify this by means of Eq. (C.2).

As depicted in Fig. C.1, silver atoms would be heated in an oven, allowed to “effuse”<sup>2</sup> through a hole in the oven and then a collimating slit. They then would pass through a nonuniform magnetic field which would exert a force on the atoms, described by Eq. (C.2), and therefore spread out the beam. Classically, the magnetic moment vectors of the atoms point in random directions, and therefore the  $z$  components would take on a continuous range of values, which means that the initially narrow beam would be spread out. However, if the quantum intuition of physicists like Niels Bohr was correct, then only certain discrete values of  $\mu_z$  would be allowed and the beam would split into two or more discrete beams, resulting in discrete lines on the detecting photograph. This, in fact, was Stern’s motivation: to “decide unequivocally between the quantum theoretical and classical views.”<sup>3</sup> It took a year to complete the experiment because the deflection of the beam was small, and the entire apparatus had to be aligned to a tolerance of 0.01 mm or the result would be inconclusive.

<sup>2</sup>effusion, n., the flow of gas through an aperture whose diameter is small as compared with the distance between the molecules of the gas.

<sup>3</sup>Quoted in Friedrich and Herschbach, *Physics Today*, 2003.

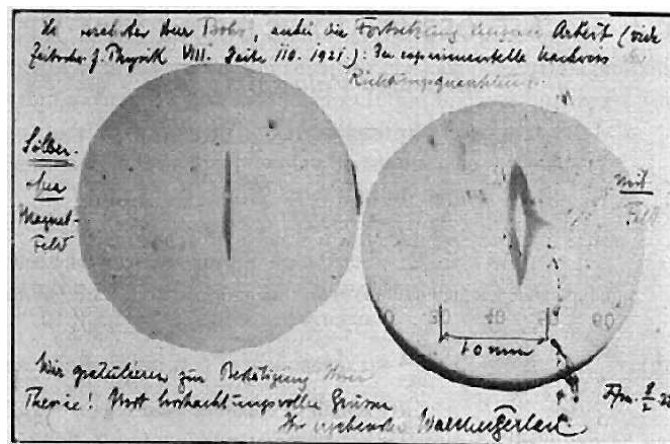


Figure C.2: Postcard from Gerlach to Bohr. The message reads “Attached [is] the experimental proof of directional quantization. We congratulate [you] on the confirmation of your theory.” Note the scale denoting 1.0 mm at the bottom right. Figure 4 from Friedrich and Herschbach, *Physics Today*, 2003.

Since silver has 47 electrons, the first 46 form a spherical cloud — and each pair of them has their spins anti-aligned — while the 47th electron, the only one in the 5s subshell, is the only one to contribute to the magnetic moment of the atom (see Problem 67 for a proof that the nucleus contributes only negligibly to the atom’s magnetic moment). Of course, Stern and Gerlach did not know about spin (it wasn’t proposed by George Uhlenbeck and Samuel Goudsmit until 1925), but they assumed the Bohr model, which stated that the unpaired electron would have a nonzero orbital angular momentum, and hence a nonzero magnetic dipole moment, as in Eq. (C.3). Of course, as we now know, the 47th electron has zero orbital angular momentum, so that the magnetic moment of the atom is solely due to its spin, and can take on the values

$$\mu_z = -gm_s\mu_B, \quad (\text{C.4})$$

as given in Eq. (2.10). When they found that the beam was split into two beams, and that the strength of the splitting implied that silver had a magnetic moment equal to  $\mu_B$  to within 10%, Gerlach sent a postcard to Bohr in congratulations (see Fig. C.2). They thought that they had confirmed Eq. (C.3) with  $n = 1$ . But in fact they had confirmed Eq. (C.4), with  $g \approx 2$  and  $m_s = \pm\frac{1}{2}$ . It wasn’t until 1927, after spin was discovered and after Schrodinger modeled the hydrogen atom, that it was recognized that they had actually measured the spin of the electron. In 1922, Stern and Gerlach were completely in the dark about the true nature of their result.

## Sequential Stern-Gerlach experiments

What is special about the  $z$  axis? Could we turn the magnets in Fig. C.1 horizontal and measure  $\mu_x$ ? Of course, but then, quantum mechanics tells us, we would have no knowledge

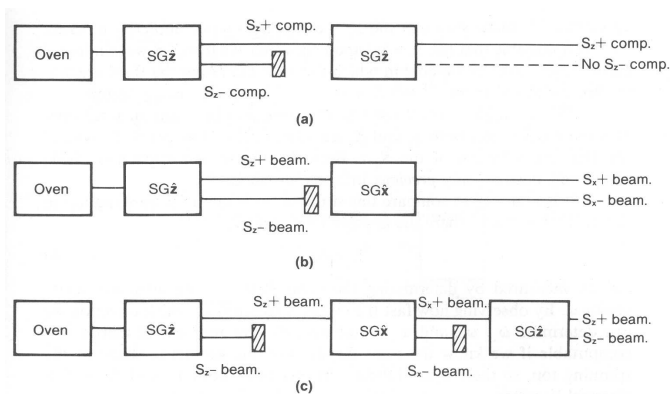


Figure C.3: Schematic of three different possibilities for Stern-Gerlach type experiments that are run in sequence, i.e., one after the other. Figure 1.3 from Sakurai, *Modern Quantum Mechanics*.

of the  $z$  component. Recall from Section 2.3 that it is possible to know simultaneously only the magnitude and *one* component of the spin vector — the other two components, as given by the angle  $\phi$ , are completely “unknowable.” Since the magnetic moment is proportional to spin, this restriction applies also to it. To illustrate this restriction, consider Fig. C.3(a). We place two identical deflecting magnetic fields that are both oriented in the  $z$  direction, so they each effectively measure the  $z$  component of the magnetic moment (or spin). After the first “SG $z$ ” apparatus, which splits the beam in two, if we block the beam that was deflected downward, and let the upward-deflected beam go through *another* SG $z$  apparatus, then there should be no downward deflected beam. This is because all the atoms entering the second apparatus must have their magnetic moments pointing in the  $+z$  direction, since they had just been measured by the first SG $z$  apparatus.

Figure C.3(b) shows a similar setup, but with an SG $x$  apparatus coming second. Again, if we block the “spin down” component and let the spin up component through, the result is that 50% of the beam is deflected right (which means it has  $S_x = +\frac{1}{2}\hbar$ ) and 50% of the beam is deflected left (which means it has  $S_x = -\frac{1}{2}\hbar$ ). Now, it is tempting to conclude that we have just measured *two* components of the spin vector, something we thought was impossible. That is, a reasonable interpretation seems to be that half of the atoms in the  $S_z = +\frac{1}{2}\hbar$  beam leaving the first, SG $z$ , apparatus have both  $S_z = +\frac{1}{2}\hbar$  and  $S_x = +\frac{1}{2}\hbar$ , and the other half have both  $S_z = +\frac{1}{2}\hbar$  and  $S_x = -\frac{1}{2}\hbar$ .

Figure C.3(c) will show that our “reasonable interpretation” above is wrong. If we now block the  $S_x = -\frac{1}{2}\hbar$  beam after the second, SG $x$ , apparatus, and then run the remaining  $S_x = +\frac{1}{2}\hbar$  beam through a third, SG $z$ , apparatus, we find that even though we started with atoms that had only  $S_z = +\frac{1}{2}\hbar$ , we now have both components. This effectively proves that we cannot measure two components of the spin vector simultaneously. Specifically, when the atoms passed through the SG $x$  apparatus that measured their  $S_x$ , it destroyed any prior knowledge about the  $z$  component. This result is not due to any experimental inaccuracy or error, but is simply a microscopic limitation, as expressed by the Heisenberg uncertainty principle. It turns out that this situation is almost identical with the *classical*

case of light passing through sequential polarizing filters. It means that one method of representing the atom's spin is to use a wave equation (since the results of the experiment with light is due to the phenomenon of superposition), and that's exactly what quantum mechanical equations are, wave equations. The Schrodinger equation is a nonrelativistic wave equation, and the Dirac and Klein-Gordon equations are relativistic wave equations.

### Collateral Reading

- Bretislav Friedrich and Dudley Herschbach, "Stern and Gerlach: How a Bad Cigar Helped Reorient Atomic Physics," *Physics Today*, **56**(12) 53-59 (December 2003).
- J. J. Sakurai, *Modern Quantum Mechanics*, Benjamin-Cummings, 1985. Chapter 1.
- see Stern's Nobel Prize lecture at <http://www.nobelprize.org>.

### Problems

1. What is meant by the term "space quantization?" Is space really quantized?
2. Derive Eq. (C.3). Recall that the magnetic moment of a current loop has a magnitude  $\mu = IA$ .
3. Estimate the separation distance of the images observed on the screen of Stern and Gerlach's experiment. Their source of atoms was an oven of temperature 1000 °C, their deflecting magnet was 3.5 cm long, and the magnetic field gradient was 10 T/cm. Make any other assumptions that you need (but be sure to state them).

**Answer** The splitting was 0.2 mm.



# Appendix D

## Blackbody Radiation

*It is a highly important task to find this function.* — Gustav Kirchhoff, referring to the blackbody spectrum

An object that reflects no light (electromagnetic radiation) that falls on it, but absorbs it all, is called a perfect *blackbody*. But, if it absorbs energy, it must also radiate energy if it is not to heat up indefinitely. The form of this radiation is crucial in many fields of physics and astrophysics, but measuring it and predicting it theoretically took a long time. Because the nature of the atom was not revealed until the early 20th century, the “laws” regarding thermal radiation were determined phenomenologically (i.e., from experimental observation). Gustav Kirchhoff (1824-1887) was one of the premier investigators, and his laws are now taught in introductory physics courses. At the fundamental level, they state that the properties of the radiation emitted by a blackbody depend only on the object’s temperature  $T$  and not on the shape, size, or composition of the blackbody. Specifically, the following three properties were well known before 1900.

1. A blackbody emits electromagnetic radiation at all wavelengths (i.e., in all regions of the spectrum).
2. The total power emitted by a blackbody is proportional to  $T^4$ , where  $T$  is the temperature (in Kelvin) of the blackbody. (This is the Stefan-Boltzmann law.)
3. The wavelength at which a blackbody emits maximum power is inversely proportional to its temperature  $\lambda_{max} \propto \frac{1}{T}$ . (This is Wien’s law.)

The first property means that blackbodies emit a “continuum” of radiation rather than the discrete spectral lines emitted by gaseous elements that are heated (see Appendix A).

### Radiative transfer

Because objects emit different amounts of energy in different regions of the spectrum, we have to keep track of that radiation as a function of wavelength  $\lambda$  (or of frequency  $\nu$ ). The quantity  $R(\lambda)d\lambda$  is defined to be the energy emitted per unit time per unit area from the

surface of a blackbody, between the wavelengths  $\lambda$  and  $\lambda + d\lambda$ .  $R(\lambda)$  is called the *spectral radiance*,<sup>1</sup> and it has SI units  $\text{W}/\text{m}^3$ .

The “energy per unit time per unit area” is called the *energy flux*. Flux is a concept that is used in many different physics disciplines, particularly those dealing with transport of some quantity. In fluid dynamics, for example, the quantity  $\rho v$  (mass density  $\times$  velocity) is called the mass flux, and is nothing but the mass per unit time that passes through a unit area. In laminar flow, the quantity  $\rho v A$  remains constant along a streamline, where  $A$  is the cross sectional area of the streamline. This is called the continuity equation. The field of radiative transfer, while dealing with energy flux, has developed its own terminology, which we will follow.

The three experimental properties above imply three important mathematical properties of  $R(\lambda)$ , listed below.

1. Since the blackbody emits at all wavelengths,  $R(\lambda) > 0$  for all  $\lambda$ . In addition, it was known that  $R(\lambda) \rightarrow 0$  as both  $\lambda \rightarrow 0$  and  $\lambda \rightarrow \infty$ .
2. The total power emitted per unit area from the blackbody’s surface is just an integral of  $R(\lambda)$  over all wavelengths. That is

$$R \equiv \int_0^\infty R(\lambda) d\lambda = \sigma T^4, \quad (\text{D.1})$$

where  $\sigma$  is the proportionality factor. This is called Stefan’s Law, because it was first deduced by Jozef Stefan in 1879 from experimental observations. It is also called the Stefan-Boltzmann law because Ludwig Boltzmann derived it theoretically in 1884, and therefore  $\sigma = 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$  is called the Stefan-Boltzmann constant.  $R$  is called the “radiance,”<sup>2</sup> and has units  $\text{W}/\text{m}^2$ .

Our Sun is not a perfect blackbody, but is a very good approximation. Its surface temperature is about  $T_\odot = 5780 \text{ K}$ , and therefore emits an energy flux  $R = \sigma T_\odot^4 = 6.33 \times 10^7 \text{ W}/\text{m}^2$ . Since the radius of the Sun is  $R_\odot = 6.96 \times 10^8 \text{ m}$ , the radiant flux, or “luminosity,” of the the Sun is  $L_\odot = (4\pi R_\odot^2) \sigma T_\odot^4 = 3.85 \times 10^{26} \text{ W}$ .

3.  $R(\lambda)$  has exactly one maximum. That is,  $dR(\lambda)/d\lambda = 0$  defines  $\lambda_{max}$ , which is found to be given by

$$\lambda_{max} T = b, \quad (\text{D.2})$$

where  $b = 2.898 \times 10^{-3} \text{ m K}$ . This is called Wien’s displacement law, which he derived in 1893. Note that  $b$  has dimensions of length  $\times$  temperature, so that ‘m K’ is ‘meters  $\cdot$  Kelvin,’ not ‘milli-Kelvin.’

<sup>1</sup>Unfortunately, there is no standard terminology in this field. The quantity  $R(\lambda)$  is sometimes written as  $R_\lambda$  or  $E_\lambda$ , and it is sometimes called radiance, or monochromatic irradiance, or spectral emissive power. You must use dimensional analysis to determine which quantity is being discussed.

<sup>2</sup>Again, terminology varies, and  $R$  is sometimes called intensity or irradiance. The product  $RA$ , where  $A$  is the surface area of the blackbody, is called the “radiant flux.”



Again for the Sun, Wien's law predicts that it emits its maximum power at the wavelength  $\lambda_{max} = b/T_{\odot} = 501$  nm, very near the center of the visible spectrum. Evolutionary biologists suggest that our eyes developed sensitivity in this spectral region simply because there is so much light available.

In 1860, Kirchhoff, speaking of  $R(\lambda)$ , said, "It is a highly important task to find this function." A laudable goal, but how to achieve it? Accurate measurements of spectral radiance over a large range of wavelengths are needed, not just the visible region of the spectrum. Ångström was able to measure visible wavelengths to a precision of  $10^{-5}$ , but because absolute intensities are more difficult, it was not until the early 1900s that measurements became precise enough to compare with theoretical predictions.

## Cavity radiation

A simple technique to compare theoretical predictions with experimental measurements is to consider *Hohlraumstrahlung*, or cavity radiation. As Kirchhoff put it

"Given a space enclosed by bodies of equal temperature, through which no radiation can penetrate, then every bundle of radiation within this space is constituted, with respect to quality and intensity, as if it came from a completely black body of the same temperature."

If you cut a hole in the cavity wall, there will be light emitted from that hole, and as Kirchhoff contends, that is "blackbody radiation."

It turns out that while it is straightforward experimentally to measure the spectral radiance  $R(\lambda)$  from the hole, it is much simpler to theoretically calculate the energy density of the radiation within the cavity. It can be shown (see Problem 8) that the relation between the two quantities is

$$R(\lambda) = \frac{c}{4} u(\lambda), \quad (\text{D.3})$$

where  $u(\lambda)d\lambda$  is the energy per unit volume between the wavelengths  $\lambda$  and  $\lambda + d\lambda$  [the SI units of  $u(\lambda)$  are obviously  $\text{J/m}^4$ ], and, similar to  $R$ , the total energy density  $U$  is a sum over all wavelengths

$$U \equiv \int_0^{\infty} u(\lambda)d\lambda. \quad (\text{D.4})$$

To understand the relation between  $R$  and  $U$ , another analogy with fluid dynamics is useful. As discussed above, the mass flux  $\rho v$  is just the mass density  $\rho$  times the flow velocity  $v$ . Here, the energy flux  $R$  is just the energy density  $U$  times the velocity  $c$ , or  $R = Uc$ . This works for  $R(\lambda)$  and  $u(\lambda)$  just as it does for  $R$  and  $U$ , because all wavelengths travel at the same speed  $c$ . However,  $R = Uc$  only holds when all the energy is also traveling *in the same direction*, as in laminar fluid flow, where all the mass is traveling in the same direction. Inside our cavity, electromagnetic waves are traveling in all different directions, and only those that happen to be heading out of the hole (and would reflect back into the cavity if there were no hole) contribute to  $R$ . This geometry is what is responsible for the factor of 4 in Eq. (D.3).

## Early theoretical attempts to determine $u(\lambda)$

The details of the different derivations of  $u(\lambda)$  can be found in most modern physics textbooks. They are based on statistical mechanical arguments that count the number of different ways the resonant waves can fit inside a cavity of a certain size. Statistical mechanics is one of the topics that I chose not to cover in detail in this book, so in this appendix I want to focus on the radiation concepts and not on the derivation of formulas.

Lord Rayleigh (1900) and James Jeans (1905), using classical arguments, derived a formula for the energy density  $u(\lambda)$

$$u_{RJ}(\lambda) = \frac{8\pi}{\lambda^4} kT. \quad (\text{D.5})$$

This classical prediction agreed with the experimental measurements that had been made up until that time, but an obvious problem was that the integral over all wavelengths diverged, Eq. (D.4). The difficulty appeared at short wavelengths, and was therefore called the “ultraviolet catastrophe.” As historian Helge Kragh notes “In spite of its prominent role in physics textbooks, the formula [Eq. (D.5)] played no part at all in the earliest phase of quantum theory. The ‘ultraviolet catastrophe’ — a name coined by Paul Ehrenfest in 1911 — only became a matter of discussion in a later phase of quantum theory.”<sup>3</sup>

In 1905, with the help of Einstein, Rayleigh added an *ad-hoc*<sup>4</sup> exponential factor to get rid of the ultraviolet catastrophe

$$u_R(\lambda) = \frac{8\pi}{\lambda^4} kT e^{-c_2/\lambda T}. \quad (\text{D.6})$$

This forces the integral  $\int_0^\infty u(\lambda)d\lambda$  to be finite; however, it did not agree with experiment.

In 1896, Wilhelm Wien had derived his own radiation law

$$u_W(\lambda) = \frac{a}{\lambda^5} e^{-b/\lambda T}, \quad (\text{D.7})$$

which unfortunately was also somewhat *ad-hoc*. With correct values for  $a$  and  $b$ , this expression agrees with Planck’s spectrum below in the limit of small wavelength. In fact, it was Planck’s desire to find a satisfactory theoretical explanation for this function that led him to his own spectrum.

In 1900, Max Planck derived a spectral formula by assuming that within the cavity, the electromagnetic waves and the walls could only exchange energy in discrete amounts  $h\nu$ .<sup>5</sup> He realized that this suggestion was not physical, but it was the only way that he was able to obtain a formula in agreement with experiment. The spectrum that Planck derived was

$$u_P(\lambda) = \frac{8\pi}{\lambda^4} \left( \frac{hc}{\lambda} \frac{1}{e^{hc/\lambda kT} - 1} \right), \quad (\text{D.8})$$

<sup>3</sup>“Max Planck: The reluctant revolutionary,” *Physics World*, December 2000, pp 31-35.

<sup>4</sup>ad-hoc, adj., made with a particular purpose, without reference to wider application.

<sup>5</sup>In the case of a neutral gas confined in a box, Boltzmann had already shown that during collisions with the walls the molecules exchange momentum (and energy) with the walls. But in order for this process to predict the ideal gas law, he showed that the energy exchanged can take on any value, i.e., a continuous set of values.

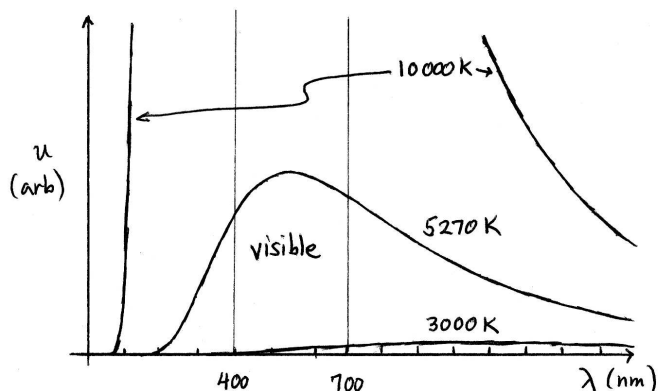


Figure D.1: Energy density  $u_P(\lambda)$  given by the Planck function for three values of the temperature, 3000 K, 5270 K, and 10 000 K. These temperatures were chosen for the following reasons. A glowing metal can have a temperature on the order of 3000 K, and it appears red due to the fact that the spectral radiance at 700 nm is greater than that at 400 nm; the Sun’s surface temperature is near 5270 K — which has a peak intensity at  $\lambda_{max} = 550$  nm, exactly in the center of the visible spectrum — and it appears white since the spectral radiance is approximately flat in the visible region; and a very massive star can have a surface temperature near 10 000K, which makes it appear blue (even though the scale does not allow us to see the curve for this temperature, Wien’s law tells us that its maximum must be at about 275 nm, well in the ultraviolet).

which I’ve written in a suggestive way. The Planck function effectively replaces  $kT$  in the Rayleigh-Jeans formula with a more complicated function of  $\lambda$  and  $T$ , and Planck showed (Problem 2) that it agreed with the Rayleigh-Jeans formula in the long wavelength limit. Also, it did not diverge at small wavelengths, i.e., there was no ultraviolet catastrophe.

During the period 1900-1905, it was not clear which of the theoretical predictions, Eqs. (D.5)-(D.8) was correct. They all agreed with each other (and with experiments) in the limit of long wavelengths,<sup>6</sup> but the experiments were not precise enough to distinguish between them in the short wavelength limit. It was only after 1905, when Einstein used the same quantization ( $E = h\nu$ ) to explain the photoelectric effect, that consensus started to back Planck’s function.

## Properties of the Planck function

The Planck function, Eq. (D.8), certainly satisfies the three experimental properties listed above. Figure D.1 shows  $u(\lambda)$  for three values of the temperature  $T$ . It is clear that as the temperature increases, the wavelength of maximum intensity decreases and the total intensity (i.e., the area under the curve) increases. Looking at the strength of the Planck function in the visible region of the spectrum, and the relative strengths in the red and

<sup>6</sup>Two sets of measurements had confirmed the blackbody spectrum in the infrared: Lummer and Rubens looked between 12-18  $\mu\text{m}$ , and Rubens and Karlbaum looked between 30-60  $\mu\text{m}$ .

blue regions, it is clear that cool blackbodies appear red (they emit more red than blue), hot blackbodies appear blue, and “medium” blackbodies (i.e., the Sun) appear white — the spectral radiancy is approximately flat across the visible spectrum.

**Self similarity** An interesting and useful mathematical property of  $u(\lambda)$ , as well as  $R(\lambda)$ , is that of *self-similarity*. Self-similarity is commonly encountered in fractal theory where a portion of an object looks the same as the entire object. In other words, an object is self-similar if it looks the same on all scales, large and small. A *function*, on the other hand, is self-similar if you can express it as a function of only one variable. For example, I can rewrite the Planck function in the following way

$$\frac{u(\lambda)}{T^5} = \frac{8\pi hc}{(\lambda T)^5} \frac{1}{e^{hc/k(\lambda T)} - 1}. \quad (\text{D.9})$$

Notice that the right-hand-side is a function of only the combination  $\lambda T$ , not of  $\lambda$  and  $T$  separately, with all other terms being constant. This means that if I know the form of the curve for one temperature, I can determine it for another temperature in the following manner. A plot of  $u$  versus  $\lambda$  for one particular value of the parameter  $T_1$  can be transformed into a plot for another value of  $T_2$  by shrinking the abscissa axis by a factor equal to the temperature ratio  $T_2/T_1$  and stretching the ordinate axis by a factor  $(T_2/T_1)^5$ .

## Problems

1. What are the dimensions and SI units of “radiant flux.”
2. Show explicitly that the Planck function, Eq. (D.8), agrees with the Rayleigh-Jeans function, Eq. (D.5), in the limit where  $\lambda \rightarrow \infty$ . Also show explicitly that the Planck function does not diverge in the  $\lambda \rightarrow 0$  limit. That is, determine an approximation that is correct in this limit.

**Solution** As usual, it is imprecise to let a physical quantity become infinitely large (or infinitesimally small), physical quantities must be large or small compared with another physical quantity. In this case, the correct approximation should be  $\lambda kT \gg hc$ . In this limit, the argument of the exponential is small, and we can use the Taylor series expansion  $e^x \approx 1 + x$ , which results in

$$u_P(\lambda) \approx \frac{8\pi hc}{\lambda^5} \frac{1}{(1 + hc/\lambda kT) - 1} = \frac{8\pi hc}{\lambda^5} \frac{\lambda kT}{hc} = \frac{8\pi}{\lambda^4} kT,$$

which is just the Rayleigh-Jeans law. Note that another way to think about this classical limit is that it is obtained by letting  $h$  become very small. This is a common method to obtain the classical limit of quantum equations.

In the opposite limit,  $\lambda kT \ll hc$ , the argument of the exponential is large, so that the  $-1$  in the denominator can be ignored, resulting in

$$u_P(\lambda) \approx \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT}} = \frac{8\pi hc}{\lambda^5} e^{-hc/\lambda kT},$$

which is just Wien's law. Note that there is no ultraviolet catastrophe.

3. For the Rayleigh-Jeans energy density in Eq. (D.5), evaluate the integral  $\int_a^b u_{RJ}(\lambda) d\lambda$ . Which limit,  $a \rightarrow 0$  or  $b \rightarrow \infty$ , causes the integral to diverge?

**Solution** Application of the power rule gives

$$\int_a^b u_{RJ}(\lambda) d\lambda = 8\pi kT \int_a^b \frac{d\lambda}{\lambda^4} = \frac{8\pi kT}{3} \left( \frac{1}{a^3} - \frac{1}{b^3} \right).$$

Letting  $b \rightarrow \infty$  is fine since then  $b^{-3} \rightarrow 0$ , but  $a^{-3} \rightarrow \infty$  when  $a \rightarrow 0$ . The ultraviolet catastrophe.

4. Rayleigh added an exponential factor  $e^{-c_2/\lambda T}$  to account for the high-frequency behavior of the measured blackbody radiation. His spectral radiance was therefore

$$R_R(\lambda) = \left( \frac{c}{4} \right) \frac{8\pi kT}{\lambda^4} e^{-c_2/\lambda T},$$

and this was called the "Rayleigh Law." Assuming that  $c_2 = hc/k$ , (a) calculate  $\sigma$  [i.e., evaluate the integral  $R_R = \int R(\lambda) d\lambda = \sigma T^4$ ], and (b) calculate the constant  $b$  in Wien's law, i.e., determine the maximum of the function. How well do these agree with the similar parameters calculated from the correct Planck law?

**Solution** (a) Changing variables to  $x = c_2/\lambda T$ , the integral becomes

$$R_R = \frac{8\pi ckT^4}{4c_2^3} \int_0^\infty x^2 e^{-x} dx.$$

Using the fact that the gamma function is defined as

$$\Gamma(z) \equiv \int_0^\infty dt t^{z-1} e^{-t},$$

where  $z$  is a complex argument, and for integer arguments,  $\Gamma(n+1) = n!$ , so that the integral becomes

$$R_R = \left( 4\pi \frac{k^4}{h^3 c^2} \right) T^4.$$

The factor in parentheses is what the Rayleigh Law would predict for the Stefan-Boltzmann constant  $\sigma$ . Note that it has the same units as the correct  $\sigma$  (i.e.,  $k^4/h^3 c^2$ ), but the numerical factor is wrong: the Rayleigh Law predicts  $4\pi \approx 12.6$ , whereas the correct  $\sigma$  has  $2\pi^2/15 \approx 40.8$ .

5. Locate the maximum of the Planck function (by taking its derivative with respect to  $\lambda$  and setting it equal to zero) and obtain a formula for  $b$  in terms of other fundamental constants.
6. Evaluate Eq. (D.1) using Planck's function in Eq. (D.8). Obtain a formula for the Stefan-Boltzmann constant  $\sigma$  in terms of other fundamental constants. This theoretical prediction of a quantity that had previously only been experimentally measured was one of the great successes of Planck's theory.

**Partial solution** The integral in Eq. (D.1) can be simplified by the change of variables  $x = hc/\lambda kT$ . It becomes

$$\frac{8\pi k^4 T^4}{h^3 c^2} \int_0^\infty \frac{x^3}{e^x - 1} dx.$$

The following definitions will help with evaluating this integral. The Riemann Zeta function is defined as

$$\zeta(s) \equiv \sum_{k=1}^{\infty} k^{-s} \quad \Re s > 1.$$

In 1740, Euler was able to evaluate  $\zeta$  for even arguments, and obtained

$$\begin{aligned} \zeta(2) &= \frac{\pi^2}{6} \\ \zeta(4) &= \frac{\pi^4}{90} \\ \zeta(6) &= \frac{\pi^6}{945} \\ &\vdots \end{aligned}$$

In addition, it can be shown that  $\zeta(0) = -\frac{1}{2}$ , and  $\zeta(1) = \infty$ . Another expression for  $\zeta$ , useful in evaluating integrals related to Planck's blackbody function (as in this problem), is

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \quad \Re s > 1$$

where  $\Gamma(s)$  is the Gamma function, and for integer arguments is related to the factorial function  $\Gamma(n) = (n-1)!$

7. Derive the expression for the spectral radiance as a function of frequency  $R(\nu)$  from a knowledge of  $R(\lambda)$ , using the Planck function, given that  $R(\nu)d\nu = -R(\lambda)d\lambda$ . This last equation simply states that the energy emitted between  $\lambda$  and  $\lambda + d\lambda$  must be the same as that emitted between  $\nu$  and  $\nu + d\nu$ . But, since the two variables are related by  $c = \lambda\nu$ , the derivative  $d\lambda/d\nu$  is needed. Finally, the negative sign just ensures that both  $R(\lambda)$  and  $R(\nu)$  are positive.

**Solution** The derivative is

$$\frac{d\lambda}{d\nu} = -\frac{c}{\nu^2}$$

so that  $R(\nu) = R(\lambda)c/\nu^2$ . For the Planck function I obtain

$$u(\nu) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1},$$

and for the Rayleigh-Jeans function I obtain

$$u(\nu) = \frac{8\pi\nu^2}{c^3} kT.$$

8. Show that Eq. (D.3) holds. HINT: see the discussion after Eq. (D.4).

# Appendix E

## The Photoelectric Effect

*It was in 1905 that Einstein made the first coupling of photo effects with quantum theory by bringing forward the bold, not to say the reckless, hypothesis of an electro-magnetic light corpuscle of energy  $h\nu$ , which energy was transferred upon absorption to an electron. – R. A. Millikan, 1916*

Heinrich Hertz (1857-1894) studied the spark discharges that occurred between two metal surfaces when they were held at different electric potentials, and in 1886 was the first to create and detect the electromagnetic waves that had been predicted by Maxwell in 1865. In addition to the waves, he noticed that charged objects would easily lose their charge when illuminated by light. Then, in 1887, in a series of experiments with spark discharges, he found that not only a large potential difference between the two electrodes (now called an anode and a cathode) was able to cause sparks, but ultraviolet light can *also* produce sparks. Figure E.1 shows a schematic of Hertz's experimental setup. He had created what we now would call a “vacuum diode.” The vacuum chamber was important, because air between the two metal plates inhibits current flow unless the electric field between the plates exceeds the “breakdown” potential of air (which is about  $3 \times 10^6$  V/m). In this case, the electric field ionizes the air and it becomes a good conductor—this is the physical mechanism of lightning.

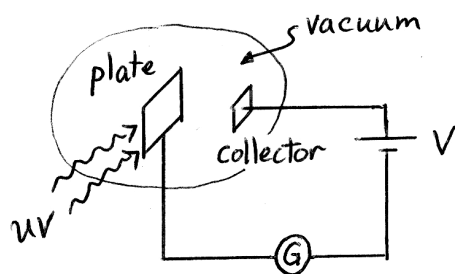


Figure E.1: The 1887 experiment by Hertz that first detected the photoelectric effect. After placing a metal plate and a collector in a vacuum chamber, he biased the plate negative with respect to the collector. If electrons can “jump” off the plate, cross the vacuum gap and reach the collector, a current will flow as measured by the galvanometer  $G$ .

If the space is evacuated, however, there are no air molecules for the electrons to collide with, and the only impediment to current flow is getting the electrons to leave the plate in the first place. Hertz realized that there were two possible methods that could induce the electrons to leave the metallic plate:

1. Heat up the plate, and the electrons would leave via *thermionic emission*.
2. Illuminate the plate, and the electrons would leave via *photo-emission*.

Both mechanisms give some of the electrons enough energy to overcome the binding energy of the metal, also known as the “work function,”  $\phi$ . That is,  $\phi$  is just the minimum energy necessary for an electron to escape from the metal. (Typically, work functions for metals are between 1 eV and 10 eV.) That a *minimum* energy exists makes sense because removing a negatively charged electron from a neutral metal plate results in a positively charged metal plate. The resulting opposite charges attract, and the electron is pulled back toward the metal plate, *unless* an external force does enough work to overcome that attractive force. As Einstein put it

Energy quanta penetrate into the surface layer of the body, and their energy is transformed, at least in part, into kinetic energy of the electrons. The simplest way to imagine this is that a light quantum delivers its entire energy to a single electron; we shall assume that this is what happens.<sup>1</sup>

## Cutoff wavelength

One of the crucial experimental results, which was a key clue in determining that the underlying physical mechanism is quantum in nature, was the existence of a cutoff wavelength. If you, as the experimenter, vary the wavelength  $\lambda$  of the light incident on the plate, while keeping the potential bias  $V$  constant, the current  $I$  measured by the galvanometer would also vary. However, if the wavelength was greater than some maximum wavelength, usually called the “cutoff” wavelength,  $\lambda > \lambda_c$ , there would be no current, *regardless* of the intensity of the incident light. (See Fig. E.2 for a typical current trace.) In this regime, the fact that there is no current implies that the electrons are not receiving enough energy to overcome the work function, and any explanation of this must be based on a theory of the interaction of light and matter.<sup>2</sup>

Maxwell’s wave theory of light predicted that matter obtains energy from light in a continuous manner, just like an ocean wave washing up on the shore. As the amplitude  $E_0$  of the light wave increases, the intensity also increases ( $I \propto E_0^2$ ), so that by increasing the light intensity an electron should be able to absorb as much energy as needed, *regardless of the light’s wavelength*. Figure E.2 shows that this is not what actually happens. In 1905, Einstein realized that an adoption of Planck’s quantum hypothesis not only correctly predicts the features of Fig. E.2, but many other discrepancies as well.

Here is Einstein’s logic. Light of frequency  $\nu$  exists in discrete packets with energy  $E = h\nu = hc/\lambda$ , called photons. If, during a “collision” with an electron, a photon is

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<sup>1</sup>Einstein, 1905.

<sup>2</sup>Refer to Problems 5 and 6 at the end of Chapter 1 for a simple theory of this interaction.



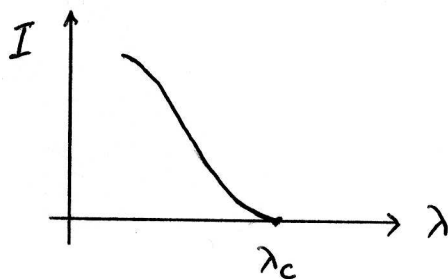


Figure E.2: The current  $I$  as measured by the galvanometer as a function of the wavelength  $\lambda$  of the light incident on the metal plate. If  $\lambda$  is small enough, this light causes photoemission of the electrons.

“annihilated,” that is, it gives up all its energy to the electron, then in order for the electron to be ejected from the metal plate, the photon’s energy must be greater than the work function of the metal,  $E > \phi$ , or

$$\lambda < \frac{hc}{\phi} \equiv \lambda_c. \quad (\text{E.1})$$

For example, if the plate is made of nickel, whose work function is about 5 eV, then  $\lambda_c$  can be calculated to be about 250 nm, which means the incident light must be in the ultraviolet. Of course, in practice it is an experimental measurement of  $\lambda_c$  that is used to determine  $\phi$ .

## The photoelectric equation

Applying the concept of energy conservation to the interaction between the photon and electron results in the following equation

$$h\nu = \phi + K_{max}, \quad (\text{E.2})$$

where  $h\nu$  is the total energy before the interaction, since the electron is assumed to be at rest, and the right hand side is the total energy of the electron after the interaction (the photon no longer exists). The quantity  $K_{max}$  is the maximum kinetic energy of the electron after it has left the metal’s surface.<sup>3</sup>



It was found experimentally in 1902 by Philipp Lenard [Nobel Prize, Physics, 1905] that  $K_{max}$  was independent of the intensity of the light, and also that  $K_{max}$  increased with the frequency of the light  $\nu$ , both of which are predicted by Einstein’s photoelectric equation (E.2). In fact, a straightforward determination of  $h$  can be made by measuring the slope

<sup>3</sup>Equation (E.2) assumes, as Einstein did, that the “light quantum delivers its entire energy to a single electron.”

of  $K_{max}$  versus  $\nu$ . This is exactly what Robert Millikan did in 1916 to obtain the most precise value for  $h$  at that time. He obtained

$$h = (6.56 \pm 0.03) \times 10^{-34} \text{ J s.}$$

All this is fine, but how is one to determine  $K_{max}$ ? It turns out that there is a very simple method. For a given frequency of incident light, reverse the polarity of the battery in Fig. E.1 until it is strong enough to *stop* the current. This value of  $V$  is called the “stopping potential,”  $V_s$ . When the battery’s polarity is reversed, the electric field now points *from* the plate *to* the collector, which serves to repel the electrons from the collector. At this critical value of the potential, the electric field is just barely large enough to repel the most energetic electrons, those with energy  $K_{max}$ , which means that  $K_{max} = eV_s$ . The experimentally measured quantities  $V_s$  and  $\nu$  are therefore related by

$$V_s = \left(\frac{h}{e}\right)\nu - \frac{\phi}{e}, \quad (\text{E.3})$$

which means that in actuality the experimenter measures a slope of  $h/e$ , rather than  $h$  directly.

## Einstein’s revolution

The photoelectric effect, and the various interpretations of Einstein’s explanation, is useful to illustrate some issues in the philosophy of science and some of the consequences of Einstein’s quote on page xv. For example, it is usually stated that the experimental facts of the photoelectric effect, and Einstein’s explanation, unambiguously suggest that light comes in discrete clumps, or photons. In fact, the situation is not that clear, as we shall see below.

Einstein’s 1905 paper, in which he explained the photoelectric effect, was primarily a study of the thermodynamics of radiation, and in particular how that applied to blackbody radiation. He limited his analysis to the so-called “Wien regime,” which can be expressed as  $h\nu \gg kT$ . This is the regime where the ultraviolet catastrophe (see App. D) rears its ugly head. That is, in this regime, “the classical theory becomes an unreliable predictor for the quantum results.”<sup>4</sup> After a study of Planck’s explanation of blackbody radiation, Einstein was prompted to make the “light-quantum hypothesis:”

Monochromatic radiation...behaves in thermodynamic respect as if it consists of mutually independent energy quanta of magnitude  $R\beta\nu/N$ .<sup>5</sup>

You can think of this hypothesis (not a theorem) as “just a curious property of pure radiation in thermal equilibrium, without any physical consequence,”<sup>6</sup> but Einstein next made a statement about *physical reality*, called the “heuristic principle:”

<sup>4</sup>Emch and Liu, *The logic of thermostistical physics*, Springer 2001, page 363.

<sup>5</sup>Einstein, 1905. In Einstein’s notation  $R\beta/N = h$ .

<sup>6</sup>Pais, *Subtle is the Lord*, page 377.

If...monochromatic radiation...behaves as a discrete medium consisting of energy quanta of magnitude  $R\beta\nu/N$ , then this suggests [that] the laws of the generation and conversion of light are also constituted as if light were to consist of energy quanta of this kind.<sup>7</sup>

That is, the “light-quantum hypothesis” describes a property of radiation, nothing more or less, but the “heuristic principle,” on the other hand, makes the stronger claim that this property can be extended to the interaction of light and matter. Einstein is now describing the underlying reason for a physical *process*, which, if not falsified, is the first step on the road to becoming a physical theory.

In 1905, there was no quantum theory for electrons — the Schrodinger equation did not make an entrance until 1926 — and so Einstein, as might be expected, treated the electrons classically. Only the light was assumed to be quantized, and this was enough to explain all the strange experimental observations. However, in 1927 after it became possible to describe the electrons using a quantum theory, Gregor Wentzel was able to explain the photoelectric effect *without photons!* Either the electron *or* the light must be quantized, but both is not necessary. We can conclude that the photoelectric effect does *not* prove the existence of photons, but is somewhat more ambiguous. In addition, the Compton effect (Appendix H) is also an experiment that claims to prove the existence of photons. However, his explanation is from 1923, again before the advent of the Schrodinger equation. In 1927, Schrodinger himself explained the Compton effect without photons, although he had to quantize the electron, just as Wentzel did.

So we are left with a conundrum: are photons real or not? This is not the proper question to ask, however. Two better questions are, “Do photons explain nature?” and “Is the concept of a photon required to explain experimental observations?” The answer to the first question is yes, but, considering only the photoelectric effect and the Compton effect, the answer to the second question is no. Are there any other observations that can decide the issue? In fact, there are. When an atom radiates light, it recoils. If we viewed the electromagnetic radiation classically, atoms would radiate a spherical wave in all directions (if the atom were spherically symmetric), and conservation of momentum would dictate that the atom would not move. However, since atoms do recoil, this implies that the radiation is emitted in a particular direction, and in fact, is a photon.

The moral of this story is that while there might be one experiment that is conceptually straightforward and that experiment comes to be known as the “proof” of a concept, it is usually several experiments that result in a “preponderance of the evidence.” That is, one experiment does not usually remove “reasonable doubt.” A slightly different viewpoint is put forth by Luis Alvarez when he was discussing the impact theory for the extinction of the dinosaurs:

A few words on how theories become accepted are appropriate here. Most laymen feel that theories can be proved or disproved, but with very few exceptions, theories can’t be proved, only disproved. For example, Newton’s extremely accurate theory of gravity was disproved by Einstein’s general theory of relativity, but Einstein’s theory wasn’t proved in that process some new

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<sup>7</sup>Einstein, 1905.

theory may prove that Einstein was wrong. So how do some theories gain nearly universal acceptance, when proofs are so rare? The answer is that every useful theory explains all known observations and makes predictions, and if the predictions turn out to be true, particularly if some of them are very surprising, then that theory becomes an accepted theory, even though someone may later find that one of its predictions doesn't correspond to reality and thereby invalidates it. An example of what I've called a surprising prediction comes from Maxwell's kinetic theory of gases, which predicted that the viscosities of gases would increase with temperature. This was counter to everyone's intuition — viscosities of liquids decrease with increasing temperature — but it turned out to be true. <sup>8</sup>

### Collateral Reading

The following articles and sections of books give a brief introduction to problems of epistemology, especially as it applies to science. That is, as scientists we want to make statements with certainty, or, barring that, at least know the degree of certainty that holds for each particular statement.

- Sam Inglis, review of *Karl Popper: Philosophy and Problems*, *Am. J. Phys.*, **65** 162-164 (1997). (ERAU: Reynolds' office)
- Bertrand Russell, *The Problems of Philosophy*, Chapter 1, 1912. (ERAU: online)
- Hans Reichenbach, *Philosophic foundations of quantum mechanics*, Sections §4-6, University of California Press, 1944 (Dover, 1998). (ERAU: Reynolds' office)

### Problems

1. In his measurement of  $h$ , Millikan used sodium metal for the material of his metal plate, and was able to determine that the minimum (cutoff) frequency was  $0.439 \times 10^{15}$  Hz. What does this imply for the work function for sodium? How does Millikan's value compare with the presently accepted value?

**Solution** Multiplication gives  $\phi = h\nu = (6.561 \times 10^{-34} \text{ J s})(0.439 \times 10^{15} \text{ Hz}) = 2.88 \times 10^{-19} \text{ J} = 1.798 \text{ eV}$ . The current values for sodium range from 1.82 eV to 2.75 eV depending on the surface cleanliness. This variable (surface cleanliness) was what Millikan spent several years trying to improve. A recent value from CRC is 2.36 eV.

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<sup>8</sup>“Mass extinctions caused by large bolide impacts,” *Physics Today*, July 1987, pp 24-33.

# Appendix F

## Reduced Mass

Consider two particles of mass  $m_1$  and  $m_2$  located at positions  $\vec{\mathbf{r}}_1$  and  $\vec{\mathbf{r}}_2$  respectively, as shown in Fig. F.1. If, in addition to the forces that they exert on each other, there is an external force  $\vec{\mathbf{F}}_{\text{ext}}$  that is exerted on each of them, then the equations of motion for each of the particles are

$$\vec{\mathbf{F}}_{\text{ext}} + \vec{\mathbf{F}}_{21} = m_1 \frac{d^2}{dt^2} \vec{\mathbf{r}}_1, \quad (\text{F.1})$$

$$\vec{\mathbf{F}}_{\text{ext}} + \vec{\mathbf{F}}_{12} = m_2 \frac{d^2}{dt^2} \vec{\mathbf{r}}_2, \quad (\text{F.2})$$

where  $\vec{\mathbf{F}}_{12}$  is the force exerted by  $m_1$  on  $m_2$ , and  $\vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}$  (by virtue of Newton's Third Law). The two (vector) dependent variables in this description that are to be solved for as functions of time are  $\vec{\mathbf{r}}_1(t)$  and  $\vec{\mathbf{r}}_2(t)$ . This “two-body problem” can be reduced to an equivalent “one-body problem” by making the following change of variables

$$\vec{\mathbf{R}} = \frac{m_1 \vec{\mathbf{r}}_1 + m_2 \vec{\mathbf{r}}_2}{m_1 + m_2}, \quad (\text{F.3})$$

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2,$$

where  $\vec{\mathbf{R}}$  is the position of the center of mass of the system and  $\vec{\mathbf{r}}$  is the relative position of the particles. It doesn't matter whether you use the original set of dependent variables,

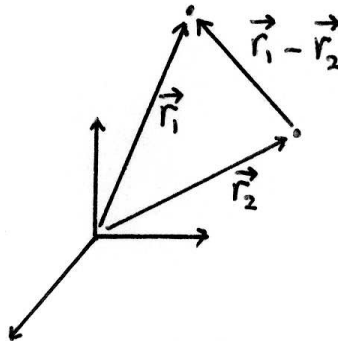


Figure F.1: Geometry for two particles moving in one coordinate system. If there is no external forces, then only the relative position vector  $\vec{\mathbf{r}}$  is needed.

$\vec{\mathbf{r}}_1(t)$  and  $\vec{\mathbf{r}}_2(t)$ , or the new set,  $\vec{\mathbf{R}}(t)$  and  $\vec{\mathbf{r}}(t)$ . Therefore, in order to recast Eqs. (F.1) and (F.2) we need to invert the transformation in Eq. (F.3) and express  $\vec{\mathbf{r}}_1$  and  $\vec{\mathbf{r}}_2$  in terms of  $\vec{\mathbf{R}}$  and  $\vec{\mathbf{r}}$ . Doing this I obtain

$$\begin{aligned}\vec{\mathbf{r}}_1 &= \frac{(m_1 + m_2)\vec{\mathbf{R}} + m_2\vec{\mathbf{r}}}{m_1 + m_2}, \\ \vec{\mathbf{r}}_2 &= \frac{(m_1 + m_2)\vec{\mathbf{R}} - m_1\vec{\mathbf{r}}}{m_1 + m_2}.\end{aligned}\tag{F.4}$$

Plugging these into the original equations of motion, (F.1) and (F.2), I obtain two new equations of motion, for  $\vec{\mathbf{R}}$  and  $\vec{\mathbf{r}}$ . First, adding (F.1) and (F.2) gives

$$2\vec{\mathbf{F}}_{\text{ext}} = (m_1 + m_2)\frac{d^2}{dt^2}\vec{\mathbf{R}},\tag{F.5}$$

and then subtracting (F.2) from (F.1) results in

$$\begin{aligned}\vec{\mathbf{F}}_{12} &= \left(\frac{m_1 - m_2}{2}\right)\frac{d^2}{dt^2}\vec{\mathbf{R}} + \frac{m_1 m_2}{m_1 + m_2}\frac{d^2}{dt^2}\vec{\mathbf{r}} \\ &= \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\vec{\mathbf{F}}_{\text{ext}} + \frac{m_1 m_2}{m_1 + m_2}\frac{d^2}{dt^2}\vec{\mathbf{r}}.\end{aligned}\tag{F.6}$$

Eq. (F.5) is simply the equation of motion for the entire system—the total force,  $2\vec{\mathbf{F}}_{\text{ext}}$ , is equal to the total mass,  $(m_1 + m_2)$ , times the acceleration of the center of mass. The first term in Eq. (F.6) is zero if  $\vec{\mathbf{F}}_{\text{ext}} = 0$ , so that in the absence of external forces the internal dynamics are determined by

$$\vec{\mathbf{F}} = \mu\frac{d^2}{dt^2}\vec{\mathbf{r}},\tag{F.7}$$

where  $\vec{\mathbf{F}} = \vec{\mathbf{F}}_{12}$  is the internal force between the two particles, and

$$\mu = \frac{m_1 m_2}{m_1 + m_2}\tag{F.8}$$

is called the “reduced mass.” Equation (F.7) is identical to that for *one* particle of mass  $\mu$  moving in a central force  $\vec{\mathbf{F}}$ . Another, more suggestive, way of expressing the reduced mass is

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}.$$

The net result of this transformation has been to reduce a two-body problem to a one-body problem. For many situations,  $m_1 \gg m_2$ , and the approximation  $\mu \approx m_2$  is valid. Such is the case, for example, for a low-mass planet orbiting a high-mass star.

# Appendix G

## Cosmic Rays and Muons

*Coming out of space and incident on the high atmosphere, there is a thin rain of charged particles known as the primary cosmic radiation. — Cecil Powell*  
[Nobel Prize, Physics, 1950]

The charged particles that make up the “primary” cosmic rays are protons,  $\alpha$  particles, heavier nuclei, and electrons, and they impact the Earth from all directions and with various energies. Most of these are protons (about 80%), second in abundance are  $\alpha$  particles (about 14%), while electrons make up less than 1%. When they impact nuclei in the atmosphere — mostly oxygen and nitrogen nuclei — their energies are such that they create “showers” of hadrons, mostly pions, along with some kaons, and anti-protons, and anti-neutrons. These then decay into photons, electrons, positrons, neutrinos, and muons (which themselves decay into electrons and neutrinos). These are all called “secondary” cosmic rays.



Where do the primary cosmic rays come from? Some come from the sun (mostly due to solar flares), most come from galactic supernovae, and a few with the highest energy are suspected to originate from outside the Milky Way. You might suspect the solar wind—a neutral plasma that consists of low energy protons, electrons, and helium nuclei—as a source of cosmic rays. Due to their low energies, however, these particles are stopped from reaching the atmosphere by the Earth’s magnetic field, except in the polar regions. While they have enough energy to cause aurora, they do not cause showers of secondary subatomic particles.

How many are there? About 1 charged particle per second per  $\text{cm}^2$  impacts the Earth.<sup>1</sup> This is a far cry from the  $6 \times 10^{10}$  neutrinos  $\text{s}^{-1} \text{cm}^{-2}$  that come from the Sun.

What are their energies? The typical kinetic energy of these particles is about 10 MeV to 100 MeV, although there are some at higher energies. Figure G.1 shows the distribution of the measured energy per particle. In fact, the cosmic ray with the highest energy has been measured at 48 J! These ultra-high energy cosmic rays are suspected to be extra-galactic, as there is no plausible mechanism of acceleration to these energies by

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<sup>1</sup>Henley and Garcia, *Subatomic Physics*, page 597.

a supernova, for example. Again, compare these energies to those of solar neutrinos that have only 0.26 MeV.

What happens to the secondary cosmic rays?

The pions decay via the following modes

$$\pi^0 \rightarrow 2\gamma \quad (\text{G.1})$$

$$\pi^\pm \rightarrow \mu^\pm + \nu, \quad (\text{G.2})$$

where the neutral pions decay electromagnetically with an average lifetime of  $8.4 \times 10^{-17}$  s, and the photons subsequently create electron-positron pairs. Most of the energy of the original cosmic ray follows this path. Some of the energy goes into charged pions, which decay into muons with an average lifetime of  $2.6 \times 10^{-8}$  s. This long lifetime indicates that the decay is due to the weak interaction, and is therefore relatively unlikely. The muons then decay into electrons (or positrons) and neutrinos

$$\mu^\pm \rightarrow e^\pm + 2\nu, \quad (\text{G.3})$$

and their average lifetime is  $2.2 \mu\text{s}$ , also a weak interaction.<sup>2</sup>

What happens to these secondary cosmic rays as they pass through the atmosphere? First of all, in addition to possible decay, the charged particles cause ionization of the atmospheric molecules and therefore lose energy. For example, a typical muon loses about 2 GeV of kinetic energy before it hits the ground (if it hasn't decayed yet), and by the time they do reach the ground, the average muon energy is about 4 GeV. Secondly, the showers spread out laterally from the direction of the primary cosmic ray. The main hadronic core (pions, etc.) covers a few meters by the time it hits the ground, and the electromagnetic particles (electrons, positrons, photons) have spread further, about 100 m. Finally, the muons have spread the furthest, almost 1 km.

## Muons as clocks

This spreading means that muons are continually bombarding the Earth's surface and, since it is not clear what direction they came from, statistical methods must be used to interpret the muon flux. That is, the muons are all "born" at different altitudes, they travel downward with different speeds, and they "live" for different intervals of time. Therefore, you might expect that the muon flux would increase with increasing altitude, at least initially, reach a maximum at some altitude, and then finally decrease. (This altitude of maximum flux is called the "Pfozter maximum.") This is precisely what is observed, but

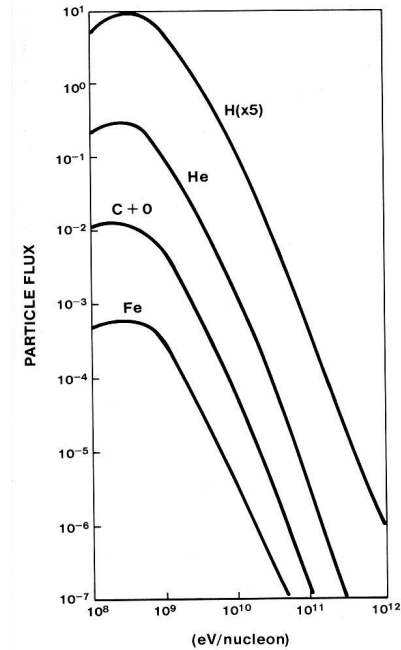


Figure G.1: The energy spectrum of the different nuclei that make up cosmic rays. Carbon and oxygen are lumped together. From Friedlander, *Cosmic Rays*, Figure 6.4.

<sup>2</sup>Recall that the weak force is responsible for changing one family of quarks or leptons into another.



the exact shape of this curve is a convolution of a source function and a decay function, and therefore requires lots of modeling to interpret.

However, for our purpose — special relativity — we want to use the muons as a clock. In Chapter 5 we assume that our muons are all created at the same altitude, and all live for the same amount of time,  $2.2 \mu\text{s}$ . You might think that we are not justified in doing this, because of the statistical spread of muon lifetimes, but that turns out not to be true. Scott and Burke state the case:

It may seem at first glance that a real particle that is formed and later decays does not constitute an accurate clock, because of the uncertain nature of the decay process. Given a number of particles, some will decay at times less than the mean life, some will decay at times greater than the mean life, and in general it is impossible to predict exactly when any given particle will decay. However, it is possible to determine the *mean* lifetime of a number of particles to any desired accuracy simply by observing a sufficient number of such particles, and in this sense, decaying particles are just as good clocks as vibrating molecules. Indeed, for a vibrating molecule it is necessary to observe it for a large number of cycles in order to determine its frequency precisely; this is analogous to observing a large number of decays in an exponentially decaying system.<sup>3</sup>

### Collateral Reading

- “The early history of cosmic ray research.” by Q. Xu and L. M. Brown, *Am. J. Phys.*, **55** 23-33 (1987).
- Michael W. Friedlander, *Cosmic Rays*, Harvard University Press, 1989. (ERAU: QC 485 .F75 1989)

### Problems

1. Calculate the energy in MeV of a 48-J proton. Also calculate  $\gamma$  and  $\beta$  for the same proton.

**Solution** The conversion factor is, of course, just the electronic charge  $e$

$$48 \text{ J} = 48 \text{ J} \left( \frac{1 \text{ eV}}{1.602 \times 10^{-19} \text{ J}} \right) = 3.00 \times 10^{14} \text{ MeV}$$

To calculate  $\gamma$ , this total energy must be divided by the proton’s rest energy

$$\gamma = \frac{E}{mc^2} = \frac{3.00 \times 10^{14} \text{ MeV}}{938.27 \text{ MeV}} = 3.19 \times 10^{11}$$

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<sup>3</sup>Scott and Burke, *Special Relativity Primer*, page 5.

For this large  $\gamma$ , the value of  $\beta$  will be extremely close to unity, so we can use the usual approximation

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2} = 1 - 4.9 \times 10^{-24}$$

2. (a) Calculate the reaction energy for a pion decaying into a muon and a neutrino.  
 (b) Using the conservation of momentum, calculate how much energy the muon has. HINT: You can assume the muon is non-relativistic (check this), but you must take relativistic effects into account for the neutrino. One approximation is to take the highly relativistic limit for the neutrino, where the relationship between its energy and momentum is  $E_\nu = p_\nu c$ . As usual, ignore the neutrino mass.
3. Why can't a  $\pi^0$  decay into a  $\mu^-$  and a  $\mu^+$ ?
4. If a muon  $\mu^-$  is "born" due to a pion decay  $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$  at an altitude of 20 km, how fast must it be traveling to reach the ground before it decays 2.2  $\mu\text{s}$  later? Express your answer in the form  $\beta = 1 - \epsilon$ , and calculate  $\epsilon$ .

# Appendix H

## The Compton Effect

In 1923, Arthur Holly Compton [Nobel Prize, Physics, 1927] performed a simple experiment concerning the scattering of X-rays. He used the  $K_\alpha$  X-rays from molybdenum (see Section 4.4 for a description of secondary X-rays) and scattered them off a carbon target. X-rays had already been scattered from crystal targets by the Braggs (father William Henry and son William Lawrence) in 1913, and they had found that the wavelength of the scattered X-rays was identical to the incident X-rays, but the *intensity* varied with scattering angle. The angles that exhibited the highest intensity were explained by constructive interference of the electromagnetic wave with itself as it was scattered by parallel planes in the crystal, and those that exhibited low intensity were the result of destructive interference. This process is called “Bragg scattering.” In these solids, however, the electrons were strongly bound to the atoms so that they did not interact individually with the incident X-rays.



Compton used carbon (in the form of graphite) as his target. In this case, the high electrical conductivity means that there are plenty of effectively “free” electrons willing and able to collide with the X-ray photons allowing them to exhibit their particle nature. In observing the scattered X-rays at different angles, Compton found that the wavelength *increased*, which means that they must have lost energy during the interaction with the graphite. This process is called “Compton scattering.” The only way that Compton was able to explain this effect is by invoking the quantum nature of light and treating the interaction as a two-body collision between photon and electron, although using relativistic dynamics. This experiment, along with Planck’s explanation of the blackbody spectrum and Einstein’s explanation of the photoelectric effect, finally convinced most physicists that photons were “real.” In fact, it wasn’t until after this experiment, in 1926, that the term “photon” was first used.

In principle, when X-rays are incident on a solid, both types of scattering, Bragg and Compton, will occur. The wave properties of light allow it to interfere with itself and show preferential scattering directions, while the particle properties allow it to interact with single electrons.

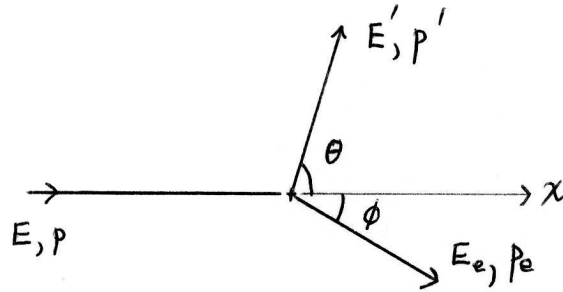


Figure H.1: Scattering geometry for an incoming particle of energy  $E$  and momentum  $p$ , moving in the  $x$  direction, and impacting a stationary particle at the origin. After the collision, the incoming particle moves off with energy  $E'$  and momentum  $p'$  at an angle  $+\theta$  with respect to the positive  $x$  axis, and the stationary particle moves off with energy  $E_e$  and momentum  $p_e$  at an angle  $-\phi$  with respect to the positive  $x$  axis.

## Two-body collisions

While the analysis is somewhat complicated algebraically, it is a straightforward application of the laws of conservation of energy and momentum. Consider the geometry as shown in Fig. H.1, where a stationary electron of mass  $m_e$  is hit by a photon of energy  $E$  and momentum  $\vec{p} = p\hat{x}$ . After the collision, the photon now has a different energy  $E'$  with a component of momentum in the  $\hat{y}$  direction, and the electron increases its energy to  $E_e$  and has a component of momentum in the  $-\hat{y}$  direction. The three conservation equations become (energy,  $x$ -momentum, and  $y$ -momentum)

$$E + m_e c^2 = E' + E_e \quad (\text{H.1a})$$

$$p = p' \cos \theta + p_e \cos \phi \quad (\text{H.1b})$$

$$0 = p' \sin \theta - p_e \sin \phi \quad (\text{H.1c})$$

where the scattering angles  $\theta$  and  $\phi$  are both taken to be positive. In principle, a knowledge of the initial conditions — the photon's initial wavelength  $\lambda$ , from which both its energy and momentum can be calculated — allows us to calculate the final conditions — the direction and momentum of both the photon and the electron. However, there are four unknowns,  $p_e$ ,  $p'$ ,  $\theta$ , and  $\phi$ , but there are only three equations. Hence the system is underdetermined, and the best we can do is to either (a) fix one of the unknowns and solve for the other three in terms of the first, or (b) eliminate two of the unknowns to obtain a relationship between the other two unknowns. Since there is no general experimental method to fix one of the unknowns, the second approach is the one that allows a comparison with experiment. To illustrate the procedure, I'll first analyze a classical, nonrelativistic collision between two point particles that you have seen before in elementary mechanics. Then I'll look at the Compton scattering experiment, where relativistic dynamics are required, but the physical principles are identical.

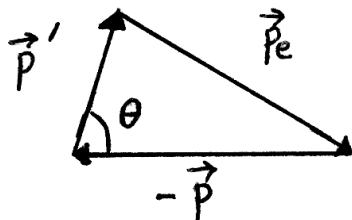


Figure H.2: Graphical representation of conservation of momentum for the Compton scattering geometry. The difference in the momenta before and after the collision must be zero, which means that a vector sum of all the momentum vectors must return to the original starting location.

**Billiard ball collision.** Let's assume that Fig. H.1 and Eqs. (H.1) applies to two billiard balls: a cue ball (instead of a photon), and an eight-ball (instead of an electron). In the nonrelativistic case, the energy of each particle is just the kinetic energy, given by  $K = mv^2/2$  plus the rest energy, and the momentum is the linear momentum  $\vec{p} = m\vec{v}$ . Assuming that the masses of the cue ball and eight-ball,  $m$  and  $m_e$  respectively, do not change, the final unknowns are just the two speeds,  $v'$  and  $v_e$ , and the two angles,  $\theta$  and  $\phi$ . Our goal is to determine these quantities in terms of the one initial quantity: the initial kinetic energy  $K$  of the cue ball. The technique is identical to that used in Problem 60.

The first step is to combine Eqs. (H.1b) and (H.1c) by moving the terms containing  $\theta$  to the left side of the equations, squaring both sides and adding the two equations. Using elementary trigonometric relations, I get

$$p^2 - 2pp' \cos \theta + p'^2 = p_e^2. \quad (\text{H.2})$$

This fundamental relation can be obtained very easily with a graphical analysis, as shown in Fig. H.2. Since the momentum *vector* is conserved, a triangle can be drawn whose three sides are the three momenta in the problem. A trivial application of the law of cosines results in Eq. (H.2). Notice that we have effectively eliminated the angle  $\phi$ .

The second step is to express the energy of each particle in terms of its momentum. Since this problem is nonrelativistic, we have

$$E = mc^2 + K \quad (\text{H.3a})$$

$$E' = mc^2 + \frac{p'^2}{2m} \quad (\text{H.3b})$$

$$E_e = m_e c^2 + \frac{p_e^2}{2m_e}, \quad (\text{H.3c})$$

where  $K$  is the initial kinetic energy of the cue ball.<sup>1</sup> Conservation of energy, Eq. (H.1a), becomes

$$K = \frac{p'^2}{2m} + \frac{p_e^2}{2m_e}$$

<sup>1</sup>This initial energy is a known quantity, so we are interested in solving for the other quantities in terms of  $K$ . If desired, we could express it as  $K = p^2/2m$  and take  $p$  and  $m$  as our known quantities.

$$= \frac{p'^2}{2m} + \frac{1}{2m_e} (p^2 - 2pp' \cos \theta + p'^2), \quad (\text{H.4})$$

where I have eliminated  $p_e$  using Eq. (H.2). This is our final result, and it is a relation between  $p'$  and  $\theta$ . That is, given the fact that the cue ball deflects at a certain angle, then Eq. (H.4) tells us what its final momentum and energy *must be*.

Of course, as you may know from playing billiards, the cue ball can deflect at any angle depending on how it impacts the eight-ball. However, if you were to measure its final momentum, you would find that  $p'$  and  $\theta$  are always related by Eq. (H.4) — see Problems 1 and 2 at the end of this appendix.

**Photon-electron collision.** How do we analyze the case of a X-ray interacting with an electron? At first glance, it appears to be a completely different situation, but in fact is remarkably similar. Energy and momentum are still conserved, so Eqs. (H.1) and (H.2) still apply. The only difference is that we must allow for the fact that the electron might be moving relativistically after the collision, and of course the photon is always moving ultra-relativistically, so we must replace Eqs. (H.3) with

$$E = pc \quad (\text{H.5a})$$

$$E' = p'c \quad (\text{H.5b})$$

$$E_e = \sqrt{(m_e c^2)^2 + p_e^2 c^2}. \quad (\text{H.5c})$$

The conservation of energy equation, Eq. (H.1a), becomes slightly more complicated

$$(pc + m_e c^2 - p'c)^2 = (p^2 - 2pp' \cos \theta + p'^2)c^2 + (m_e c^2)^2. \quad (\text{H.6})$$

Again, I have eliminated  $p_e$  using Eq. (H.2), so this is our final result, a relation between  $p'$  and  $\theta$ .

There are two key differences between this case and the billiard-ball collision. First, the initial condition is given by the photon momentum  $p$  rather than the cue ball kinetic energy  $K$ . Second, we eliminated  $\phi$  (rather than  $\theta$ ) because the electron remains in the target so that only  $\theta$  is measurable. With billiard balls, in principle we could measure either  $\theta$  or  $\phi$ .

Compton wanted to compare this theoretical prediction, Eq. (H.6), with experimentally observable quantities. It turns out that photon momentum is not simple to measure, but photon wavelength is, so using the de Broglie relation  $p = h/\lambda$  and  $p' = h/\lambda'$ , where  $\lambda$  and  $\lambda'$  are the photon wavelength before and after the collision, results in the famous Compton scattering formula

$$\lambda' - \lambda = \frac{h}{m_e c} (1 - \cos \theta). \quad (\text{H.7})$$

The quantity

$$\lambda_C \equiv \frac{h}{m_e c}, \quad (\text{H.8})$$

is called the *Compton wavelength* of the electron. It is *not* the de Broglie wavelength of the electron matter wave, but rather indicates the magnitude of the wavelength *shift* of

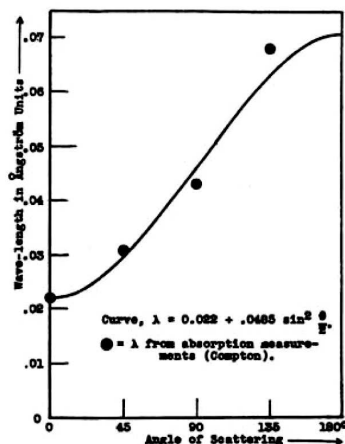


Figure H.3: Wavelength of the scattered  $\gamma$ -rays as a function of angle of deflection  $\theta$ . The four dots are experimental measurements, and the curve is Eq. (H.7). Figure 5 from Compton, 1923.

the photon when it collides with an electron. Equation (H.7) predicts that the wavelength of the scattered light is longer than the wavelength of the incident light — how much longer depends on the angle of deflection. Figure H.3 shows some of Compton’s original data clearly showing this effect.

This process is fundamental to the operation of NASA’s Compton Gamma Ray Observatory, which was in Earth orbit from 1991 through 2000. The NASA web site states, “The Observatory was named in honor of Dr. Arthur Holly Compton, who won the Nobel prize in physics for work on scattering of high-energy photons by electrons — a process which is central to the gamma-ray detection techniques of all four instruments [on the CGRO].”

## Problems

1. A cue ball of mass  $m$  with kinetic energy  $K$  collides elastically with an eight-ball of mass  $m_e$  at rest. Derive the relationship between the deflection angle of the cue ball ( $\theta$ ) and its kinetic energy *after* the collision ( $p'^2/2m$ ). That is, express  $\theta$  as a function of  $K'$  and  $K$ .

**Solution** Manipulating Eq. (H.4) by expressing the momenta in terms of the kinetic energy ( $p = \sqrt{2mK}$ ) gives

$$\cos \theta = \frac{(m + m_e)K' + (m - m_e)K}{m\sqrt{KK'}}.$$

2. For the situation described in Problem 1, if  $m < m_e$  then it is possible for  $m$  to bounce directly backwards (i.e., with a deflection angle of  $\theta = \pi$ ). However, if  $m > m_e$ , then there is a maximum deflection angle. Find this angle.

3. Derive Eq. (H.7).

**Solution** Expanding the square on the left-hand-side of Eq. (H.6) and canceling terms gives

$$m_e c(p - p') = pp'(1 - \cos \theta).$$

Expressing the momenta in terms of the wavelength gives

$$\frac{p - p'}{pp'} = \frac{1}{h} \lambda \lambda' \left( \frac{1}{\lambda} - \frac{1}{\lambda'} \right) = \frac{\lambda' - \lambda}{h}.$$

Simplification results in Eq. (H.7).

4. If the incident photon has energy  $E$  (or  $h\nu$ ), what is the maximum possible kinetic energy imparted to the scattered electron? At what angle  $\phi$  does this electron scatter?

**Solution** The electron will have maximum kinetic energy if the photon loses a maximum amount of energy. This occurs when the photon is backscattered, i.e.,  $\theta = \pi$ . For this angle, the Compton scattering formula can be written

$$\frac{1}{E'} - \frac{1}{E} = \frac{2}{m_e c^2}.$$

Solving this for  $E'$  and then forming the quantity  $E - E'$ , I get

$$E - E' = \frac{2E^2}{m_e c^2 + 2E}.$$

Since this is the energy lost by the photon, it is also the energy gained by the electron.

5. Obtain a formula that gives the electron's kinetic energy as a function of the photon's scattering angle  $\theta$ .

**Solution** Following the method of solution for the previous problem, but retaining an arbitrary angle  $\theta$ , the Compton scattering formula is

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{m_e c^2} (1 - \cos \theta),$$

and forming  $E - E'$  again (which is just the electron kinetic energy)

$$K_e = E - E' = \frac{E^2(1 - \cos \theta)}{E(1 - \cos \theta) + m_e c^2}.$$

Since the quantity  $(1 - \cos \theta)$  is positive definite, the electron kinetic energy is also positive definite. Also, you can show that  $K_e$  is zero when  $\theta = 0$ , and that it is a maximum when  $\theta = \pi$ .



6. Determine the maximum wavelength shift in the Compton scattering of photons from *protons*.
7. What is the maximum possible kinetic energy of a recoiling Compton electron? Express your answer in terms of the incident photon energy  $h\nu$  and the electron's rest energy  $m_e c^2$ .
8. From Figure H.3, can you calculate the Compton wavelength  $\lambda_C$ ? The units of the  $y$  axis are Ångstroms.



# Appendix I

## Relativistic quantum mechanics

### The Klein-Gordon Equation

The pseudo-derivation of the Schrodinger equation in Sec. 7.1 started with the nonrelativistic relation between energy and momentum,  $E = p^2/2m + U$ . If we use instead the correct relativistic equation, Eq. (5.62),

$$E^2 = (pc)^2 + (mc^2)^2, \quad (\text{I.1})$$

and make the same assumptions, i.e., that the wave function is a traveling plane wave and the wave-particle duality relations hold, Eqs. (7.2), then we can make the usual replacements

$$\begin{aligned} E &\rightarrow i\hbar \frac{\partial}{\partial t} \\ \vec{p} &\rightarrow -i\hbar \nabla. \end{aligned} \quad (\text{I.2})$$

Since each power of  $E$  represents one time derivative, these replacements result in a second order (in time) differential equation

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \Psi(\vec{r}, t) = -\hbar^2 c^2 \nabla^2 \Psi(\vec{r}, t) + m^2 c^4 \Psi(\vec{r}, t). \quad (\text{I.3})$$

This is the Klein-Gordon equation, named after Oskar Klein and Walter Gordon, who were only one of many in 1926 that proposed this equation to describe relativistic electrons. In fact, this equation was first developed by Schrodinger in 1925, but he didn't publish it because he couldn't make it correctly describe the hydrogen atom. It turned out that the Klein-Gordon equation describes *spinless* particles, like the pion, and the Dirac equation is needed to relativistically describe the spin  $\frac{1}{2}$  electron.

To see that this equation can also describe photons (even though they are spin 1), we can rewrite it more compactly as

$$\frac{\partial^2 \Psi}{\partial t^2} = c^2 \nabla^2 \Psi - \left( \frac{m^2 c^4}{\hbar^2} \right) \Psi, \quad (\text{I.4})$$

where it reduces to the scalar wave equation for  $m = 0$ . Clearly, then, any particle with zero mass must travel at speed  $c$ .

## Solutions to the Klein-Gordon equation and the Schrodinger equation

From the viewpoint of 1926, Eq. (I.4) *should* be able to describe a particle at rest — of course if  $m = 0$  it cannot, but that is precisely the role of the  $m^2$  term. One simple, spatially independent, solution is

$$\Psi \sim e^{-i(mc^2/\hbar)t}. \quad (\text{I.5})$$

This wave function is not moving, but is simply oscillating in time. Using our knowledge that  $E = \hbar\omega$  and  $E_0 = mc^2$ , the coefficient of  $t$  can be defined as

$$\omega_0 \equiv \frac{mc^2}{\hbar}, \quad (\text{I.6})$$

the “rest frequency” of the particle.<sup>1</sup>

A more interesting solution can be obtained by making the trial solution

$$\Psi(\vec{r}, t) = \psi(\vec{r}, t) e^{-i(mc^2/\hbar)t}. \quad (\text{I.7})$$

This is similar to the technique we have used to obtain the time-independent Schrodinger equation by separation of variables, except that here  $\psi$  still retains a time dependence. Although we are making no assumptions about the time dependence of  $\psi$  (yet), this technique is often used to obtain approximate solutions to a differential equation when the time dependence of  $\psi$  is much slower than  $\omega_0$ .<sup>2</sup> Plugging this trial function into the Klein-Gordon equation results in a differential equation for  $\psi$

$$\frac{\partial^2 \psi}{\partial t^2} - 2i\omega_0 \frac{\partial \psi}{\partial t} = c^2 \nabla^2 \psi. \quad (\text{I.8})$$

Since this is a linear equation, the usual technique of assuming an exponential solution  $\psi \sim e^{i(\vec{k}\cdot\vec{r}-\omega t)}$  turns the differential equation into an algebraic equation

$$\omega^2 + 2\omega\omega_0 = k^2 c^2. \quad (\text{I.9})$$

This algebraic equation is, of course, just the dispersion relation. In the limit  $\omega \gg \omega_0$ , Eqs. (I.8) and (I.9) are just the wave equation and its associated dispersion relation, respectively, because the second terms on the left-hand-sides are ignorable. However,

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<sup>1</sup>For an electron, this rest frequency has the numerical value

$$\omega_0 = \frac{mc^2}{\hbar} \approx 7.5 \times 10^{20} \text{ s}^{-1},$$

which are extremely fast oscillations. This, of course, is just the angular frequency of the photons created in an electron-positron annihilation event.

<sup>2</sup>That is, it is common to make the following approximation

$$\frac{1}{\psi} \frac{\partial \psi}{\partial t} \ll \omega_0.$$

if  $\omega \ll \omega_0$ , then the first terms on the left-hand-sides are small, and Eq. (I.8) is the Schrodinger equation for a free particle and Eq. (I.9) is the non-relativistic dispersion relation that we have already derived in Eq. (6.15).

What does this low-frequency approximation really mean? From the non-relativistic dispersion relation

$$\omega = \frac{k^2 c^2}{2\omega_0} = \frac{\hbar}{2m} k^2 \quad (\text{I.10})$$

we can obtain the group velocity, Eq. (6.17)

$$v_g = \frac{\partial \omega}{\partial k} = \frac{\hbar k}{m}.$$

Comparing this speed to  $c$ , we have

$$\left(\frac{v_g}{c}\right)^2 = \frac{\hbar^2 k^2}{m^2 c^2} = \frac{2\hbar\omega}{mc^2} \ll \frac{2\hbar\omega_0}{mc^2} = 2. \quad (\text{I.11})$$

So, this implies that we have found a non-relativistic equation — the Schrodinger equation!

## Exchange particles

In addition to being the starting point for a derivation of the Schrodinger equation, the Klein-Gordon equation is also a way to understand the physics of the exchange particles, and is similar to the way that Yukawa originally derived the properties of the “meson,” the mediator of the strong nuclear force (see page 26). The  $\pi$ -meson is spin 0 and therefore is governed by the Klein-Gordon equation. Let’s assume we are in the rest frame of the pion and take the time-independent version ( $\partial/\partial t = 0$ ) which can be written

$$\nabla^2 \Psi - \left(\frac{m^2 c^2}{\hbar^2}\right) \Psi = 0. \quad (\text{I.12})$$

This is called the “screened Poisson equation.” If we make a further approximation and look for a spherically symmetric solution, i.e., no dependence on the angular coordinates  $\theta$  or  $\phi$ , we obtain the equation

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Psi}{\partial r} \right) - k_0^2 \Psi = 0, \quad (\text{I.13})$$

where  $k_0 = \omega_0/c$ . You can show that a solution to this equation is

$$\Psi \sim \frac{1}{r} e^{-k_0 r}, \quad (\text{I.14})$$

which is the so-called “Yukawa potential.” In contrast with the electric potential for a point charge, this potential is “screened” — i.e., it decays exponentially — with a screening distance equal to

$$d \sim \frac{1}{k_0} = \frac{\hbar}{mc} = \lambda_C. \quad (\text{I.15})$$

This screening distance is just the Compton wavelength of the exchange particle (see page 260). This means that, unlike a massless photon whose spatial range is infinite, the pion with nonzero mass can only make its effect felt out to a distance  $d$ . Since the  $\pi^0$  has a mass of 135 MeV, its effective distance is only about 1 fm. Because the strong nuclear force was known to have a range of only 1 fm, this correspondence led Yukawa to his suggestion that a “meson,” with a mass of about 200 MeV, must be responsible for the strong force. Hence, an exchange particle with a large mass represents a force with a short range ( $d \sim 1/m$ ), while those with zero mass (e.g., a photon) describe forces with an infinite range.

This result can be obtained in a more approximate way by using the energy-time uncertainly relation

$$\Delta E \Delta t \sim \hbar. \quad (\text{I.16})$$

As mentioned several times in Chapter 2, virtual exchange particles can only last for a time that is consistent with Eq. (I.16). This means that if they are traveling at near the speed of light, then that time is  $\Delta t \sim d/c$ , where  $d$  is the maximum distance they can travel. And since the uncertainty in their energy must be equal to their rest energy (since the particle is appearing out of “thin air”),  $\Delta E \sim mc^2$ . Inserting these expressions into Eq. (I.16) results in Eq. (I.15).

## Technical details

The reason why Eq. (I.14) appears to be a solution to Eq. (I.12) is because

$$\nabla^2 \left( \frac{1}{r} \right) = 0. \quad (\text{I.17})$$

However, this is *only* true when  $r \neq 0$ . When  $r = 0$ , you can show that the derivative diverges. The correct equation and solution are given by

$$\nabla^2 \Psi - k_0^2 \Psi = -G \delta(\vec{r}), \quad (\text{I.18})$$

and

$$\Psi = \frac{G}{4\pi r} e^{-k_0 r}, \quad (\text{I.19})$$

where  $\delta(\vec{r}) = \delta(x)\delta(y)\delta(z)$  is the three-dimensional Dirac delta function, and  $G$  is the “strength” of the charge at the origin. Note that this is an inhomogeneous linear equation and therefore the solution is known exactly, without any arbitrary multiplicative constant.

This turns out to be the same formalism that describes a point electric charge in a plasma, and  $k_0$  is the inverse of the Debye length. That is, the plasma particles “screen” out the potential of the point charge, so that beyond a distance  $1/k_0$  the charge’s effects are not felt.

# Appendix J

## Particle Discovery Timeline

**1874** electron coined by Stoney

**1876** cathode rays (Kathodenstrahlen) coined by Goldstein

**1897** electron discovered by Thomson

**1905** light quantum proposed by Einstein

**1911** nucleus discovered by Rutherford

**1913** isotope coined by Soddy

**1920** proton named

**1925** Rhenium [Re] discovered by Walter Noddack and Ida Tacke (later Noddack), by concentrating it from gadolinite. They claimed they discovered technetium [Tc] (they named it masurium) but this is controversial.

**1926** photon coined by Gilbert Lewis

**1928** positron predicted by Dirac (discovered 1933)

**1930** neutrino proposed by Pauli (observed 1956)

**1931** deuterium discovered by Harold Urey ( $^1\text{H}$  called 'protium')

**1932** neutron discovered by Chadwick

**1933** positron discovered by Anderson, and Blackett & Ochialini using a cloud chamber.

**1935** meson predicted by Yukawa

**1937** muon discovered by J. C. Street and E. C. Stevenson in a cloud chamber at Harvard. Also Carl Anderson and Seth Neddermeyer saw the muon.

**1937** Technetium [Tc] discovered by Emilio Segré [Nobel Prize, Physics, 1959] and Carlo Perrier.

- 1940**  $^{14}\text{C}$  discovered by Martin Kamen and Sam Ruben at the University of California, Berkeley, Radiation Laboratory.
- 1941** nucleon invented by Christian Møller
- 1946** lepton invented by Pais and Møller
- 1947** pions (all three) discovered by Lattes, Occhialini and Cecil Powell [Nobel Prize, Physics, 1950]
- 1947**  $\Lambda^0$  (uds) in cosmic rays; first “strange” baryon; long lifetime ( $10^{-8}$  s), hence weak force.
- 1949**  $\Delta$  resonances by Fermi and H.L. Anderson
- 1950**  $\pi^0$  discovered
- 1950s** Technetium ( $^{43}\text{Tc}$ ) identified in stellar spectra by Paul Merrill at Mt Wilson.
- 1954** baryon coined
- 1955** anti-proton discovered by Owen Chamberlain and Emilio Segré, for which they earned the Nobel Prize in Physics, 1959.
- 1956** anti-neutron discovered
- 1956** electron neutrino observed by Cowan and Reines
- 1961**  $\rho$ ,  $\omega$ ,  $\eta$ , and  $K^*$  discovered
- 1962** muon neutrino discovered by Lederman, Schwartz, and Steinberger at Columbia University
- 1962** hadron coined
- 1964** quarks discovered by Friedman, Kendall and Taylor (1990 Nobel)
- 1964**  $\Omega^-$  discovered at Brookhaven
- 1964** charm quark proposed
- 1968** quarks experimentally confirmed by Friedman, Kendall, and Taylor; a SLAC-MIT collaboration
- 1974** charm quark discovered by Richter and Ting at SLAC and Brookhaven (the  $J/\psi = c\bar{c}$  was discovered simultaneously in November: the “November revolution”)
- 1975** tau lepton discovered by Martin Perl at Stanford
- 1977** bottom quark discovered by Lederman at Fermilab ( $\Upsilon = b\bar{b}$ )
- 1983**  $W^\pm$  and  $Z^0$  by Carlo Rubbia and Simon van der Meer at CERN



**1995** top quark discovered at Fermilab, CDF and D0 collaborations

**2000** tau neutrino discovered at Fermilab



# CODATA RECOMMENDED VALUES OF THE FUNDAMENTAL PHYSICAL CONSTANTS: 2002

NIST SP 961 (Dec/2005) Values from: P. J. Mohr and B. N. Taylor, Rev. Mod. Phys. **77**, 1 (2005).

A more extensive listing of constants is available in the above references and on the NIST Physics Laboratory Web site [physics.nist.gov/constants](http://physics.nist.gov/constants).  
The number in parenthesis is the one-standard-deviation uncertainty in the last two digits of the given value.

Quantity	Symbol	Numerical value	Unit	Quantity	Symbol	Numerical value	Unit
speed of light in vacuum	$c$	299 792 458 (exact)	$\text{m s}^{-1}$	muon $g$ -factor $-2(1 + a_\mu)$	$g_\mu$	$-2.002\,331\,839\,6(12)$	
magnetic constant	$\mu_0$	$4\pi \times 10^{-7}$ (exact) $= 12.566\,370\,614\dots \times 10^{-7}$	$\text{N A}^{-2}$	muon-proton magnetic moment ratio	$\mu_\mu/\mu_p$	$-3.183\,345\,118(89)$	
electric constant $1/\mu_0 c^2$	$\epsilon_0$	$8.854\,187\,817\dots \times 10^{-12}$	$\text{F m}^{-1}$	proton mass	$m_p$	$1.672\,621\,71(29) \times 10^{-27}$	kg
Newtonian constant of gravitation	$G$	$6.6742(10) \times 10^{-11}$	$\text{m}^3 \text{kg}^{-1} \text{s}^{-2}$	in u		$1.007\,276\,466\,88(13)$	u
Planck constant	$h$	$6.626\,0693(11) \times 10^{-34}$	$\text{J s}$	energy equivalent in MeV	$m_p c^2$	$938.272\,029(80)$	MeV
in eV s	$h$	$4.135\,667\,43(35) \times 10^{-15}$	$\text{eV s}$	proton-electron mass ratio	$m_p/m_e$	$1836.152\,672\,61(85)$	
in eV s	$h$	$1.054\,571\,68(18) \times 10^{-34}$	$\text{eV s}$	proton magnetic moment	$\mu_p$	$1.410\,606\,71(12) \times 10^{-26}$	$\text{J T}^{-1}$
elementary charge	$e$	$1.602\,176\,53(14) \times 10^{-19}$	$\text{C}$	to nuclear magneton ratio	$\mu_p/\mu_N$	$2.792\,847\,351(28)$	
magnetic flux quantum $h/2e$	$\phi_0$	$2.067\,833\,72(18) \times 10^{-15}$	$\text{Wb}$	proton magnetic shielding correction $1 - \mu_p'/\mu_p \sigma_p'$	$\sigma_p'$	$25.689(15) \times 10^{-6}$	
Josephson constant $2e/h$	$K_J$	$483\,597.879(41) \times 10^9$	$\text{Hz V}^{-1}$	proton gyromagnetic ratio $2\mu_p/h$	$\gamma_p$	$2.675\,222\,05(23) \times 10^8$	$\text{s}^{-1} \text{T}^{-1}$
von Klitzing constant $h/e^2 = \mu_0 c/2\alpha$	$R_K$	$25\,812.807\,449(86)$	$\Omega$	( $\text{H}_2\text{O}$ , sphere, 25 °C)	$\gamma_p/2\pi$	$42.577\,4813(37)$	$\text{MHz T}^{-1}$
Bohr magneton $eh/2m_e$	$\mu_B$	$927.400\,949(80) \times 10^{-26}$	$\text{J T}^{-1}$	shielded proton gyromagnetic ratio $2\mu_p'/h$	$\gamma_p'$	$2.675\,153\,33(23) \times 10^8$	$\text{s}^{-1} \text{T}^{-1}$
in $\text{eV T}^{-1}$	$\mu_B$	$5.788\,381\,804(39) \times 10^{-5}$	$\text{eV T}^{-1}$	neutron mass in u		$1.008\,664\,915\,60(55)$	u
nuclear magneton $eh/2m_p$	$\mu_N$	$5.050\,783\,43(43) \times 10^{-27}$	$\text{J T}^{-1}$	energy equivalent in MeV	$m_n c^2$	$939.565\,360(81)$	MeV
in $\text{eV T}^{-1}$	$\mu_N$	$3.152\,451\,259(21) \times 10^{-8}$	$\text{eV T}^{-1}$	neutron-proton mass ratio	$m_n/m_p$	$1.001\,378\,418\,70(58)$	
fine-structure constant $e^2/4\pi\epsilon_0 hc$	$\alpha$	$7.297\,352\,568(24) \times 10^{-3}$	$\text{eV T}^{-1}$	neutron-proton magnetic moment	$\mu_n$	$-0.966\,236\,45(24) \times 10^{-26}$	$\text{J T}^{-1}$
inverse fine-structure constant	$\alpha^{-1}$	$137.035\,999\,11(46)$		to nuclear magneton ratio	$\mu_n/\mu_N$	$-1.913\,042\,73(45)$	
Rydberg constant $\alpha^2 m_e c/2h$	$R_\infty$	$10\,973\,731.568\,525(73)$	$\text{m}^{-1}$	deuteron mass in u	$m_d$	$2.013\,553\,212\,70(35)$	u
energy equivalent in eV	$R_\infty hc$	$3.289\,841\,960\,360(22) \times 10^{15}$	$\text{Hz}$	energy equivalent in MeV	$m_d c^2$	$1875.612\,82(16)$	MeV
Bohr radius $\alpha/4\pi R_\infty = 4\pi\epsilon_0 \hbar^2/m_e e^2$	$a_0$	$0.529\,177\,2108(18) \times 10^{-10}$	m	deuteron-proton mass ratio	$m_d/m_p$	$1.999\,007\,500\,82(41)$	
Hartree energy $e^2/4\pi\epsilon_0 a_0 = 2R_\infty hc = \alpha^2 m_e c^2$	$E_h$	$4.359\,744\,17(75) \times 10^{-18}$	$\text{J}$	deuteron magnetic moment	$\mu_d$	$0.433\,073\,482(38) \times 10^{-26}$	$\text{J T}^{-1}$
in eV	$E_h$	$27.211\,3845(23)$	$\text{eV}$	to nuclear magneton ratio	$\mu_d/\mu_N$	$0.857\,438\,2329(92)$	
electron mass	$m_e$	$9.109\,3826(16) \times 10^{-31}$	kg	helion ( $^3\text{He}$ nucleus) mass in u	$m_h$	$3.014\,932\,243(58)$	u
in u	$m_e$	$5.485\,799\,0945(24) \times 10^{-4}$	u	energy equivalent in MeV	$m_h c^2$	$2808.391\,42(24)$	MeV
energy equivalent in MeV	$m_e c^2$	$0.510\,998\,918(44)$	MeV	shielded helion magnetic moment (gas, sphere, 25 °C)	$\mu_h$	$-1.074\,553\,024(93) \times 10^{-26}$	$\text{J T}^{-1}$
electron-proton mass ratio	$m_e/m_p$	$4.836\,331\,67(13) \times 10^{-3}$		to Bohr magneton ratio	$\mu_h'/\mu_B$	$-1.158\,671\,474(14) \times 10^{-3}$	
electron-photon mass ratio	$m_e/m_\gamma$	$5.446\,170\,2173(25) \times 10^{-4}$	$\text{C kg}^{-1}$	to nuclear magneton ratio	$\mu_h'/\mu_N$	$-2.127\,497\,723(25)$	
electron charge to mass quotient	$-e/m_e$	$-1.758\,820\,12(15) \times 10^{11}$		alpha particle mass in u	$m_\alpha$	$4.001\,506\,179\,149(56)$	u
Compton wavelength $h/m_e c$	$\lambda_C$	$2.426\,310\,238(16) \times 10^{-12}$	m	energy equivalent in MeV	$m_\alpha c^2$	$3727.379\,17(32)$	MeV
$\lambda_C/2\pi = \alpha a_0 = \alpha^2/4\pi R_\infty$	$\lambda_C$	$386.159\,2678(26) \times 10^{-15}$	m	Avogadro constant	$N_A$	$6.022\,1415(10) \times 10^{23}$	$\text{mol}^{-1}$
classical electron radius $\alpha^2 a_0$	$r_e$	$2.817\,940\,325(28) \times 10^{-15}$	$\text{m}$	atomic mass constant $\frac{1}{12}m(^{12}\text{C}) = 1$ u	$m_u$	$1.660\,538\,86(28) \times 10^{-27}$	kg
Thomson cross section $(8\pi/3)r_e^2$	$\sigma_e$	$0.665\,245\,873(13) \times 10^{-28}$	$\text{m}^2$	energy equivalent in MeV	$m_u c^2$	$931.494\,043(80)$	MeV
electron magnetic moment	$\mu_e$	$-928.476\,412(80) \times 10^{-26}$	$\text{J T}^{-1}$	Faraday constant $N_A e$	$F$	$96\,485.3383(83)$	$\text{C mol}^{-1}$
to Bohr magneton ratio	$\mu_e/\mu_B$	$-1.001\,159\,652\,1859(38)$		molar gas constant	$R$	$8.314\,472(15)$	$\text{J K}^{-1}$
to nuclear magneton ratio	$\mu_e/\mu_N$	$-1838.281\,971\,07(85)$		Boltzmann constant $R/N_A$	$k$	$1.380\,6505(24) \times 10^{-23}$	$\text{J K}^{-1}$
electron magnetic moment anomaly $ \mu_e /\mu_B - 1$	$a_e$	$1.159\,652\,1859(38) \times 10^{-3}$		in $\text{eV K}^{-1}$	$k$	$8.617\,343(15) \times 10^{-5}$	$\text{eV K}^{-1}$
electron $g$ -factor $-2(1 + a_e)$	$g_e$	$-2.002\,319\,304\,3718(75)$		molar volume of ideal gas $RT/p$	$V_m$	$22.413\,996(39) \times 10^{-3}$	$\text{m}^3 \text{mol}^{-1}$
electron-proton magnetic moment ratio	$\mu_e/\mu_p$	$-658.210\,6862(66)$		Stefan-Boltzmann constant $\pi^2 k^4/60\pi^3 c^2$	$\sigma$	$5.670\,400(40) \times 10^{-8}$	$\text{W m}^{-2} \text{K}^{-4}$
energy equivalent in MeV	$m_\mu$	$0.113\,428\,9264(30)$	u	When displacement law constant $b = \lambda_{\text{max}} T = c_2/4.965\,114\,231\dots$	$c_1$	$3.741\,771\,38(64) \times 10^{-16}$	$\text{W m}^2$
muon-electron mass ratio	$m_\mu/m_e$	$105.658\,3692(94)$	MeV	second radiation constant $hc/k$	$c_2$	$1.438\,7752(25) \times 10^{-2}$	$\text{m K}$
muon magnetic moment	$\mu_\mu$	$206.768\,2838(54)$	$\text{J T}^{-1}$	When displacement law constant $b = \lambda_{\text{max}} T = c_2/4.965\,114\,231\dots$	$b$	$2.897\,7685(51) \times 10^{-3}$	$\text{m K}$
to Bohr magneton ratio	$\mu_\mu/\mu_B$	$-4.490\,447\,99(40) \times 10^{-26}$		Cu x unit: $\lambda(\text{Cu K}\alpha_1)/1.537\,400$	$\lambda(\text{Cu K}\alpha_1)$	$1.002\,077\,10(29) \times 10^{-13}$	m
to nuclear magneton ratio	$\mu_\mu/\mu_N$	$-8.890\,596\,98(23)$		Mo x unit: $\lambda(\text{Mo K}\alpha_1)/707.831$	$\lambda(\text{Mo K}\alpha_1)$	$1.002\,099\,66(53) \times 10^{-13}$	m
muon magnetic moment anomaly $ \mu_\mu /(eh/2m_\mu) - 1$	$a_\mu$	$1.165\,919\,81(62) \times 10^{-3}$					
Energy equivalents							
$(1 \text{ m}^{-1})c$	$= 299\,792\,458 \text{ Hz}$	$(1 \text{ Hz})/h/k = 4.799\,237\,4(84) \times 10^{-11} \text{ K}$	$(1 \text{ J}) = 6.241\,509\,47(53) \times 10^{18} \text{ eV}$	$(1 \text{ eV})/c^2 = 1.073\,544\,171(92) \times 10^{-9} \text{ u}$			
$(1 \text{ m}^{-1})hc/k$	$= 1.438\,7752(25) \times 10^{-2} \text{ K}$	$(1 \text{ Hz})/h = 4.135\,667\,43(35) \times 10^{-15} \text{ eV}$	$(1 \text{ eV}) = 1.602\,176\,53(14) \times 10^{-19} \text{ J}$	$(1 \text{ kg}) = 6.022\,1415(10) \times 10^{26} \text{ u}$			
$(1 \text{ m}^{-1})hc$	$= 1.239\,841\,91(11) \times 10^{-6} \text{ eV}$	$(1 \text{ K})k/hc = 69.503\,56(12) \text{ m}^{-1}$	$(1 \text{ eV})/hc = 8.065\,544\,45(69) \times 10^5 \text{ m}^{-1}$	$(1 \text{ u}) = 1.660\,538\,86(28) \times 10^{-27} \text{ kg}$			
$(1 \text{ m}^{-1})h/c$	$= 1.331\,025\,0506(89) \times 10^{-15} \text{ u}$	$(1 \text{ K})k/h = 2.083\,664(36) \times 10^{10} \text{ Hz}$	$(1 \text{ eV})/hc = 2.417\,989\,40(21) \times 10^{14} \text{ Hz}$	$(1 \text{ u})c/h = 7.513\,006\,608(50) \times 10^{14} \text{ m}^{-1}$			
$(1 \text{ Hz})/c$	$= 3.335\,640\,951\dots \times 10^{-9} \text{ m}^{-1}$	$(1 \text{ K})k = 8.617\,343(15) \times 10^{-5} \text{ eV}$	$(1 \text{ eV})/h = 1.160\,4505(20) \times 10^4 \text{ K}$	$(1 \text{ u})c^2 = 931.494\,043(80) \times 10^6 \text{ eV}$			



# Los Alamos National Laboratory Chemistry Division

## Periodic Table of the Elements

1A <b>1</b> H hydrogen 1.008	2A <b>4</b> Be beryllium 9.012											3A <b>5</b> B boron 10.81	4A <b>6</b> C carbon 12.01	5A <b>7</b> N nitrogen 14.01	6A <b>8</b> O oxygen 16.00	7A <b>9</b> F fluorine 19.00	8A <b>2</b> He helium 4.003						
<b>3</b> Li lithium 6.941	<b>12</b> Mg magnesium 24.31											<b>13</b> Al aluminum 26.98	<b>14</b> Si silicon 28.09	<b>15</b> P phosphorus 30.97	<b>16</b> S sulfur 32.07	<b>17</b> Cl chlorine 35.45	<b>18</b> Ar argon 39.95						
<b>11</b> Na sodium 22.99	<b>20</b> Ca calcium 40.08											<b>19</b> K potassium 39.10	<b>28</b> Ni nickel 58.69	<b>29</b> Cu copper 63.55	<b>30</b> Zn zinc 65.39	<b>31</b> Ga gallium 69.72	<b>32</b> Ge germanium 72.64	<b>33</b> As arsenic 74.92	<b>34</b> Se selenium 78.96	<b>35</b> Br bromine 79.90	<b>36</b> Kr krypton 83.79		
<b>19</b> K potassium 39.10	<b>38</b> Sr strontium 87.62											<b>47</b> Ag silver 107.9	<b>48</b> Cd cadmium 112.4	<b>49</b> In indium 114.8	<b>50</b> Sn tin 118.7	<b>51</b> Sb antimony 121.8	<b>52</b> Te tellurium 127.6	<b>53</b> I iodine 126.9	<b>54</b> Xe xenon 131.3				
<b>37</b> Rb rubidium 85.47	<b>39</b> Y yttrium 88.91											<b>45</b> Rh rhodium 102.9	<b>46</b> Pd palladium 106.4	<b>47</b> Ag silver 107.9	<b>48</b> Cd cadmium 112.4	<b>49</b> In indium 114.8	<b>50</b> Sn tin 118.7	<b>51</b> Sb antimony 121.8	<b>52</b> Te tellurium 127.6	<b>53</b> I iodine 126.9	<b>54</b> Xe xenon 131.3		
<b>55</b> Cs cesium 132.9	<b>56</b> Ba barium 137.3											<b>76</b> Os osmium 190.2	<b>77</b> Ir iridium 192.2	<b>78</b> Pt platinum 195.1	<b>79</b> Au gold 197.0	<b>80</b> Hg mercury 200.5	<b>81</b> Tl thallium 204.4	<b>82</b> Pb lead 207.2	<b>83</b> Bi bismuth 209.0	<b>84</b> Po polonium (209)	<b>85</b> At astatine (210)	<b>86</b> Rn radon (222)	
<b>87</b> Fr francium (223)	<b>88</b> Ra radium (226)											<b>108</b> Hs hassium (277)	<b>109</b> Mt meitnerium (288)	<b>110</b> Ds darmstadtium (271)	<b>111</b> Rg roentgenium (272)	<b>112</b> Cn copernicium (285)	<b>113</b> Uut ununtrium (286)	<b>114</b> Fl flerovium (289)	<b>115</b> Uup ununpentium (289)	<b>116</b> Lv livermorium (291)	<b>117</b> Uus ununseptium (294)	<b>118</b> Uuo ununoctium (294)	
<b>21</b> Sc scandium 44.96	<b>22</b> Ti titanium 47.88											<b>25</b> Mn manganese 54.94	<b>26</b> Fe iron 55.85	<b>27</b> Co cobalt 58.93	<b>28</b> Ni nickel 58.69	<b>29</b> Cu copper 63.55	<b>30</b> Zn zinc 65.39	<b>31</b> Ga gallium 69.72	<b>32</b> Ge germanium 72.64	<b>33</b> As arsenic 74.92	<b>34</b> Se selenium 78.96	<b>35</b> Br bromine 79.90	<b>36</b> Kr krypton 83.79
<b>39</b> Y yttrium 88.91	<b>40</b> Zr zirconium 91.22											<b>43</b> Tc technetium (98)	<b>44</b> Ru ruthenium 101.1	<b>45</b> Rh rhodium 102.9	<b>46</b> Pd palladium 106.4	<b>47</b> Ag silver 107.9	<b>48</b> Cd cadmium 112.4	<b>49</b> In indium 114.8	<b>50</b> Sn tin 118.7	<b>51</b> Sb antimony 121.8	<b>52</b> Te tellurium 127.6	<b>53</b> I iodine 126.9	<b>54</b> Xe xenon 131.3
<b>72</b> Hf hafnium 178.5	<b>73</b> Ta tantalum 180.9											<b>75</b> Re rhenium 186.2	<b>76</b> Os osmium 190.2	<b>77</b> Ir iridium 192.2	<b>78</b> Pt platinum 195.1	<b>79</b> Au gold 197.0	<b>80</b> Hg mercury 200.5	<b>81</b> Tl thallium 204.4	<b>82</b> Pb lead 207.2	<b>83</b> Bi bismuth 209.0	<b>84</b> Po polonium (209)	<b>85</b> At astatine (210)	<b>86</b> Rn radon (222)
<b>104</b> Rf rutherfordium (261)	<b>105</b> Db dubnium (262)											<b>107</b> Bh bohrium (264)	<b>108</b> Hs hassium (277)	<b>109</b> Mt meitnerium (288)	<b>110</b> Ds darmstadtium (271)	<b>111</b> Rg roentgenium (272)	<b>112</b> Cn copernicium (285)	<b>113</b> Uut ununtrium (286)	<b>114</b> Fl flerovium (289)	<b>115</b> Uup ununpentium (289)	<b>116</b> Lv livermorium (291)	<b>117</b> Uus ununseptium (294)	<b>118</b> Uuo ununoctium (294)
<b>106</b> Sg seaborgium (266)	<b>107</b> Bh bohrium (264)											<b>109</b> Mt meitnerium (288)	<b>110</b> Ds darmstadtium (271)	<b>111</b> Rg roentgenium (272)	<b>112</b> Cn copernicium (285)	<b>113</b> Uut ununtrium (286)	<b>114</b> Fl flerovium (289)	<b>115</b> Uup ununpentium (289)	<b>116</b> Lv livermorium (291)	<b>117</b> Uus ununseptium (294)	<b>118</b> Uuo ununoctium (294)		

### Lanthanide Series\*

<b>57</b> La lanthanum 138.9	<b>58</b> Ce cerium 140.1	<b>59</b> Pr praseodymium 140.9	<b>60</b> Nd neodymium 144.2	<b>61</b> Pm promethium (145)	<b>62</b> Sm samarium 150.4	<b>63</b> Eu europium 152.0	<b>64</b> Gd gadolinium 157.2	<b>65</b> Tb terbium 158.9	<b>66</b> Dy dysprosium 162.5	<b>67</b> Ho holmium 164.9	<b>68</b> Er erbium 167.3	<b>69</b> Tm thulium 168.9	<b>70</b> Yb ytterbium 173.0	<b>71</b> Lu lutetium 175.0
<b>89</b> Ac actinium (227)	<b>90</b> Th thorium 232.0	<b>91</b> Pa protactinium 231	<b>92</b> U uranium 238	<b>93</b> Np neptunium (237)	<b>94</b> Pu plutonium (244)	<b>95</b> Am americium (243)	<b>96</b> Cm curium (247)	<b>97</b> Bk berkelium (247)	<b>98</b> Cf californium (251)	<b>99</b> Es einsteinium (252)	<b>100</b> Fm fermium (257)	<b>101</b> Md mendelevium (258)	<b>102</b> No nobelium (259)	<b>103</b> Lr lawrencium (262)

### Actinide Series\*\*