Philosophic Foundations of Quantum Mechanics

HANS REICHENBACH

DOVER PUBLICATIONS, INC.
Mineola, New York

Part I

GENERAL CONSIDERATIONS

§ 1. Causal Laws and Probability Laws

The philosophical problems of quantum mechanics are centered around two main issues. The first concerns the transition from causal laws to probability laws; the second concerns the interpretation of unobserved objects. We begin with the discussion of the first issue, and shall enter into the analysis of the second in later sections.

The question of replacing causal laws by statistical laws made its appearance in the history of physics long before the times of the theory of quanta. Since the time of Boltzmann's great discovery which revealed the second principle of thermodynamics to be a statistical instead of a causal law, the opinion has been repeatedly uttered that a similar fate may meet all other physical laws. The idea of determinism, i.e., of strict causal laws governing the elementary phenomena of nature, was recognized as an extrapolation inferred from the causal regularities of the macrocosm. The validity of this extrapolation was questioned as soon as it turned out that macrocosmic regularity is equally compatible with irregularity in the microcosmic domain, since the law of great numbers will transform the probability character of the elementary phenomena into the practical certainty of statistical laws. Observations in the macrocosmic domain will never furnish any evidence for causality of atomic occurrences so long as only effects of great numbers of atomic particles are considered. This was the result of unprejudiced philosophical analysis of the physics of Boltzmann.[1]

With this result a decision of the question was postponed until it was possible to observe macrocosmic effects of individual atomic phenomena. Even with the use of observations of this kind, however, the question is not easily answered, but requires the development of a more profound logical analysis.

Whenever we speak of strictly causal laws we assume them to hold between idealized physical states; and we know that actual physical states never cor-

[1] It is scarcely possible to say who was the first to formulate this philosophical idea. We have no published utterances of Boltzmann indicating that he thought of the possibility of abandoning the principle of causality. In the decade preceding the formulation of quantum mechanics the idea was often discussed. F. Exner, in his book, *Vorlesungen über die physikalischen Grundlagen der Naturwissenschaften* (Vienna, 1919), is perhaps the first to have clearly stated the criticism which we gave above: "Let us not forget that the principle of causality and the need for causality has been suggested to us exclusively by experiences with macrocosmic phenomena and that a transference of the principle to microcosmic phenomena, viz. the assumption that every individual occurrence be strictly causally determined, has no longer any justification based on experience."—p. 691. With reference to Exner, E. Schrödinger has expressed similar ideas in his inaugural address in Zurich, 1922, published in *Naturwissenschaften*, 17:9 (1929).

respond exactly to the conditions assumed for the laws. This discrepancy has often been disregarded as irrelevant, as being due to the imperfection of the experimenter and therefore negligible in a statement about causality as a property of nature. With such an attitude, however, the way to a solution of the problem of causality is barred. Statements about the physical world have meaning only so far as they are connected with verifiable results; and a statement about strict causality must be translatable into statements about observable relations if it is to have a utilizable meaning. Following this principle we can interpret the statement of causality in the following way.

If we characterize physical states in observational terms, i.e., in terms of observations as they are actually made, we know that we can construct probability relations between these states. For instance, if we know the inclination of the barrel of a gun, the powder charge, and the weight of the shell, we can predict the point of impact with a certain probability. Let A be the so-defined initial conditions and B a description of the point of impact; then we have a probability implication

$$A \underset{p}{\rightarrow} B \qquad (1)$$

which states that if A is given, B will happen with a determinate probability p. From this empirically verifiable relation we pass to an ideal relation by considering ideal states A' and B' and stating a logical implication

$$A' \supset B' \qquad (2)$$

between them, which represents a law of mechanics. Since we know, however, that from the observational state A we can infer only with some probability the existence of the ideal state A', and that similarly we have only a probability relation between B and B', the logical implication (2) cannot be utilized. It derives its physical meaning only from the fact that in all cases of applications to observable phenomena it can be replaced by the probability implication (1). What then is the meaning of a statement saying that if we knew exactly the initial conditions we could predict with certainty the future states resulting from them? Such a statement can be meaningfully said only in the sense of a transition to a limit. Instead of characterizing the initial conditions of shooting only by the mentioned three parameters, the inclination of the barrel, the powder charge, and the weight of the shell, we can consider further parameters, such as the resistance of the air, the rotation of the earth, etc. As a consequence, the predicted value will change; but we know that with such a more precise characterization also the probability of the prediction increases. From experiences of this kind we have inferred that the probability p can be made to approach the value 1 as closely as we want by the introduction of further parameters into the analysis of physical states. It is in this form that we must state the principle of causality if it is to have physical meaning. The statement that nature is governed by strict causal laws means that we can predict the future with a determinate probability and that we can push this probability as

close to certainty as we want by using a sufficiently elaborate analysis of the phenomena under consideration.

With this formulation the principle of causality is stripped of its disguise as a principle *a priori*, in which it has been presented within many a philosophical system. If causality is stated as a limit of probability implications, it is clear that this principle can be maintained only in the sense of an empirical hypothesis. There is, logically, no need for saying that the probability of predictions can be made to approach certainty by the introduction of more and more parameters. In this form the possibility of a limit of predictability was recognized even before quantum mechanics led to the assertion of such a limit.[2]

The objection has been raised that we can know only a finite number of parameters, and that therefore we must leave open the possibility of discovering, at a later time, new parameters which lead to better predictions. Although, of course, we have no means of excluding with certainty such a possibility, we must answer that there may be strong inductive evidence against such an assumption, and that such evidence will be regarded as given if continued attempts at finding new parameters have failed. Physical laws, like the law of conservation of energy, have been based on evidence derived from repeated failures of attempts to prove the contrary. If the existence of causal laws is denied, this assertion will always be grounded only in inductive evidence. The critics of the belief in causality will not commit the mistake of their adversaries, and will not try to adduce a supposed evidence *a priori* for their contentions.

The quantum mechanical criticism of causality must therefore be considered as the logical continuation of a line of development which began with the introduction of statistical laws into physics within the kinetic theory of gases, and was continued in the empiricist analysis of the concept of causality. The specific form, however, in which this criticism finally was presented through Heisenberg's principle of indeterminacy was different from the form of the criticism so far explained.

In the preceding analysis we have assumed that it is possible to measure the independent parameters of physical occurrences as exactly as we wish; or more precisely, to measure the *simultaneous values* of these parameters as exactly as we wish. The breakdown of causality then consists in the fact that these values do not strictly determine the values of dependent entities, including the values of the same parameters at later times. Our analysis therefore contains an assumption of the measurement of simultaneous values of independent parameters. It is this assumption which Heisenberg has shown to be wrong.

The laws of classical physics are throughout *temporally directed laws*, i.e., laws stating dependences of entities at different times and which thus establish causal lines extending in the direction of time. If simultaneous values of differ-

[2] Cf. the author's "Die Kausalstruktur der Welt," *Ber. d. Bayer. Akad., Math. Kl.* (Munich, 1925), p. 138; and his paper, "Die Kausalbehauptung und die Möglichkeit ihrer empirischen Nachprüfung," which was written in 1923 and published in *Erkenntnis* 3 (1932), p. 32.

ent entities are regarded as dependent on one another, this dependence is always construed as derivable from temporally directed laws. Thus the correspondence of various indicators of a physical state is reduced to the influence of the same physical cause acting on the instruments. If, for instance, barometers in different rooms of a house always show the same indication, we explain this correspondence as due to the effect of the same mass of air on the instruments, i.e., as due to the effect of a common cause. It is possible, however, to assume the existence of *cross-section laws,* i.e., laws which directly connect simultaneous values of physical entities without being reducible to the effects of common causes. It is such a cross-section law which Heisenberg has stated in his relation of indeterminacy.

This cross-section law has the form of a *limitation of measurability.* It states that the simultaneous values of the independent parameters cannot be measured as exactly as we wish. We can measure only one half of all the parameters to a desired degree of exactness; the other half then must remain inexactly known. There exists a coupling of simultaneously measurable values such that greater exactness in the determination of one half of the totality involves less exactness in the determination of the other half, and vice versa. This law does not make half of the parameters functions of the others; if one half is known, the other half remains entirely unknown unless it is measured. We know, however, that this measurement is restricted to a certain exactness.

This cross-section law leads to a specific version of the criticism of causality. If the values of the independent parameters are inexactly known, we cannot expect to be able to make strict predictions of future observations. We then can establish only statistical laws for these observations. The idea that there are causal laws "behind" these statistical laws, which determine exactly the results of future observations, is then destined to remain an unverifiable statement; its verification is excluded by a physical law, the cross-section law mentioned. According to the verifiability theory of meaning, which has been generally accepted for the interpretation of physics, the statement that there are causal laws therefore must be considered as physically meaningless. It is an empty assertion which cannot be converted into relations between observational data.

There is only one way left in which a physically meaningful statement about causality can be made. If statements of causal relations between the exact values of certain entities cannot be verified, we can try to introduce them at least in the form of *conventions* or *definitions;* that is, we may try to establish arbitrarily causal relations between the strict values. This means that we can attempt to assign definite values to the unmeasured, or not exactly measured, entities in such a way that the observed results appear as the causal consequences of the values introduced by our assumption. If this were possible, the causal relations introduced could not be used for an improvement of predictions; they could be used only after observations had been made in the sense of a causal construction *post hoc.* Even if we wish to follow such a procedure, however, we must answer the question of whether such a *causal supplementation of observable data by interpolation of unobserved values* can be consistently done. Although the interpolation is based on conventions, the answer to the latter question is not a matter of convention, but depends on the structure of the physical world. Heisenberg's principle of indeterminacy, therefore, leads to a revision of the statement of causality; if this statement is to be physically meaningful, it must be made as an assertion about a possible causal supplementation of the observational world.

With these considerations the plan of the following inquiry is made clear. We shall first explain Heisenberg's principle, showing its nature as a cross-section law, and discuss the reasons why it must be regarded as being well founded on empirical evidence. We then shall turn to the question of the interpolation of unobserved values by definitions. We shall show that the question stated above is to be answered negatively; that the relations of quantum mechanics are so constructed that they do not admit of a causal supplementation by interpolation. With these results the principle of causality is shown to be in no sense compatible with quantum physics; causal determinism holds neither in the form of a verifiable statement, nor in the form of a convention directing a possible interpolation of unobserved values between verifiable data.

✓ § 2. The Probability Distributions

Let us analyze more closely the structure of causal laws by means of an example taken from classical mechanics and then turn to the modification of this structure produced by the introduction of probability considerations.

In classical physics the physical state of a free mass particle which has no rotation, or whose rotation can be neglected, is determined if we know the *position* q, the *velocity* v, and the *mass* m of the particle. The values q and v, of course, must be corresponding values, i.e., they must be observed at the same time. Instead of the velocity v, the *momentum* $p = m \cdot v$ can be used. The future states of the mass particle, if it is not submitted to any forces, is then determined; the velocity, and with it, the momentum, will remain constant, and the position q can be calculated for every time t. If external forces intervene, we can also determine the future states of the particle if these forces are mathematically known.

If we consider the fact that p and q cannot be exactly determined, we must replace strict statements about p and q by probability statements. We then introduce *probability distributions*

$$d(q) \quad \text{and} \quad d(p) \tag{1}$$

which coordinate to every value q and to every value p a probability that this value will occur. The symbol $d(\ )$ is used here in the general meaning of *distri-*

bution of; the expressions $d(q)$ and $d(p)$ denote, therefore, different mathematical functions. As usual, the probability given by the function is coordinated, not to a sharp value q or p, but to a small interval dq or dp such that only the expressions

$$d(q)dq \quad \text{and} \quad d(p)dp \tag{2}$$

represent probabilities, whereas the functions (1) are probability *densities.* This can also be stated in the form that the integrals

$$\int_{q_1}^{q_2} d(q)dq \quad \text{and} \quad \int_{p_1}^{p_2} d(p)dp \tag{3}$$

represent the probabilities of finding a value of q between q_1 and q_2, or a value of p between p_1 and p_2.

Probability distributions can be determined only for sets of measurements, not for an individual measurement. When we speak of the exactness of a measurement we therefore mean, more precisely, the exactness of a type of measurement made in a certain type of physical system. In this sense we can say that every measurement ends with the determination of probability functions d. Usually d is a Gauss function, i.e., a bell-shaped curve following an exponential law (cf. figure 1); the steeper this curve, the more precise is the measurement. In classical physics we make the assumption that each of these curves can be made as steep as we want, if only we take sufficient care in the elaboration of the measurement. In quantum mechanics this assumption is discarded for the following reasons.

Whereas, in classical physics, we consider the two curves $d(q)$ and $d(p)$ as independent of each other, quantum mechanics introduces the rule that they are not. This is the cross-section law mentioned in § 1. The idea is expressed through a mathematical principle which determines both curves $d(q)$ and $d(p)$, at a given time t, as derivable from a mathematical function $\psi(q)$; the derivation is so given that a certain logical connection between the shapes of the curves $d(q)$ and $d(p)$ follows. This contraction of the two probability distributions into one function ψ is one of the basic principles of quantum mechanics. It turns out that the connection between the distributions established by the principle has such a structure that if one of the curves is very steep, the other must be rather flat. Physically speaking, this means that measurements of p and q cannot be made independently and that an arrangement which permits a precise determination of q must make any determination of p unprecise, and vice versa.

The function $\psi(q)$ has the character of a wave; it is even a complex wave, i.e., a wave determined by complex numbers ψ. Historically speaking, the introduction of this wave by L. de Broglie and Schrödinger goes back to the struggle between the wave interpretation and the corpuscle interpretation in the theory of light. The ψ-function is the last offspring of generations of wave concepts stemming from Huygens's wave theory of light; but Huygens would

scarcely recognize his ideas in the form which they have assumed today in Born's probability interpretation of the ψ-function. Let us put aside for the present the discussion of the physical nature of this wave; we shall be concerned with this important question in later sections of our inquiry. In the present section we shall consider the ψ-waves merely as a mathematical instrument used to determine probability distributions; i.e., we shall restrict our presentation to show the way in which the probability distributions $d(q)$ and $d(p)$ can be derived from $\psi(q)$.

The derivation which we are going to explain coordinates to a curve $\psi(q)$ at a given time the curves $d(q)$ and $d(p)$; this is the reason that t does not enter into the following equations. If, at a later time, $\psi(q)$ should have a different shape, different functions $d(q)$ and $d(p)$ would ensue. Thus, in general, we have functions $\psi(q,t)$, $d(q,t)$, and $d(p,t)$. We omit the t for the sake of convenience.

The derivation will be formulated in two rules, the first determining $d(q)$, and the second determining $d(p)$. We shall state these rules here only for the simple case of free particles. The extension to more complicated mechanical systems will be given later (§ 17). We present first the rule for the determination of $d(q)$.

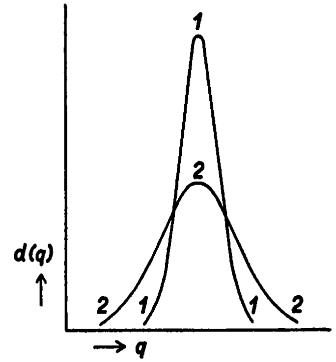

Fig. 1. Curve 1-1-1 represents a precise measurement, curve 2-2-2 a less precise measurement of q. Both curves are Gauss distributions, or *normal curves.*

Rule of the squared ψ-function: The probability of observing a value q is determined by the square of the ψ-function according to the relation

$$d(q) = |\psi(q)|^2 \tag{4}$$

The explanation of the rule for the determination of $d(p)$ requires some introductory mathematical remarks. According to Fourier a wave of any shape can be considered as the superposition of many individual waves having the form of sine curves. This is well known from sound waves, where the individual waves are called *fundamental tone* and *overtones,* or *harmonics.* In optics the individual waves are called *monochromatic waves,* and their totality is called the *spectrum.* The individual wave is characterized by its frequency ν, or its wave length λ, these two characteristics being connected by the relation $\nu \cdot \lambda = w$, where w is the velocity of the waves. In addition, every individual wave has an amplitude σ which does not depend on q, but is a constant for the whole individual wave. The general mathematical form of the Fourier expansion is explained in § 9; for the purposes of the present part it is not necessary to introduce the mathematical way of writing.

The Fourier superposition can be applied to the wave ψ, although this wave is considered by us, at present, not as a physical entity, but merely as a mathematical instrument. In case the wave ψ consists of periodic oscillations extended over a certain time, such as in the case of sound waves produced by musical instruments, the spectrum furnished by the Fourier expansion is *discrete*. Thus the individual waves of musical instruments have the wave lengths $\lambda, \frac{\lambda}{2}, \frac{\lambda}{3}, \frac{\lambda}{4}, \cdots$ where λ is the wave length of the fundamental tone and the other values represent the harmonics. In case the wave ψ consists of only one simple impact moving along the q-axis, i.e., in case the function ψ is not periodic, the Fourier expansion furnishes a continuous spectrum, i.e., the frequencies of the individual waves constitute, not a discrete, but a continuous set. As before, each of these individual waves possesses an amplitude σ, which can be written $\sigma(\lambda)$, since it depends on the wave length λ but is independent of q.

It is the amplitudes $\sigma(\lambda)$ which are connected with the momentum. We shall not try to explain here the trend of thought which led to this connection and which is associated with the names of Planck, Einstein, and L. de Broglie. Such an exposition may be postponed to a later section (§ 13). Let us suppress therefore any question of *why* this connection holds true, and let us rely, instead, upon the authority of the physicist who says that this is the case. Suffice it to say, therefore, that every wave of the length λ is coordinated to a momentum of the amount

$$p = \frac{h}{\lambda} \tag{5}$$

where h is Planck's constant. The probability of finding a momentum p then is connected with the amplitude σ belonging to the coordinated wave λ. This is expressed in the following rule.[1]

Rule of spectral decomposition: The probability of observing a value p is determined by the square of the amplitude $\sigma(\lambda)$ occurring within the spectral decomposition of $\psi(q)$, in the form

$$d(p) = \frac{1}{h^3} |\sigma(\lambda)|^2 \tag{6}$$

The factor $\frac{1}{h^3}$ results from the relation between p and λ expressed in (5).[2]

The two rules show clearly the connection which the ψ-function establishes between the two distributions $d(q)$ and $d(p)$, so far as it reduces these two distributions to one root. We shall later show that this kind of connection is not

[1] The name "principle of spectral decomposition" has been introduced by L. de Broglie, *Introduction à l'Etude de la Mécanique ondulatoire* (Paris, 1930), p. 151. In his later book, *La Mécanique ondulatoire* (Paris, 1939), p. 47, he uses also the name "principle of Born," since this principle was introduced by Born. For the rule of the squared ψ-function he uses the name "principle of interference" and in his later book the name "principle of localization".

[2] Mathematically speaking, this factor corresponds to a density function r as introduced in (22), § 9. The third power in h originates from the fact that we assume the waves to be three-dimensional.

restricted to the simple case of one mass particle, and that the same logical pattern is established by quantum mechanics for the analysis of all physical situations. For every physical situation there exists a ψ-function, and the probability distributions of the entities involved are determined by two rules of the kind described. This is one of the basic principles of quantum mechanics. We shall now construct the implications of this principle, returning once more to the simple case of the mass particle.

√ § 3. The Principle of Indeterminacy

It can be shown that the derivation of the two distributions $d(q)$ and $d(p)$ from a function ψ leads immediately to the principle of indeterminacy. Let us consider a particle moving in a straight line, and let us assume that the func-

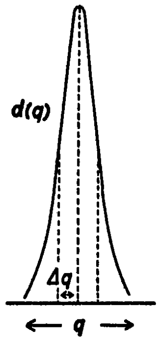

Fig. 2

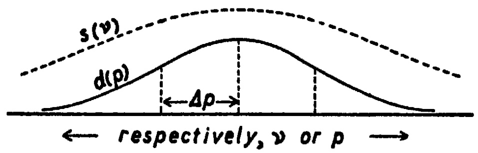

Fig. 3

Fig. 2. Distribution of the position q, in the form of a Gauss curve.

Fig. 3. The dotted line indicates the direct Fourier expansion of the curve of fig. 2. The solid line is constructed through the Fourier expansion of a ψ-function from which the curve $d(q)$ of fig. 2 is derivable, and represents the distribution $d(p)$ of the momentum, coordinated to $d(q)$.

tion ψ is practically equal to zero except for a certain interval along the line. The function $|\psi(q)|^2$, i.e., the function $d(q)$, then will have the same property; let us assume that it is a Gauss curve such as is shown in figure 2. The shape of the curve means that we do not know the location of the particle exactly; with practical certainty it is within the interval where the curve is noticeably different from zero, but for a given place within this interval we know only with a determinate probability that the particle is there. Our diagram, of course, represents the situation only for a given time t; for a later time, when the particle has moved to the right, we shall have a similar curve, but it will be shifted to the right.[1]

[1] The curve will also gradually change its form. This, however, is irrelevant for the present discussion.

Now let us apply the principle of spectral decomposition. This decomposition, it is true, is to be applied to the complex function $\psi(q)$, and not to the real function $|\psi(q)|^2$ of our diagram. For the study of the mathematical relations in the decomposition, however, we shall first apply it to the real function of the diagram; the results can then be transferred to the complex case.

The Fourier expansion constructs the curve of figure 2 out of an infinite set of individual waves. Each of these individual waves is a pure harmonic wave of infinite length; i.e., its oscillations have a sine form and extend along the whole infinite line. Their amplitudes differ, however. The maximal amplitude will be associated with a certain mean frequency ν_0; for frequencies greater and smaller than ν_0 the amplitude will be smaller, and outside a certain range on each side of ν_0 the amplitude of the individual waves will be practically zero. Let us call the range within which the amplitudes have a noticeable size, the practical range. A bunch of harmonic waves of the kind described is also called a *wave packet*, since the superposition of all these harmonic waves results in one packet of the form given in figure 2.

Now it is one of the theorems of Fourier analysis that the practical range of a packet of harmonic waves is great when the curve of figure 2 is rather steep, but is small when this curve is flat. We can illustrate this by a diagram when we draw as abscissae the frequencies ν, and as ordinates the corresponding amplitudes $s(\nu)$ of the harmonic analysis, such as is done for the dotted line in figure 3. We choose the notation $s(\nu)$ for these amplitudes of the Fourier expansion of the real function $d(q)$, or $|\psi(q)|^2$, in order to distinguish them from the amplitudes $\sigma(\nu)$ of the Fourier expansion of the complex function $\psi(q)$. In our case, since we had assumed the curve $d(q)$ to be a Gauss curve, the curve $s(\nu)$ is also a Gauss curve, but of a much flatter shape,[2] as shown by the dotted-line curve of figure 3. This illustrates the theorem mentioned; it can be proved generally that the steeper the curve of figure 2, the flatter will be the dotted-line curve of figure 3, and vice versa. We therefore have an inverse correlation between the shape of the original curve and the shape of the curve expressing its harmonic analysis. We call this the law of *inverse correlation of harmonic analysis*.

An instructive illustration of this law is found in some problems of radio transmission. If the wave of a radio transmitter does not carry any sound, it is a pure sine wave of a sharply defined frequency. If it is *modulated*, however, i.e., if its amplitude varies according to the intensity of impressed sound waves, it no longer represents one sharp frequency, but a spectrum of frequencies varying continuously within a certain range. This range is given by the highest pitch of the sound frequencies. Consequently, a receiver with a sharp resonance system will pick out only a narrow domain of the transmitted waves; it will therefore drop the higher sound frequencies and reproduce transmitted music in a distorted form. On the other hand, if the resonance curve of the receiver is sufficiently flat for a high fidelity reproduction of music, the receiver will not sufficiently separate two radio stations transmitting adjacent wave lengths. The principle of inverse correlation is here expressed in the fact that it is impossible to unite high fidelity and high selectivity in the same adjustment of the receiver.

[2] The maximum of the $s(\nu)$ is at the place $\nu = 0$, and the curve is symmetrical for positive and negative frequencies ν. This results from the fact that we have assumed the curve $d(q)$ in the form of a Gauss curve. For the curve $d(p)$ there exists no such restriction, because the function $\psi(q)$ is not determined by $d(q)$, but left widely arbitrary; the maximum of $d(p)$ may therefore be situated at any value $\nu = \nu_0$, or correspondingly, at any value p.

The application of these considerations to the determination of the probability distribution of the momentum of the particle involves some complications, which, however, in principle, do not change the result. As explained above, the spectral decomposition which furnishes the momenta must be applied, not to the probability curve $d(q)$ of figure 2, but to a complex ψ-curve from which this curve is derivable by the rule of the squared ψ-function. The complex amplitudes $\sigma(\nu)$ of the resulting harmonic waves then must be squared according to the rule of spectral decomposition. It is only by means of this detour through the complex domain that we arrive at the probability distribution $d(p)$ of the momentum as shown in the solid line of figure 3. Like the dotted line, this curve is also a rather flat Gauss distribution, although not quite so flat. But it can be shown that the law of inverse correlation holds as well for the two curves $d(q)$ and $d(p)$. We therefore shall speak of the *law of inverse correlation of the probability distributions of momentum and position.*

More precisely, this inverse correlation is to be understood as follows. If only the curve $d(q)$ is given, the curve $d(p)$ is not determined; it can have various forms depending on the shape of the function $\psi(q)$ from which $d(q)$ is derived. But there is a limit to the steepness of $d(p)$, represented by the solid line of figure 3. Should $d(q)$ be derived from another ψ-function than that assumed for the diagram, the resulting curve $d(p)$ can only be flatter. This general theorem can be stated without a further discussion of the question concerning the practical determination of the function $\psi(q)$ in a given physical situation. The answer to the latter question, which requires the mathematical apparatus of quantum mechanics, must be postponed to a later section (§ 20).

We said that the results obtained for the mass particle can be extended to all physical situations. Now the difference between q and p is transferred to situations in general as a difference between *kinematic* and *dynamic* parameters. We therefore shall speak generally of the *law of inverse correlation of kinematic and dynamic parameters*. It is in the form of this law of inverse correlation that we express the principle of indeterminacy. The universality of this principle follows from the fact that, whatever be the physical situation, our observational knowledge of it is summarized in a ψ-function.

The law of inverse correlation can be extended to the two parameters *time* and *energy*. This extension can be made clear as follows. A measurement of time is analogous to a determination of position. When we speak of the position

q of a particle we mean the position at a given time t; inversely we can ask for the time t at which the particle will be at a given space point q. This value t will be determinable only with a certain probability, and we can therefore introduce a probability distribution $d(t)$ analogous to $d(q)$. Similarly we can introduce a probability function $d(H)$ stating the probability that the particle will have a certain energy H. H is connected with the frequency of the harmonics by the Planck relation

$$H = h \cdot \nu \tag{1}$$

corresponding to (5), § 2, and the probability $d(H)$ is therefore determined by the principle of spectral decomposition. Hence the two curves $d(t)$ and $d(H)$ are subject to the principle of inverse correlation in the same way as the curves $d(q)$ and $d(p)$. We shall therefore include time in the category of kinematic parameters, and energy in that of the dynamic parameters. The general law of inverse correlation of kinematic and dynamic parameters then includes the inverse correlation of time and energy.

This general law can be formulated somewhat differently when we use the concept of *standard deviation*. Let q_0 represent the mean value of q, that is, the value of the abscissa for which the curve $d(q)$ reaches its maximum; and let Δq be the standard deviation. Then the area between the curve and the axis of abscissae is divided by the ordinates $q_0 - \Delta q$ and $q_0 + \Delta q$ (drawn in figure 2) in such a way that approximately $\frac{2}{3}$ of the whole area is situated between these two ordinates. It is shown in the theory of probability that this ratio is independent of the shape of the Gauss curve. The probability of finding a value q within the interval $q_0 \pm \Delta q$ is therefore approximately $= \frac{2}{3}$. Because of these properties, the quantity Δq represents a measure of the steepness of the Gauss curve, and is therefore used as an expression characterizing the exactness of the distribution of measurements. If the standard deviation is small, the measurements are exact; if it is great, the measurements are inexact. In figure 2 and figure 3 the standard deviations Δq and Δp belonging, respectively, to the curves $d(q)$ and $d(p)$ are indicated on the axis. Now it can be shown that for the case of curves of this kind, which are derived from the same ψ-function, the relation

$$\Delta q \cdot \Delta p \geqq \frac{h}{4\pi} \tag{2}$$

holds where h is the Planck constant. For time and energy we have the corresponding relation:

$$\Delta t \cdot \Delta H \geqq \frac{h}{4\pi} \tag{3}$$

The inequalities (2) and (3) represent the form in which the relation of indeterminacy has been established by Heisenberg. (2) expresses the inverse correlation of measurements of position and momentum by stating that a small standard deviation in q implies a great standard deviation in p, and vice versa. (3) states the corresponding relation between Δt and ΔH. These relations show at the same time the significance of the constant h. Since h has a very small value, the indeterminacy will be visible only in observations within the microcosmic domain; there, however, the indeterminacy cannot be neglected. The case of classical physics corresponds to the assumption that $h = 0$.

Relation (2) can be interpreted in the form: When the position of a particle is well determined, the momentum is not sharply determined, and vice versa. (3) can be interpreted in a similar way. This form makes it clear that the cross-section law of inverse correlation between kinematic and dynamic parameters states a limitation of measurability.

We now are in a position to answer the question raised above about the legitimacy of this cross-section law. If the basic principles of quantum mechanics are correct, the principle of indeterminacy must hold because it is a logical consequence of these basic principles. Furthermore, it must hold for all physical situations, because it is derivable directly from the rules of the squared ψ-function and of spectral decomposition, without reference to any special form of the ψ-function. The issue of legitimacy is reduced with this to the validity of the basic principles of quantum mechanics. Now these principles are, of course, empirical principles, and no physicist claims absolute truth for them. But what can be claimed for them is the truth of a well-established theory. Since it is a consequence of the limitation of measurability that all relations between observational data are restricted to statistical relations, we can therefore say: *With the same right with which the physicist maintains any one of his fundamental theorems, he is entitled to assert a limitation of predictability.* We may add that the same limitation follows for the determination of past data in terms of given observations, and that we therefore must also speak of a *limitation of postdictability.*

It has sometimes been said that quantum mechanics possesses a mathematical proof of the limitation of predictability. Such a statement can reasonably be meant only in the sense that there is a mathematical proof deriving the statement of the limitation from the basic principles of quantum mechanics. The principle of indeterminacy is an empirical statement; all that can be said mathematically in its favor is that it is supported by the very evidence on which the basic principles of quantum mechanics are founded. This is, however, very strong evidence.

We occasionally meet with the objection that the laws of quantum mechanics, perhaps, hold only for a certain kind of parameter; that at a later stage of science other parameters may be found for which the relation of uncertainty does not hold; and that the new parameters may enable us to make strict predictions. Logically speaking, such a possibility cannot be denied. It then might be possible, for instance, to combine a measurement of the new parameters with a measurement of the kinematic parameters in such a way that the results of measurements of the dynamic parameters could be predicted. The law of inverse correlation between kinematic and dynamic parameters then still would hold for the old parameters so long as the new ones were not used; but when the new observables were to be applied for the selection of types of physical systems, the law of inverse correlation would no longer hold within the assemblages so constructed, even for the old parameters, and therefore their values could be strictly predicted. This would mean, in other words, that we could empirically define types of physical systems for which the statistical relations controlling their parameters were not expressible in terms of ψ-functions.[3]

In such a case quantum mechanics would be considered as a statistical part of science imbedded in a universal science of causal character. Although, as we said, we cannot adduce logical reasons excluding such a further development of physics, and, although some eminent physicists believe in such a possibility, we cannot find much empirical evidence for such an assumption. If a physical principle embracing all known entities has been established, it seems plausible to assume that it holds universally, and that there is no unknown class of physical entities which do not conform to this principle. Such an inductive inference from *all known* entities to *all* entities has always been considered legitimate. The principle of describing all physical situations in terms of ψ-functions is a well-established principle, and though, certainly, quantum mechanics is still confronted by many unsolved problems and may experience important improvements, nothing indicates that the principle of the ψ-function will be abandoned. Since the relation of uncertainty and the limitation of predictability follow directly from the principle of the ψ-function, these theorems must be regarded as being as well founded in their universal claims as all other general theorems of physics.

§ 4. The Disturbance of the Object by the Observation

We now turn to considerations involving the second main issue confronting the philosophy of quantum mechanics—the issue of the interpretation of unobserved objects. This question finds a first answer in the statement that the object is disturbed by the means of observation. Heisenberg, who recognized this feature in combination with his discovery of the principle of indeterminacy, used it as an explanation of the latter principle; he maintained that the indeterminacy of all measurements is a consequence of the disturbance by the means of observation.

[3] We use the term "expressible in terms of ψ-functions" in order to include both the pure case and the mixture; cf. § 23. In his book, *Mathematische Grundlagen der Quantenmechanik* (Berlin, 1932), p. 160–173, J. v. Neumann has given a proof that no "hidden parameters" can exist. But this proof is based on the assumption that for all kinds of statistical assemblages the laws of quantum mechanics, expressed in terms of ψ-functions, are valid. If the indeterminism of quantum mechanics is criticized, this assumption will be equally questioned. J. v. Neumann's proof therefore cannot exclude the case to which we refer in the text. It shows only that the assumption of hidden parameters is not compatible with a universal validity of quantum mechanics.

This statement has aroused a wave of philosophical speculation. Some philosophers, and some physicists as well, have interpreted Heisenberg's statement as the confirmation, in terms of physics, of traditional philosophical ideas concerning the influence of the perceiving subject on its percepts. They have iterated this idea by seeing in Heisenberg's principle a statement that the subject cannot be strictly separated from the external world and that the line of demarcation between subject and object can only be arbitrarily set up; or that the subject creates the object in the act of perception; or that the object seen is only a thing of appearance, whereas the thing in itself forever escapes human knowledge; or that the things of nature must be transformed according to certain conditions before they can enter into human consciousness, etc. We cannot admit that any version of such a philosophical mysticism has a basis in quantum mechanics. Like all other parts of physics, quantum mechanics deals with nothing but relations between physical things; all its statements can be made without reference to an observer. The disturbance by the means of observation—which is certainly one of the basic facts asserted in quantum mechanics—is an entirely physical affair which does not include any reference to effects emanating from human beings as observers.

This is made clear by the following consideration. We can replace the observing person by physical devices, such as photoelectric cells, etc., which register the observations and present them as data written on a strip of paper. The act of observation then consists in reading the numbers and signs written on the paper. Since the interaction between the reading eye and the paper is a macrocosmic occurrence, the disturbance by the observation can be neglected for this process. It follows that all that can be said about the disturbance by the means of observation must be inferable from the linguistic expressions on the paper strip, and must therefore be statable in terms of physical devices and their interrelations. Quantum mechanics should not be misused for attempts to revive philosophical speculations which are not on a level with the clarity and precision of the language of physics. The solution of its philosophical problems can only be given within a scientific philosophy such as has been developed in the analysis of science and in symbolic logic.

There was a similar period in the discussion of Einstein's theory of relativity in which the relativity of time and motion was ascribed to the subjectivity of the observer. Later analysis has shown that the dependence of statements about space and time on the system of reference is in no way connected with the privacy of every person's sense data, but represents the expression of arbitrary definitions involved in every description of the physical world. We shall see that a similar solution can be given to the problems of quantum mechanics, although the situation there is even more complicated than in the case of the theory of relativity. The difference is that on the arbitrariness of definitions is superimposed, in quantum mechanics, an uncertainty in the prediction of observable results, a feature which has no analogue in the theory of relativity.

We must begin our analysis with a revision of Heisenberg's statement that the uncertainty of predictions is a consequence of the disturbance by the means of observation. We do not think that the statement is correct in this form, although it is true that there is a disturbance by the observation and that there is a logical connection between this principle and the principle of indeterminacy. This connection should rather be stated inversely, namely, in the form that the principle of indeterminacy implies the statement of a disturbance of objects by the means of observation.

To say that the indeterminacy of predictions originates from the disturbance by the instruments of observation means that whenever there is a non-negligible disturbance by observation there will always be a limitation of predictability. A consideration of classical physics shows that this is not true. There are many cases in classical physics where the influence of the instrument of measurement cannot be neglected, and where, nevertheless, exact predictions are possible. Such cases are dealt with by the establishment of a physical theory which includes a theory of the instrument of measurement. When we put a thermometer into a glass of water we know that the temperature of the water will be changed by the introduction of the thermometer; therefore we cannot interpret the reading taken from the thermometer as giving the water temperature before the measurement, but must consider this reading as an observation from which we can determine the original temperature of the water only by means of inferences. These inferences can be made when we include in them a theory of the thermometer.

Why is it not possible to apply this logical procedure to the case of quantum mechanics? Heisenberg has shown that for a precise determination of the position of a particle we need light waves of a very short wave length, that is, waves which carry rather large quanta of energy and which change the velocity of the particle by their impacts, with the consequence that this velocity cannot be measured by the same experiment. If, on the other hand, we wish to determine the velocity of a particle, we must use rather long wave lengths in order not to change the velocity to be measured; but then we shall not be able to ascertain precisely the position of the particle. If, however, the observation of a particle by illuminating it with a light ray produces an impact which throws the particle off its path, why can we not construct a theory which tells us by means of inferences starting from the result of the observation what the original velocity of the particle was? It is here that the cross-section law of Heisenberg intervenes. This principle states that whatever be the observational results, the corresponding distributions of position and momentum must be derivable from a ψ-function, and therefore must be inversely correlated. Thus, a measurement of position involves physical processes of such a kind that, relative to the observational effects of these processes, the velocity distribution is a rather flat curve. This is the reason that we cannot determine exactly the velocity of the particle in the experiment mentioned. The relation between disturbance through observation and indeterminacy must therefore be stated as follows: The disturbance by the observation is the reason that the determination of the physical entity considered is not immediately given with the measurement, but requires inferences using physical laws; since these inferences are bound to the use of a ψ-function, they are limited by the principle of indeterminacy, and therefore it is impossible to come to an exact determination. This formulation makes clear that the disturbance by the observation, in itself, does not lead to the indeterminacy of the observation. It does so only in combination with the principle of indeterminacy.[1]

In view of such objections Heisenberg's principle has sometimes been formulated as meaning: We have no exact knowledge of physical states because the observation disturbs *in an unpredictable way*. In this form the statement is correct; but then it can no longer be interpreted as substantiating the principle of indeterminacy. It *states* this principle, but does not *give a reason* for it. And we recognize that the "disturbance in an unpredictable way" is but a special case of a general cross-section law of nature stating the inverse correlation of all physical data available. The instrument of measurement disturbs, not because it is an instrument used by human observers, but because it is a physical thing like all other physical things. Instruments of measurement do not represent exceptions to physical laws; the general limitation of inferences leading to simultaneous values of parameters includes the case of inferences referring to the effects of instruments of measurement. It is in this form that we must state the principle of the disturbance by the means of observation.

This is, however, only the first step in our analysis of the disturbance by the observation. We have so far taken it for granted that we know what we mean by saying that the observation disturbs the object. In order to come to a deeper understanding of the relations involved here, we must first construct a precise formulation of this statement.

§ 5. The Determination of Unobserved Objects

When we say that the object is disturbed by the observation, or that the unobserved object is different from the observed object, we must have some knowledge of the unobserved object; otherwise our statement would be unjustifiable. Before we can enter into an analysis of the particular situation in

[1] A mathematical proof that the disturbance of the object by the observation does not entail the principle of indeterminacy will be given later (p. 104). The idea that it is not the disturbance in itself which leads to the indeterminacy was first expressed by the author in "Ziele und Wege der physikalischen Erkenntnis," *Handbuch der Physik*, Vol. IV (ed. by Geiger-Scheel, Berlin, 1929), p. 78. The same idea has also been expressed by E. Zilsel, *Erkenntnis* 5 (1935), p. 59. The precise formulation of the principle of uncertainty requires a qualification which will be explained in § 30.

quantum mechanics we must therefore discuss in general the problem of our knowledge of unobserved things. How do things look when we do not look at them? This is the question to which we must find an answer.

It has sometimes been said that this problem is specific for quantum mechanics, whereas for classical physics there is no such problem. This is, however, a misunderstanding of the nature of the problem. Even in classical physics we meet with the problem of the nature of unobserved things; and only after giving a correct treatment of this problem on classical grounds shall we be able to answer the corresponding question for quantum mechanics. The logical methods by which the answer is to be formulated are the same in both cases.

To begin our analysis with an example, let us assume we look at a tree, and then turn our head away. How do we know that the tree remains in its place when we do not look at it? It would not help us to answer that we can easily turn our head forward and thus "verify" that the tree did not disappear. What we thus verify is only that the tree is always there when we look at it; but this does not exclude the possibility that it always disappears when we do not look at it, if only it reappears when we turn our head toward it. We could make an assumption of the latter kind. According to this assumption the observation produces a certain change of the object in such a way that there *appears* to be no change. We have no means to prove that this assumption is false. If it is suggested that another person may observe the tree when we do not see it and thus confirm the statement that the tree does not disappear, we may restrict our assumption to cases in which no person looks at the tree, thus ascribing the power of reproducing the tree to the observation of any human being. If it is suggested that we may derive the existence of the tree from certain effects remaining observable even when we do not see the tree, such as the shadow of the tree, we may answer that we can assume a change in the laws of optics such that there is a shadow although there is no tree. The argument therefore proves only that an assumption concerning the existence, or disappearance, of the unobserved object is to be connected with an assumption concerning the laws of nature in both cases.

It would be a mistake to say that there is inductive evidence for the assumption that the tree does not disappear when we do not see it, and that this assumption is at least highly probable. There is no such inductive evidence. We cannot say: "We have so often found the unobserved tree to be unchanged that we assume this to hold always". The premise of this inductive inference is not true, since, in fact, we never have seen an unobserved tree. What we have often seen is that when we turned our head to the tree it was there; from this set of facts we can inductively infer that the tree will always be there when we look at it, but there is no inductive inference leading from these facts to statements about the unobserved tree. We therefore cannot even say that the unchanged existence of the unobserved object is at least probable.

We are inclined to discard considerations of the given kind as "nonsense",

because it seems so obvious that the tree is not created by the observation. Such an answer, however, does not meet the problem. The correct answer requires deeper analysis.

We must say that there is more than one true description of unobserved objects, that there is a *class of equivalent descriptions*, and that all these descriptions can be used equally well. The number of these descriptions is not limited. Thus we can easily introduce an assumption according to which the tree splits into two trees every time we do not look at it; this is permissible if only we change the optics of unobserved things in a corresponding way, such that the two trees produce only one shadow. On the other hand, we see that not all descriptions are true. Thus it is false to say that there are two unobserved trees, *and* that the laws of ordinary optics hold for them. It follows that the statements about unobserved things are to be made in a rather complicated way. Descriptions of unobserved things must be divided into *admissible* and *inadmissible* descriptions; each admissible description can be called true, and each inadmissible description must be called false. Looking for general features of unobserved things, we must not try to find *the* true description, but must consider the whole class of admissible descriptions; it is in properties of this class as a whole that the nature of unobserved things is expressed.

In the case of classical physics this class contains one description which satisfies the following two principles:

1) The laws of nature are the same whether or not the objects are observed.
2) The state of the objects is the same whether or not the objects are observed.

Let us call this descriptional system the *normal system*. It is this system which we usually consider as the "true" system. We see that this interpretation is incorrect. We may, however, make the following statement. In case a class of descriptions contains a normal system, each of the descriptions is equivalent to the normal system. If we now consider one of the unreasonable descriptions of the class, such as the statement that a tree splits into two trees whenever it is not observed, we see that these anomalies are harmless. They result from the use of a different language, whereas the description as a whole says the same as the normal system. This is the reason that we can select the normal system as the only description to be used.

The convention that the normal system be used is always tacitly assumed in the language of daily life when we speak of inductive evidence for or against changes of unobserved objects. This convention is understood when we say that our house remains in its place as long as we are absent; and the same convention is understood when we say that the girl is not in the box of the magician while he is sawing the box into two pieces, although we have seen the girl in it before. It is because of the use of this convention that ordinary statements about unobserved things are testable. The same convention is used in scientific language; it simplifies the language considerably. We must, however, realize that this choice of language has the character of a definition and that the simplicity of the normal system does not make this system "more true" than the others. We are concerned here only with what has been called a difference in *descriptive simplicity*,[1] such as we find in the case of the metrical system as compared with the yard-inch system.

In stating that a class of descriptions includes a normal system, we make a statement about the whole class. This way of stating a property of the class by means of a statement about the existence of a normal system may be illustrated by an example from differential geometry. Properties of curvature are statable in terms of systems of coordinates and their properties. Thus the surface of the sphere can be characterized by the statement that it is not possible to introduce on it a system of orthogonal straight-line coordinates which covers large areas. Only for an infinitesimal area is this possible; i.e., for small areas it is possible to introduce approximately orthogonal straight-line coordinates, and the degree of approximation increases for smaller areas. For the plane, however, such a system covering the whole plane can be introduced. It is not necessary, though, to use this "normal system" of coordinates for the plane, since any kind of curved coordinates can be used equally well; but the fact that *there is* such a normal system distinguishes the class of possible systems of coordinates holding for the plane from the corresponding class holding for a curved surface.

Similar considerations have been developed for Einstein's theory of relativity, which is the classical domain of application for the theory of classes of equivalent descriptions. Every system of reference, including systems in different states of motion, furnishes a complete description, and we have therefore in the class of systems of reference a class of equivalent descriptions. If the class of such systems includes one for which the laws of special relativity hold, we say that the considered space does not possess a "real" gravitational field. This is true, although we can introduce in such a world unreasonable systems which contain pseudogravitational fields; they are pseudogravitational because they can be "transformed away".[2]

§ 6. Waves and Corpuscles

Turning from these general considerations to quantum mechanics, we first must clarify what is to be meant by observable and by unobservable occurrences. Using the word "observable" in the strict epistemological sense, we must say that none of the quantum mechanical occurrences is observable; they are all inferred from macrocosmic data which constitute the only basis accessible to observation by human sense organs. There is, however, a class of

[1] Cf. the author's *Experience and Prediction* (Chicago, 1938), § 42. Descriptive simplicity is distinguished from inductive simplicity; the latter involves predictional differences.
[2] Cf. the author's *Philosophie der Raum-Zeit-Lehre* (Berlin, 1928), p. 271.

occurrences which are so easily inferable from macrocosmic data that they may be considered as observable in a wider sense. We mean all those occurrences which consist in coincidences, such as coincidences between electrons, or electrons and protons, etc. We shall call occurrences of this kind *phenomena*. The phenomena are connected with macrocosmic occurrences by rather short causal chains; we therefore say that they can be "directly" verified by such devices as the Geiger counter, a photographic film, a Wilson cloud chamber, etc.

We then shall consider as unobservable all those occurrences which happen between the coincidences, such as the movement of an electron, or of a light ray from its source to a collision with matter. We call this class of occurrences the *interphenomena*. Occurrences of this kind are introduced by inferential chains of a much more complicated sort; they are constructed in the form of an *interpolation* within the world of phenomena, and we can therefore consider the distinction between phenomena and interphenomena as the quantum mechanical analogue of the distinction between observed and unobserved things.

The determination of phenomena is practically unambiguous. Speaking more precisely, this means that in the inferences leading from macrocosmic data to phenomena we use only the laws of classical physics; the phenomena are therefore determinate in the same sense as the unobserved objects of classical physics. Putting aside as irrelevant for our purposes the problem of the unobserved things of classical physics, we therefore can consider the phenomena as verifiable occurrences. It is different with the interphenomena. The introduction of the interphenomena can only be given within the frame of quantum mechanical laws; it is in this connection that the principle of indeterminacy leads to some ambiguities which find their expression in the duality of waves and corpuscles.

The history of the theories of light and matter since the time of Newton and Huygens shows a continuous struggle between the interpretation by corpuscles and the interpretation by waves. Toward the end of the nineteenth century this struggle had reached a phase in which it seemed practically settled; light and other kinds of electromagnetic radiation were regarded as consisting of waves, whereas matter was assumed to consist of corpuscles. It was Planck's theory of quanta which, in its further development, conferred a serious shock to this conception. In his theory of needle radiation Einstein showed that light rays behave in many respects like particles; later L. de Broglie and Schrödinger developed ideas according to which material particles inversely are accompanied by waves. The wave nature of electrons then was demonstrated by Davisson and Germer in an experiment of a type which, a dozen or so years before, had been made by M. v. Laue with respect to X-rays, and which had been considered at that time as the definitive proof that X-rays do not consist of particles. With these results the struggle between the conceptions of waves and corpuscles seemed to be revived, and once more physics seemed to be confronted by the dilemma of two contradictory conceptions each of which seemed to be equally demonstrable. One sort of experiment seemed to require the wave interpretation, another the corpuscle interpretation; and in spite of the apparent inconsistency of the two interpretations, physicists displayed a certain skill in applying sometimes the one, sometimes the other, with the fortunate result that there was never any disagreement with facts so far as verifiable data were concerned.

An attempt to reconcile the two interpretations was made by Born who introduced the assumption that the waves do not represent fields of a kind of matter spread through space, but that they constitute only a mathematical instrument of expressing the statistical behavior of particles; in this conception the waves formulate the probabilities for observations of particles. It is this interpretation which we have used in § 2. It has turned out, however, that even this ingenious combination of the two interpretations cannot be carried through consistently. We shall describe in § 7 experiments which do not conform with the Born conception. On the other hand, the latter conception has been incorporated into quantum physics so far as it has been made the definitive form of the corpuscle interpretation. Whenever we speak of corpuscles we assume them to be controlled by *probability waves*, i.e., by laws of probability formulated in terms of waves. The duality of interpretations, therefore, is given by a wave interpretation according to which matter consists of waves; and a corpuscle interpretation, according to which matter consists of particles controlled by probability waves. As to the waves the struggle between the two interpretations, therefore, amounts to the question whether the waves have *thing-character* or *behavior-character*, i.e., whether they constitute the ultimate objects of the physical world or only express the statistical behavior of such objects, the latter being represented by atomic particles.

The decisive turn in the evaluation of this state of affairs was made by Bohr in his *principle of complementarity*. This principle states that both the wave conception and the corpuscle conception can be used, and that it is impossible ever to verify the one and to falsify the other. This indiscernability was shown to be a consequence of the principle of indeterminacy, which with this result appeared to be the key unlocking the door through which an escape from the dilemma of two equally demonstrable and contradictory conceptions was possible. The contradictions disappear, since it can be shown that they are restricted to occurrences situated inside the range of indeterminacy; they are therefore excluded from verification.

Although we should like to consider this Bohr-Heisenberg interpretation as ultimately correct, it seems to us that this interpretation has not been stated in a form which makes sufficiently clear its grounds and its implications. In the form so far presented it leaves a feeling of uneasiness to everyone who wants to consider physical theories as complete descriptions of nature; the path towards this aim seems either to be barred by rigorous rules forbidding us to ask questions of a certain kind, or open only to vague pictures which have no claim to be regarded as an adequate expression of reality. It seems to us that this state of affairs is due, not so much to mistakes in the quantum mechanical part of the interpretation, as to an erroneous interpretation of the corresponding problem of classical physics which has not been seen in the full extent of its logical complications. In the following considerations we shall try to give a solution of these problems which follows the outlines of the ideas of Bohr and Heisenberg, but which seems to us to avoid the unsatisfactory parts of this conception.

We shall use for our analysis a formulation of the ideas of Bohr and Heisenberg which has been developed by Landé.[1] Landé states the duality of the two interpretations, by waves and by corpuscles, in the following form. It is incorrect to say that some experiments require the wave interpretation and others require the corpuscle interpretation; such a conception, which represents the state of affairs before the Bohr-Heisenberg theory, is not permissible, because it would make physical theory inconsistent. Instead, we must say that *all* experiments can be explained through *both* interpretations. It will never be possible to construct an experiment which is incompatible with one of the interpretations.

Combining this formulation with our theory of equivalent descriptions, and using the terminology explained above, we can state Landé's conception as follows. Given the world of phenomena, we can introduce the world of interphenomena in different ways; we then shall obtain a class of equivalent descriptions of interphenomena, each of which is equally true, and all of which belong to the same world of phenomena. In other words: In the class of equivalent descriptions of the world the interphenomena vary with the descriptions, whereas the phenomena constitute the invariants of the class. With this, the arbitrariness of descriptions is eliminated from the world of phenomena and restricted to the world of interphenomena; but there it is harmless, as we know that a similar arbitrariness of descriptions holds for the unobserved things of classical physics. Nowhere do we find an unambiguous supplementation of observations; interpolation of unobserved values can be given only by a class of equivalent descriptions.

From this result we turn to the question whether the class of equivalent descriptions of quantum mechanics contains a normal system, i.e , a description which satisfies the two principles established on page 19. Now it is obvious that the second principle will be violated by all descriptions, as there is always a disturbance of the object by the observation. We must therefore modify our definition of the normal system and restrict it to the requirement that at least the first principle be satisfied.[2] The question must therefore be asked in the

[1] A. Landé, *Principles of Quantum Mechanics* (Cambridge, England, 1937).

[2] Professor W. Pauli drew my attention to the fact that even in classical physics the second principle is mostly to be suspended for the introduction of normal systems. When we see a physical object, the fact producing the observation is the entering of light rays

form: Is there a normal system in the wider sense, i.e., a system at least satisfying the first principle?

We must not believe that the existence of a normal system can be postulated by philosophical considerations. We cannot admit that there is any *synthetic a priori* principle, i.e., a principle which is not logically empty and which physical theory is bound to satisfy. The question of whether there is a normal system can only be answered by experience. If there is such a system, the world of interphenomena is revealed to be of a rather simple structure; if there is no such system, this world turns out to be more complicated than we would perhaps like it to be. But by no means is it permissible to refuse an answer to the question as meaningless, or to evade it by drawing the attention to other sides of the problem. The general properties of the world of interphenomena, as it is constructable on the basis of quantum mechanics, are expressed in the answer we give to this question.

§ 7. Analysis of an Interference Experiment

To find an answer to the question of whether there is a normal system, let us analyze some experiments which can be considered as typical for various kinds of logical situations. First, let us consider (figure 4) the case of a diaphragm containing one slit B through which radiation of light, or electrons, or other particles of matter passes towards a screen. We then shall obtain an interference pattern on the screen. We know, however, that if we use very low intensities of the radiation, we obtain on the screen, not the whole pattern at once, but individual flashes in strictly localized areas, for instance, in C. These flashes could be verified, say, by a Geiger counter in C. If we let the experiment go on for a certain time, the distribution of the flashes, occurring one after the other, will follow the interference pattern mentioned above; it is this sum of individual flashes which a photographic film placed on the screen would show us.

The phenomena of this experiment are given by the individual flashes on the screen; besides, we have the macrocosmic objects consisting in the source of radiation, the diaphragm, and the screen. Let us ask which kind of interphenomena we can introduce here by the method of interpolation. First, we

into the retina of the human eye; but in this action the light ray is absorbed and thus changed by the interaction of the means of observation. The statement that there is a physical object which is undisturbed by the observation is therefore, physically speaking, obtained by an inference based on an observation which does not satisfy the second principle. Only psychologically speaking is this not true, since the inference is performed automatically by the sensory mechanism; the eye is gauged in stimulus language. This is the reason why it appears advisable in classical physics to interpret the term "observation" in such a way that the second principle can be maintained. It would be equally possible to cancel the second principle even for classical physics. This corresponds to our view that the abandonment of the second principle in quantum mechanics is rather irrelevant and that a normal system in a wider sense can be defined including the case that the second principle is violated. It is the first principle which expresses the *conditio sine qua non* of the normal system.

can use the corpuscle interpretation.[1] We then say that individual particles are emitted from the source of radiation, and move on straight lines such as indicated in figure 4; at the place B these particles are subject to impacts or other forms of interaction imparted to them by the particles of which the substance of the diaphragm is composed. They thus deviate from their paths. These impacts follow statistical laws in such a way that some parts of the screen are frequently hit, others less frequently. The interference pattern of the photographic film indicates, therefore, the probability distribution of the impacts given in B to the passing particles. Of course, there are also other particles

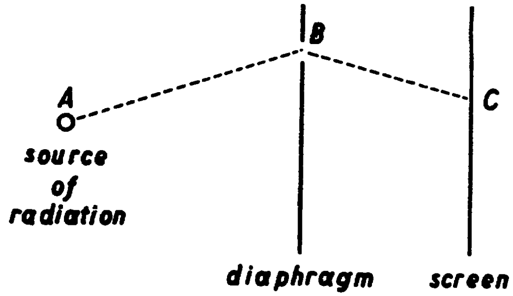

Fig. 4. Diffraction of radiation at a slit B.

leaving the source A; but when they reach the diaphragm in places other than the slit B they are absorbed or reflected and will not appear on the screen.

In this interpretation we have a certain probability

$$P(A,B) \tag{1}$$

that a particle leaving the source A will arrive at B, and a probability

$$P(A.B,C) \tag{2}$$

that a particle leaving A and passing through B will arrive at C. Both values can be statistically determined by counting on a surrounding screen all particles leaving A, then all those particles arriving on the screen (the number arrived at meaning the number of particles passing through the slit B), and then all particles arriving at C.

We see that we have here an interpretation of *interphenomena* which satisfies the first principle required for a normal system. The only deviation from classical physics consists in the fact that we have merely a probability law governing the transition from B to C; but this extension of the concept of causality holds equally for the *phenomena* of quantum mechanics. Therefore, in this interpretation both phenomena and interphenomena are governed by the same laws.

[1] The way of describing this experiment in corpuscle interpretation is represented in A. Landé, *Principles of Quantum Mechanics* (Cambridge, England, 1937), § 9.

Now let us use the wave interpretation. We then say that spherical waves leave A, that only a small part of these waves passes through the slit B and then spreads toward the screen. This part of the waves consists of different trains of waves, each of which has a different center; all these centers lie on points within the slit B (principle of Huygens). The superposition of these different trains furnishes the interference pattern on the screen.

So long as we consider only the results of the process obtained during a long time, for instance, in the pattern obtained on a photographic film, this explanation leads to no difficulties. It has even the advantage over the corpuscle interpretation in that it does not use statistical laws, but strictly causal laws. The numerical values of the probabilities (1) and (2) appear here as the intensities of waves which determine directly the amount of blackening in the various points of the screen. It is different as soon as we consider the individual flashes which we know can be verified on the screen. Let us assume, for instance, that the screen is replaced by a set of Geiger counters; the statistics of this system of counters then will be equivalent to the interference pattern on the film, with the addition, however, that it reveals the process as being composed of individual impacts. In face of these facts the assumption of waves leads into difficulties which were first pointed out by Einstein. So long as the wave has not yet reached the screen, it covers an extended surface, namely, a hemisphere, with its center in B; but when it reaches the screen it will produce a flash only at one point, say, at C, and will then automatically disappear at all other points. The wave is swallowed, so to speak, by the flash at C. This process of the disappearance of the wave constitutes a *causal anomaly* so far as it contradicts the laws established for observable occurrences. We see that in this description the laws of interphenomena are different from the laws of phenomena; the given description therefore does not represent a normal system.

With the intention of escaping disagreeable consequences of the wave interpretation, the suggestion has been advanced that it should be forbidden to ask questions about what becomes of the wave after a flash on the screen has been observed. We shall later discuss an interpretation in which such an interdiction of questions is carried through; with such an interpretation, however, we have abandoned the wave theory. *Within the wave interpretation* the legitimacy of such questions cannot be denied. The different reasons which have been offered for ruling out such questions do not stand a logical test. Thus it has been said that within the wave description we cannot speak of effects localized in space. This, however, is incorrect; the wave itself is conceived as a function of space, and if a certain effect is observed in one place on the screen, it is perfectly correct to ask what effects are caused by the wave in other places. It has been said, furthermore, that the flash on the screen belongs to the corpuscle interpretation, and that therefore we cannot incorporate it into the wave interpretation. This is incorrect because the flash on the screen is a verifiable phenomenon, and therefore is not included in the duality of interpretations

which concerns only the interphenomena. The flash belongs neither to the one nor to the other of the interpretations, but is one of the verifiable data on which both interpretations are based. Now we must demand that each interpretation of interphenomena be compatible with the given set of phenomena; and if the wave interpretation is used, it must be extended to include a statement concerning the transformation of a wave into a localized flash.

It appears understandable that in the case of such an experiment as the one considered we prefer the corpuscle interpretation, since it does not include causal anomalies, and therefore represents a normal system. But the wave interpretation is as true as the other; it is true in the same sense as is the interpretation in the example given above, according to which the tree always splits

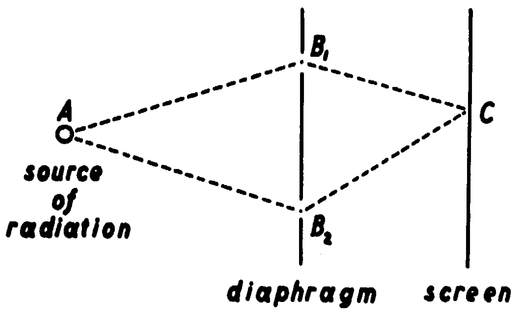

Fig. 5. Diffraction of radiation at two slits, B_1 and B_2.

into two trees at the moment it is not observed. This anomaly need not bother us because we know that it can be "transformed away" by the use of another description. In the same sense the anomaly of the wave description should not bother us because we know that it can be transformed away by the use of the corpuscle description. But if the wave interpretation is used, it includes the consequence of the disappearance of the whole wave with the appearance of the flash in one point, however distant this point may be from other points of the wave. We must have the courage to face this consequence which is necessarily combined with this interpretation of interphenomena.

Let us now turn to a second experiment, which we indicate in figure 5. We use the same arrangement as before, with the difference that the diaphragm has two slits, at B_1 and B_2. We know that in this case we shall also obtain an interference pattern on the screen, which is, however, different from the pattern obtained in the first experiment. Let us consider this experiment in different interpretations.

We first use the corpuscle interpretation. Assuming, as before, low intensities of the radiation, we know that there will be individual flashes on the screen. We can explain this by the assumption that individual particles leave the source intermittently; sometimes a particle passes through the slit B_1, sometimes through the slit B_2, sometimes it is absorbed by the diaphragm—all this being determined by the direction in which the particle leaves A. If there occurs a flash in C, we shall say that the particle has passed either through B_1 or B_2.

The probability that a particle reaches C can be stated by the rule of elimination established in the calculus of probability:

$$P(A,C) = P(A,B_1) \cdot P(A.B_1,C) + P(A,B_2) \cdot P(A.B_2,C) \tag{3}$$

The meaning of these terms follows from the explanation given with respect to (1) and (2). We should be inclined to assume that the numerical values of the probabilities on the right hand side of (3) are the same as obtained in experiments of the first type; it turns out, however, that this assumption is wrong.

This can be proved in the following way. We first close the slit at B_2, and let the process of radiation go on for a certain time; then we close the slit at B_1, and let the process go on for an equal time. Using a film as a screen we then obtain a superposition of both interference patterns. The question arises: Is this pattern of interference the same as that which will result when both slits are opened simultaneously? If it is, we can assume that the values of the occurring probabilities are the same as in the first experiment. If it is not, the probabilities $P(A.B_1,C)$ and $P(A.B_2,C)$ must have changed.

It is well known that the experiment decides in favor of the second alternative. We therefore must assume that the probability with which a particle passing through B_1 reaches C depends on whether the slit B_2 is open.[2] This is a causal anomaly; it states that there is an effect originating in B_2 and being spread to B_1 such that it influences the impacts given in B_1 to passing particles. We see that it is in this case the corpuscle interpretation which leads to causal anomalies.[3]

It would be erroneous to say that because of these anomalies the corpuscle interpretation is *false*. Some physicists who have uttered opinions of this kind have based their judgments on the fact that we have no means of knowing through which of the two slits, B_1 or B_2, the particle has passed after the flash in C has been observed. The latter statement, of course, is true, since an observation in B_1, or in B_2, would disturb the experiment. We are not even able to determine the probability $P(A.C,B_1)$ that the particle, observed in the flash C, has passed through B_1. This inverse probability could be determined by means of the rule of Bayes[4] in the form

$$P(A.C,B_1) = \frac{P(A,B_1) \cdot P(A.B_1,C)}{P(A,C)} \tag{4}$$

if we knew the value of the forward probability $P(A.B_1,C)$; but since this probability is different from the value $P(A.B,C)$ obtained with the use of only one slit, it cannot be determined. Every such determination would require observations of the particle in B_1, and therefore would lead to a disturbance of the experiment. Applying the verificability theory of meaning, even in the modified form of probability meaning,[5] we therefore must say that the sentence "the particle has passed through B_1" is meaningless if we consider it as a *statement about physical facts*. It is permissible, however, to use this sentence if we consider it as a *definition*. In order to make the corpuscular description complete we coordinate by definition to every flash in C a path of a particle, passing either through B_1 or B_2; the choice of these paths is arbitrary. Even if we follow the requirement (which in itself has only the character of a definition) that the probability $P(A,B_1)$ and $P(A,B_2)$ be the same as obtained in experiments with one slit, the choice of the paths will remain arbitrary within wide limits.

The situation presented reminds us of a similar situation in the problem of simultaneity. If a light signal leaving a point Q at the time t_1 is reflected at a point R, and then returns to Q at the time t_3, we may coordinate to the time of its arrival in R every numerical value between t_1 and t_3. With this choice we make a definition of simultaneity at the points Q and R. The sentence: "one of the events in Q between t_1 and t_3 is simultaneous with the arrival of the light signal in R" is meaningless if it is considered as an empirical statement, since it is not verifiable; but it is meaningful if we introduce it as a definition. In the same sense we can use the sentence: "the particle passed through B_1" as a definition. In both cases such a definition is necessary in order to make our description complete.

It follows that the corpuscular interpretation can be carried through consistently, and that there is nothing incorrect in it. Its only disadvantage is that it leads to causal anomalies as explained. The influence of the slit B_2 on the occurrences in B_1 is of such a kind as to violate the principle of *action by contact*. There is no spreading of the effect from B_2 to B_1; this can be seen from the fact that changes in the material of the diaphragm between B_1 and B_2, or changes of its shape such as would result from corrugating the diaphragm, would not influence the effect.

If we want to construct for the experiment described an interpretation free

[2] It can be easily seen that this consideration is a special case of the consideration given in § 22, according to which an intermediate measurement of an entity v influences the probability leading from u to w, even if the result of the measurement is not included in the statement of the latter probability. Closing the slit B_2 is equivalent to making a measurement of position in B_1.

[3] Further anomalies arise if we bring the screen closer to the diaphragm. If we choose the point C on the screen always in such a way that the direction B_1C remains the same, we find that the probability $P(A.B_1,C)$ cannot be considered as remaining constant; i.e., this probability is not a function depending only on the direction in which the particle leaves B_1. This follows from considerations using waves.

[4] Cf. any textbook on probability, or the author's *Wahrscheinlichkeitslehre* (Leiden, 1935), formula (6), § 21.

[5] Cf. the author's *Experience and Prediction* (Chicago, 1938), § 7.

from anomalies, we must use a wave interpretation. It would not be sufficient, however, to use an interpretation in which the wave spreads through all the open space; this would lead to the same anomalies as explained for the wave interpretation of the first experiment. We must use a wave interpretation according to which the wave is limited to two narrow canals such as indicated in figure 6. It can be shown that any changes made at a point inside the "two-canal element" would produce a change in the probability $P(A,C)$ that a flash in C occurs, whereas changes outside do not influence this probability. We could, for instance, fill the space outside the strips included in the dotted lines with absorbing matter, without changing $P(A,C)$.[6] This can be proved through

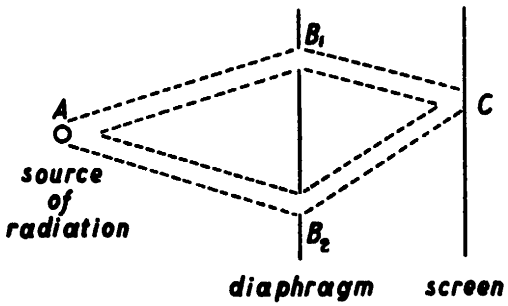

Fig. 6. The "two-canal element" representing the radiation.

the principle of superposition which we used above; the interference pattern on the screen can be considered as the superposition of the blackenings, produced one after the other, by two-canal elements having their top ends in different points C on the screen. We therefore obtain a normal system for the considered experiment if we use a description in which the interphenomena are regarded as two-canal waves spreading from A to various points C on the screen, and following each other intermittently.

We may add here the remark that for the case of only one slit, i.e., for the experiment of figure 4, we have also a wave interpretation which is free from causal anomalies; we then speak of a one-canal wave, such as Einstein originally assumed in his needle radiation. We therefore have here two normal systems. But the difference between these two descriptions is not very great, and we therefore usually speak only of the corpuscle interpretation. Every statement about corpuscles can therefore be replaced by a statement about needle radiation.

[6] Strictly speaking, this is only approximately true. The degree of approximation increases when the canals chosen are wider in their middle parts, while the widths at B_1, B_2, and C remain unchanged. The mathematical theorem to which we refer here consists in the statement that it is possible to introduce two sets of waves starting from B_1 and B_2 interfering on the screen in such a way that only in a small area at C an intensity is left. Our "two-canal element" may be considered as a simplified illustration of these two sets of waves.

It has sometimes been said that the differences of wave and corpuscle interpretation are combined with an alternative concerning the use of space-time concepts and causal concepts. The wave interpretation, according to this conception, satisfies the principle of causality so far as the waves are controlled by a differential equation, the Schrödinger equation (cf. § 13), but does not allow us to give a spatio-temporal description of physical objects. The corpuscle interpretation, on the other hand, is said to satisfy the requirements of a spatio-temporal description but to violate the principle of causality. Apart from the latter statement we cannot consider this conception as correct. It is not true that the wave description satisfies throughout the conditions of causality. It does so only so far as the wave field, conceived as a physical reality, spreads through space in a form expressible by a differential equation; in this respect it represents an action by contact, at least when free particles are concerned.[7] But, as we have shown, there are other points in which the latter principle is violated. Thus, the disappearance of the wave after the occurrence of a flash on the screen is a process not following the Schrödinger equation, and therefore not conforming to the principle of action by contact. In addition, we must say that a wave interpretation which does not satisfy the conditions of a space-time description *eo ipso* cannot satisfy the postulates of a normal causality either. Spatio-temporal order is closely connected with causal order, as has been brought to light in the analysis of the theory of relativity.[8] If the wave cannot be imagined as imbedded in a space-time manifold in which *every* part of the wave process satisfies the principle of action by contact, it cannot be said to conform to the requirements of a normal causality.

Our exposition has shown that the problems involved in quantum mechanics cannot be reduced to the alternative space-time versus causality. The space-time order to be assumed is always the ordinary one, and can be ascertained in many cases by the usual macrocosmic methods; thus the distance between the two slits B_1 and B_2 may be large enough to be measured by macrocosmic appliances. In both interpretations it is violations of the requirements of normal causality which are concerned. The kind of these violations only varies with the interpretation.

Such violations will occur also within a third interpretation which we must mention now and which is given by a combination of both wave and corpuscle interpretation. According to this interpretation we have a wave field spreading through the whole space and being diffracted at the slits B_1 and B_2 in the usual way; in addition to this field we have corpuscles whose movement is controlled

[7] For systems composed of several particles the waves spread, not in a three-dimensional space, but in the n-dimensional configuration space. But even if we were to consider the configuration space as "real" space, the waves would not satisfy the requirements of normal causality. In such a space the disappearance of the wave after the flash would lead to the same difficulties as in ordinary space.

[8] Cf. the author's *Philosophie der Raum-Zeit-Lehre* (Berlin, 1928), §§ 27, 42.

by the field in such a way that the intensity of the field determines the probability of finding a corpuscle. Using a term introduced by L. de Broglie, we speak here of *pilot waves*. This interpretation has its anomalies in the existence of a field which follows laws different from those holding for other kinds of waves; in particular, this wave field possesses no energy, since the energy is supposed to be concentrated in the particles. Furthermore, the influence of the field on the particles is governed by unusual laws. If the particles are assumed to move in straight lines, these laws would violate the principle of action by contact, since, then, the probability that a particle passing through B_1 will turn in the direction of C would depend, not on the intensity of the field near B_1, but on its intensity at C.[9] Assuming that the particles move in oscillating lines we encounter other anomalies.[10]

This description is therefore not a normal system, but it has some advantages which make its application advisable in many cases. It is not necessary to assume that the pilot waves appear intermittently with the particles, and that these waves disappear with the flash in C. The waves may be assumed to go on continuously; their existence, however, will be verifiable only at the moments when there are particles traveling through them.

§ 8. Exhaustive and Restrictive Interpretations

The above analysis shows that neither the corpuscle interpretation nor the wave interpretation can be carried through without causal anomalies. Using the particle interpretation we can explain some experiments in such a way that the laws of phenomena and interphenomena are the same; but then we encounter anomalies in other cases. Using the wave interpretation we can explain these other cases in such a way that the laws of phenomena and interphenomena are the same; but then anomalies appear in the explanation of experiments of the first kind. Finally, a combination of the two into an interpretation of pilot waves shows other anomalies.

The question arises whether there is another interpretation, perhaps unknown to us, which is free from causal anomalies. The preceding investigations cannot be considered as a proof that there is no such interpretation. Such a proof cannot be given by trying out one interpretation after another; we then are never sure whether a better interpretation which escaped our attention remains. The proof must be based on a general theory of the relations between quantum mechanical entities. We shall give this proof in § 26; the conception of causality assumed for it, which takes account of possible modifications of this notion, will be explained at the end of § 24. Our results can be formulated as follows: It is impossible to give a definition of interphenomena in such a way that the postulates of causality are satisfied. *The class of descriptions of interphenomena contains no normal system.* This can be proved to be a consequence of the basic principles of quantum mechanics. We shall call this result the *principle of anomaly*.

In view of this negative result two different conceptions can be carried through. The first calls for a *duality of interpretations*. Among the class of equivalent descriptions we have two, the corpuscle interpretation and the wave interpretation, which are more expedient than the others; since we have no normal system, we can use, instead, either of these two interpretations as a *minimum system*, i.e., a system for which the deviations from a normal system constitute a minimum. In this conception causal anomalies cannot be avoided; but they can at least be reduced to a minimum.

The second conception represents a more radical remedy. Since no normal description of interphenomena exists, it has been suggested we should renounce any description of interphenomena; we should restrict quantum mechanics to statements about phenomena—then no difficulties of causality will arise. The impossibility of a normal system is construed, in this conception, as a reason for abandoning all descriptions of interphenomena. We shall call conceptions of this kind *restrictive interpretations* of quantum mechanics, since they restrict the assertions of quantum mechanics to statements about phenomena. The rule expressing this restriction can assume various forms, and we shall therefore have several restrictive interpretations. Interpretations which do not use restrictions, like the corpuscle and the wave interpretation, will be called *exhaustive interpretations*, since they include a complete description of interphenomena.

The adherents of restrictive interpretations have maintained that a description of interphenomena is unnecessary; for the purpose of observational predictions, they say, it is sufficient to have an interpretation which refers only to phenomena. The latter statement is true; but it cannot be considered as proof that exhaustive descriptions should be abandoned. We should clearly keep in mind that neither of the two conceptions can be proved to be true. These conceptions represent volitional decisions concerning the form of physics; either of them is as justifiable as the other.

Speaking in terms of the class of equivalent descriptions, the situation can be characterized as follows. The system of phenomena is the same for each description of this class; it is therefore the *invariant* of this class. Now the class is dependent on its invariant; so far, any restrictive interpretation determines the whole class of exhaustive descriptions. The latter descriptions, however, reveal a feature which we would not know if we knew only a restrictive description: this is the fact that no interpretation free from causal anomalies can be given. Since this is a property of the class of exhaustive descriptions, it represents an inherent property of every restrictive interpretation. This property is expressed in the restrictive interpretations through the fact that they

[9] This follows from considerations similar to those indicated in fn. 3, p. 28.
[10] The anomalies of this interpretation are very clearly presented in L. de Broglie's book, *Introduction à l'Etude de la Mécanique ondulatoire* (Paris, 1930), chap. 9.

rule out certain statements; but the reason for this rule can only be formulated in terms of a statement about the properties of the class of exhaustive descriptions.

We therefore shall turn now to a further analysis of the class of exhaustive interpretations, while we postpone the discussion of restrictive interpretations. Within the first class, we said, the two interpretations in terms of corpuscles and waves hold a special position so far as they represent minimum systems. To this we now must add a second statement which secures a unique position to these two interpretations, and which at the same time attenuates the consequences resulting from the absence of a normal system.

Although we have no exhaustive description free from anomalies holding for *all* interphenomena, we can construct such a description for *every* interphenomenon by using either the wave or the corpuscle interpretation. It is this fact which we express in speaking of the *duality* of wave and corpuscle interpretation. We mean by this that *for a given experiment* at least one of the two will be a normal description and will thus define interphenomena in such a way that they follow the same laws as the phenomena; it is only in other experiments that the interpretation so chosen will lead to causal anomalies. Let us call this statement the *principle of eliminability of causal anomalies.* The difference between *all* and *every*, which we used to formulate this principle, is well known to symbolic logic. Using this grammatical distinction in another form we may also say: It is false to say that *all* interphenomena follow the laws holding for phenomena; but it is correct to say that *every* interphenomenon does so. *We do not have one normal system for all interphenomena, but we do have a normal system for every interphenomenon.*

As before, an analogy from differential geometry may illustrate these formulations. When we use a system of orthogonal coordinates on the sphere, such as that given in the circles of longitude and latitude (such a system is possible because it does not consist throughout of straightest lines, the circles of latitude not being greatest circles), this system has singularities at the North Pole and the South Pole; i.e., these points do not have a definite longitude. These singularities, however, are due only to the system of coordinates; the poles themselves are not distinguished geometrically from any other point of the sphere. The singularities can therefore be "transformed away" by the introduction of another system of coordinates; thus, the sailor will use, near the poles, a notation of points which determines positions relative to a chosen initial point and two chosen directions rectangular to each other. These coordinates could even be produced and used to cover the whole sphere, at least if one set of lines is not assumed to consist of straightest lines; but then singularities will appear in other points of the system. We may call a system of orthogonal coordinates without singularities a normal system. Then we may say that we can introduce a normal system for *every* extended area on the sphere, but we cannot introduce one normal system for *all* areas, i.e., for the whole sphere. We thus express a statement about the sphere in terms of a statement concerning the class of possible systems of coordinates.

The difference between this case and the case considered above, which concerns orthogonal straight-line coordinates, is as follows. A system of coordinates being both orthogonal and straight-lined is possible only for infinitesimal areas; for extended areas of some size it cannot even be carried through approximatively. If we renounce the requirement of straight lines we can construct a system which covers great areas of the sphere, and which is strictly orthogonal; but such a system will lead to singularities in two points. The advantage of this case over the first is that, with this definition of the normal system, we obtain a normal system for extended areas, not for infinitesimal areas only.

Returning to quantum mechanics, we must say that the situation there corresponds to the second case. The causal anomalies can be transformed away strictly for "extended areas", i.e., for a whole experiment, by a suitable description. They will reappear only for other experiments or for questions in which experiments of different kinds are compared; for the answer to such questions we then can introduce a new description such that once more the anomalies disappear. The reason that this is always possible is given in the relation of indeterminacy. If we could observe a particle passing through the slit B_1 in the experiment of figure 5, we could not introduce a wave description, and therefore would have no normal description of the experiment, i.e., no description free from anomalies. On the other hand, if we could prove that a wave arrived simultaneously at different points C of the screen in the experiment of figure 4, we could not introduce a corpuscle description, and therefore would have no normal description of this experiment. We see that *the principle of eliminability of anomalies is made possible through the principle of indeterminacy*, since the latter principle makes it impossible ever to construct a crucial experiment between wave interpretation and corpuscle interpretation.[1]

The ultimate root of the duality of wave interpretation and corpuscle interpretation is therefore given in the principle of indeterminacy; but this principle also points the way out of the dilemma of causal anomalies, a result stated by us in the principle of eliminability. We spoke above of the skill displayed by physicists in applying sometimes the wave interpretation and sometimes the corpuscle interpretation; we now see that we can give a justification of this change of interpretations which proves that the switching over to a normal interpretation is a legitimate means of physical analysis. When the physicist, in face of a particular experiment, introduces a suitable description which eliminates causal anomalies within the frame of his question, he may be compared

[1] It may be questioned whether it is actually possible in all cases to eliminate causal anomalies by a suitable description. What can be shown is that the principle of eliminability holds, at least for single particles or for swarms of particles which do not interact with each other such as electron swarms or light rays. Difficulties arise for complicated structures composed of several particles. Cf. § 27.

to the sailor who, at the North Pole, discontinues determining his position in terms of longitude, and prefers to use another system of coordinates free from singularities. Such a procedure is permissible because nature has not determined one normal system for all interphenomena, but only a separate normal system for each interphenomenon.

Let us consider some examples. Using the wave interpretation, we arrive at the question why the whole wave disappears after a flash has been observed on one point of the screen. We eliminate the causal anomaly presented in this description of the interphenomenon by introducing the particle interpretation. The wave then is transformed into a probability, and, instead of the disappearance of a wave, we have the simple statement that although the probability $P(A,C_2)$ of finding a flash on the screen in a point C_2 has a certain positive value, the probability $P(A.C_1,C_2)$ of finding a flash in C_2 after a flash has been observed in C_1, is zero. Instead of the contraction of a wave into a point, we have here the trivial logical fact that probabilities are relative. It was by considerations of this kind that Born was led to the introduction of the statistical interpretation of the waves which originally had been conceived by Schrödinger as waves of electrical density.

Another example where the normal description is given by the particle interpretation is represented by the following consideration. The probability that a particle leaving the source A will pass through either of the slits is the sum of the probabilities that a particle will pass through one of the slits; we may indicate this by the symbolic expression:

$$P(A, B_1 \vee B_2) = P(A,B_1) + P(A,B_2) \qquad (1)$$

(The logistic sign "$\vee$" means "or".) This relation follows from the rules of probability because the particles can go through only one of the holes at a time. It can be tested by observations, as follows: To ascertain the value of the left hand side we count all flashes occurring on the screen when both slits are open; to determine the two values of the right hand side we count all flashes on the screen occurring, respectively, when one of the slits is closed. Since statistics so compiled show that (1) holds, this relation must hold also for the wave interpretation. Here, however, the explanation leads to causal anomalies. We then must assume that whenever a wave leaves A in the direction of B_1 there is another wave leaving simultaneously in the direction of B_2, but that the wave going toward the open slit B_1 will sometimes disappear due to an influence of the closed slit B_2 (namely, in all those cases when, in the corpuscle interpretation, a particle is emitted only in the direction of B_2), and is thus controlled by an influence which represents an action at a distance. The physicist who wishes to explain the well-confirmed relation (1) will therefore prefer the corpuscle interpretation by the use of which he can derive the relation without the assumption of anomalies.

On the other hand, there are questions which only by the use of the wave interpretation can be answered without reference to anomalies. We saw that the corpuscle interpretation of the experiment indicated in figure 5, § 7, involves an action at a distance between the two slits B_1 and B_2. In the wave interpretation this action at a distance is eliminated and replaced by a statement about a phase relation between the waves arriving in B_1 and B_2, which is due to their common origin from the source A. In answering questions of this kind the physicist will therefore prefer the wave interpretation.

Our examples show that it is even preferable to speak, not of the normal system for every interphenomenon, but, of the normal system *for every question* concerning interphenomena. It is the question which determines the normal system, and, relative to the same experimental arrangement, different questions can be asked which require different normal systems. Thus the question concerning the probability relation (1) is asked with respect to an experimental arrangement which, for other questions, necessitates a wave interpretation. When we say "for every question", we mean, of course, that the question is sufficiently limited, and not constructed as an "and"-combination of different questions. With this qualification we can formulate the principle of eliminability as stating: We have no normal system for all questions concerning interphenomena, but we do have a normal system for every such question.

The switching over from one interpretation to another is justifiable, we said, as a means of eliminating causal anomalies. This is, however, its only justification, and it would be incorrect to adduce reasons of another kind. We sometimes read that questions like "what becomes of the wave after a flash on the screen has been observed" or "why does a particle going through the slit B_1 move differently according as the slit B_2 is closed or open" must be forbidden because they are not *adequate* to the respective interpretation. But this word "adequate" means only that the answer to such questions leads to causal anomalies. The occurrence of such anomalies does not make the question, or the answer, unreasonable. If we have decided to use one of the exhaustive interpretations, such questions are not meaningless. We then must become accustomed to the fact that for a given interpretation there are always questions which can only be answered by the assumption of causal anomalies. If we prefer to use in the answer to such questions an interpretation which is free from anomalies, we have good reasons to do so; but we should not believe that the answer so constructed is *the only meaningful* answer, or *the only admissible* answer, or *the only true* answer. All the merits of such an interpretation consist in the fact that it is free from causal anomalies for the interphenomenon considered; but neither does this fact make it more true than others, nor is it a necessary condition of meaning within an exhaustive interpretation. Judgments of this kind are based on a confusion of exhaustive and restrictive interpretations. Only for the latter will the mentioned questions be meaningless; but for such an interpretation the complete description of the experiment which is free from anomalies is meaningless as well. Within an exhaustive description, however, we shall be entitled to state causal anomalies as well as cases of normal causality.

On the other hand, the opposite mistake has been frequently made, i.e., the mistake of saying that we have no true description at all of the interphenomena; and that the duality of interpretations proves that we can construct only pictures of the interphenomena, correct in some features and incorrect in others. Whereas the previously criticized attitude appears dogmatic, the latter attitude must be called too modest. All admissible descriptions are equally true in all their details, including the anomalies. If somebody wishes to call descriptions of interphenomena *pictures* he may do so in order to stress the possibility of choice; but he must not forget that then it is also a use of pictures to say that the tree on the street remains in its place when nobody looks at it. We saw that in this case also we have a choice of descriptions, and that the question of the unobserved tree can be unambiguously answered by inductive evidence only after the postulate of unchanged laws of nature has been introduced. It is only the combination of this postulate with a description of the unobserved object which can be empirically verified. For the interphenomena of quantum mechanics the situation is the same; an unambiguous statement concerning interphenomena can be made only after the postulate of unchanged laws of nature has been introduced, and the combination of a description of interphenomena with this postulate can be empirically verified. It will determine one interpretation for every experiment, but not one for all.

If once the problem of the description of interphenomena has been formulated in this way, it is clear that even in the macrocosm there is no logical need for the existence of a normal system. Imagine that whenever we turn our eyes away from a tree and observe only its shadow we see two equally shaped shadows, whereas when we see both the tree and its shadow there is only one tree and one shadow. We then have the choice of saying either that there are always two trees when we do not look at a tree, or that there is only one unobserved tree for which, however, the known laws of optics do not hold. Therefore one of the two principles defining the normal system in the classical sense must be abandoned in such a case. Imagine, furthermore, observations showing that all changes happening to one of the shadows take place in the other shadow as well; for instance, if we feel a blast of wind and see a certain branch of one of the tree shadows being moved, we see an equal movement in the corresponding branch of the other tree shadow; if the shadow of a bird appears on one of the tree shadows, an equal shadow of a bird appears on the other tree shadow, etc. If, in this case, we want to use a description in which the laws of optics are unchanged, we must assume a duplication of occurrences, which would represent a preëstablished harmony, or an action at a distance, which produces duplicates of occurrences in different places. Since this assumption signifies a causal anomaly, we have, in the case considered, no normal description in the sense of a description satisfying the first of our principles.

A macrocosmic analogy resembling more closely the experiment of figure 5, § 7, can be constructed as follows. Imagine in A a machine gun which is turned irregularly by a machine so that it shoots bullets in all directions in an irregular sequence. Let the diaphragm have a certain thickness so that on the walls of the short slit canals the bullets can be reflected. We then shall observe an irregular distribution of bullets on the screen which may consist of lead so that the bullets are caught in it. Imagine, furthermore, that the bullets move so fast that we cannot see through which hole they pass, and that we have no other means to verify this. Let us now assume the following observations to be made. If both slits are open, the number of bullets hitting the screen is twice as large as in the case when only one slit is open. On certain spots of the screen, however, we find no hits whenever both slits are open, whereas we do find hits on these spots when only one of the slits is open. In such a case we have the same choice of interpretations as in the case of the radiation experiment described above. We may assume that the bullets on their path through the air remain individual particles, but that there is an action at a distance between the two slits; or that the interphenomena consist in waves spreading through both holes and uniting later to form the bullets found in the lead of the screen.

Such analogies may make it clear that there is nothing unimaginable in the state of affairs ascertained by quantum mechanics, and that it is possible to construct macrocosmic models of it. The physics of these models, of course, will be different from that of our actual macrocosm. But should our macrocosm follow a similar pattern, we should after some time get accustomed to it; we should consider it as a matter of course that we could not give one normal description for all interphenomena, and should learn to use, for the purpose of answering a certain question, the description which at least for that question does not involve anomalies. Fortunately our daily world does not show this kind of structure. It is different with the atomic world; quantum mechanics has shown that its structure is of the kind depicted in these analogies.

This means that in the world of atomic dimensions the postulate of unchanged laws of nature cannot be carried through for the totality of interphenomena, and therefore does not determine one interpretation as the normal interpretation of all interphenomena, although it determines one normal interpretation for every interphenomenon. This result must be considered as the most general statement which physics, in its present status, can make about the structure of the physical world. It may seem strange that the physical world cannot be caught in the network of one normal description; that the idea of uniformity of nature, so often claimed to be the ultimate result of science, cannot be extended to include the interphenomena of the world of quanta. We are dealing here, however, with a question which must be answered, not by wishful thinking, but by experimental inquiry. Since physics has come to the result as stated we must now take it seriously, and not palliate it by calling it a breakdown of human capacities for inventing pictures.

In order to be complete in our statement of general properties of the physical world, we must add to this negative result the positive statement formulated in the principle of eliminability. Nature allows us to construct, at least partly, the world of interphenomena in agreement with the laws of phenomena. This fact has consequences of great bearing. One is that we can answer all questions by constructing suitable interphenomena which follow normal laws. Another is that the anomalies incurred with other descriptions can never be used to produce anomalous effects in the world of phenomena. Thus we cannot use the action at a distance existing between the two slits B_1 and B_2 in the corpuscle description in order to send signals from one slit to the other. No such anomalous effect is possible because otherwise it would also occur in the normal description of the experiment; there, however, it is excluded through the normal behavior of the interphenomena. The causal anomalies which we encounter in anomalous descriptions may therefore be considered as pseudo anomalies; they are due to the form of the chosen description and can be eliminated. We see here the far-reaching significance of the principle of indeterminacy: It reveals the discrepancy between the laws of phenomena and the laws of interphenomena as not being of a malignant nature. The causal anomalies have a ghostlike existence; they can always be banished from the part of the world in which we happen to be interested, although they cannot be banished from the world as a whole.

It is this specious character of the causal anomalies which suggests the use of restrictive interpretations. Every exhaustive interpretation states too much so far as it speaks of causal anomalies which have no bearing upon the world of observable phenomena. It may therefore seem advisable to renounce exhaustiveness, and to prefer a restrictive interpretation which is free from statements involving such causal anomalies. We are thus led back to the problem of restrictive interpretations to which we must now turn for closer consideration.

A restrictive interpretation has been introduced by Bohr and Heisenberg. The *rule of restriction* states that only statements about measured entities, i.e., about phenomena, are admissible; statements about unmeasured entities, or interphenomena, are called meaningless. This has the immediate consequence that statements about the simultaneous values of complementary entities cannot be made. The interpretation so introduced is neither a corpuscle nor a wave interpretation; since it leaves the status of interphenomena widely indeterminate, we cannot say whether these interphenomena consist of particles or of waves. It resembles a corpuscle interpretation when the measured entity is the position, since then one rather sharply defined space point is attributed to the entity. But since a statement about the momentum is left open, we do not know whether the entity so determined is a particle. If we consider the whole spectrum of possible momenta as simultaneously realized, the entity might as well be a wave packet. On the other hand, if a measurement of momentum is made, we can consider this value either as the momentum of a particle or as the frequency of a wave; the restrictive interpretation leaves this question open.

Let us consider an example in order to show the rule of restriction at work. The interference experiment of figure 5, § 7, is an arrangement which allows us to measure a frequency, and therefore the momentum of a particle; so, all statements about the position of the particle are ruled out. This means that not only a statement of the kind "the particle went through slit B_1" is inadmissible, but that even a statement of the form "the particle went either through slit B_1 or through slit B_2" is forbidden. It is clear that this rule works. We then cannot say that *if* the particle went through slit B_1 it must have been influenced by the existence of slit B_2; the clause with "if" belongs to the forbidden domain. The causal anomaly has therefore disappeared from the domain of admissible statements. The rule of restriction, like a surgical operation, cuts off all unhealthy parts of quantum mechanical language. Unfortunately, like all such operations, it also cuts off some sound parts. Thus it is hard to abandon a statement like the one concerning the particle's going through one slit or the other. All that can be said against this statement is that it leads to undesired consequences.

It should be realized that the elimination of causal anomalies is the only justification of the restrictive interpretation of Bohr and Heisenberg. If it were possible to construct an exhaustive interpretation free from causal anomalies, no one would question the legitimacy of the definitions used in such an interpretation, even if the relation of indeterminacy should hold and make it impossible to replace these definitions by verifiable statements. Thus, if the experiment should show that the interference pattern on the screen in figure 5, § 7, were equal to the superposition of the two interference patterns resulting when first one slit is open and then the other, no one would doubt that the particles went either through one slit or the other, although, because of the disturbance by the observation, it could not be known through which slit an individual particle went.[2] It then would be considered a reasonable supplementation of observable data to say that a percentage of the particles given by $P(A,B_1)$ went through slit B_1, and a percentage given by $P(A,B_2)$ went through slit B_2. When the restrictive interpretation rules this statement out, it does so only

[2] The disturbance by the observation, in this case, may be of the same kind as is actually assumed in quantum mechanics: If we observe a particle passing through a slit, it will be pushed off its path by the light ray. It is true that if in this case the particle is not observed at the slit where the observation is made, we know that it passed through the other slit; therefore we have here a knowledge of the particle's position without having disturbed its path. But this knowledge is acquired by inference, not by observation, since what is observed is the absence of the particle at one slit, not its passage at the other. Our fictitious experiment therefore represents a case where, although the observation disturbs, a normal supplementation of observable data by interpolation can be given. This is possible because, in this case, the interference pattern on the screen is of such a kind that it permits us to consider the motion of the particle independent of what happens at a slit through which the particle did not pass.

because the experiment shows that the interference pattern on the screen is not a superposition of the two individual patterns and thus makes it necessary to assume that the path of a particle passing through one slit will depend on whether the other slit is open or not; i.e., it does so because of the causal anomalies derivable from the statement. We often read that the principles of empiricism laid down in the verifiability theory of meaning lead to the rule of restriction formulated by Bohr and Heisenberg. This argumentation is incorrect. Meaning is a matter of definition, and various definitions of meaning can be given; all that can be asked by the philosopher is: Which are the consequences to which a given definition of meaning leads?[3] The restricted meaning of Bohr and Heisenberg's interpretation has the advantage that it eliminates causal anomalies; this is a strong argument in its favor, but it is the only argument. The exhaustive interpretations given in the corpuscle and the wave interpretation are equally compatible with the principles of empiricism if they are conceived as being based on definitions.

It must be added that even a restrictive interpretation is not free from definitions. What we call *phenomena* are certainly not immediate objects of observations; they are inferred from observations by indirect methods (cf. p. 21). These inferences contain a definition, and we shall give in § 29 the exact form of this definition. The logical difference between the physics of phenomena and the physics of interphenomena is therefore a matter of degree; the latter contains more definitions than the former. It is a question of volitional decision which of these two systems we prefer; none can be said to be completely restricted to observational data.

Whereas the Bohr-Heisenberg interpretation uses a *restricted meaning*, we can construct a second form of restrictive interpretation which uses a *three-valued logic*. Ordinary logic is written in terms of the two truth values *true* and *false*. To these we shall add, for the purposes of quantum mechanics, a third truth value which we call *indeterminate*. Statements about unobserved entities then are considered as meaningful; but they are neither true nor false, they are indeterminate. This means that it is impossible to verify or falsify such statements.

The interpretation so constructed is superior to the interpretation by a restricted meaning because it possesses a system of rules which makes it possible to connect statements about unobserved entities with statements about observed entities and thus to manipulate all these statements by means of strictly logical operations. It can be shown that owing to these rules statements expressing causal anomalies always will obtain the truth value of indeterminacy, and therefore can never be asserted as true. On the other hand, part of the statements about unobserved entities are even retained as true, or considered as true in a somewhat wider meaning; they cannot be used, however, for the derivation of causal anomalies because the rules of three-valued logic, differing from those of two-valued logic, make such derivations impossible. We shall show, for instance, that the statement: "The particle passes either through slit B_1 or B_2" need not be completely abandoned, but can be retained in a somewhat wider sense, whereas a statement about an action at a distance between the slits is not derivable (cf. § 33). Three-valued logic is therefore the adequate form of quantum mechanics once the decision for the use of a restrictive interpretation has been made.

[3] Cf. the author's *Experience and Prediction* (Chicago, 1938), chap. I.

We have, therefore, good reasons to say that the language of quantum mechanics is written in terms of a three-valued logic. We must not forget, however, that the subject matter of a science, without the addition of further qualifications, does not determine a particular form of logic. Quantum mechanics can be constructed in the form of a two-valued logic; this is demonstrated by the existence of exhaustive interpretations. Only when we introduce the postulate that causal anomalies be not derivable are we obliged to turn to a three-valued logic. It is in this form that the structure of the subject matter expresses itself in the structure of its language. When we apply the same postulate to classical physics we arrive at a two-valued logic. The nature of quantum mechanical occurrences is of such a kind that statements of causal anomalies can be eliminated from the domain of true statements only if a three-valued logic is used; this is the form in which we must express the causal structure of the microcosm.

The three-valued language appears adequate to quantum mechanics because the causal anomalies, as formulated in exhaustive interpretations, appear to be superfluous complications; they need not be taken into account so far as predictions of observable phenomena are concerned. In consideration of this fact a restrictive interpretation will appear natural to one who works in quantum mechanical problems. To this logical insight, habit will add its leveling influence; the desire to ask questions transcending the limits of the restrictive interpretation will disappear; and the restrictive form of quantum mechanics may finally seem to answer everything that can be reasonably asked. This attitude will be as much help to the physicist as the above-mentioned switching over from one interpretation to the other to which he resorts in exhaustive interpretations. Similarly, a man in a world in which bullets behaved as unreasonably as electrons, might learn to restrict his questions in such a way as to obtain only reasonable answers. We should not forget, however, that such an attitude, advisable as it may be, means making a virtue of necessity. When we exclude some kinds of statements about interphenomena and admit others, we have no other reason to do so than that the first statements lead to causal anomalies, whereas the others do not.

Our final judgment concerning the logical significance of restrictive interpretations can therefore be stated as follows. The peculiar form of the causal structure of the microcosm, visible in the causal anomalies of the exhaustive interpretations, finds a corresponding expression in the rule of restriction, or the existence of indeterminate statements, in the restrictive interpretations. In other words: The physical status of the quantum mechanical world, expressed through a restrictive interpretation, is the same as the status expressed through exhaustive interpretations with causal anomalies which can be transformed away locally. The restrictive interpretations do not *say* the causal anomalies, but they do not *remove* them.

The causal anomalies cannot be removed because they are inherent in the nature of the physical world. The *principle of indeterminacy* formulates only one part of this nature; it states that it is impossible to *verify* certain statements about interphenomena. To this is added, by the system of quantum mechanics, another principle which we have called the *principle of anomaly*. It states that no definition of interphenomena can be given which satisfies the requirement of a normal causality; it therefore maintains the impossibility of a normal supplementation of the world of phenomena by interpolation. This includes the restrictive interpretations, since they do not establish a normal causality either.

The limitations of scientific interpretations of the world of quanta, expressed in these two principles, must not be considered as limitations of the power of the human intellect. It is not human ignorance, nor lack of knowledge, which leads to the conditions imposed upon descriptions of the physical world expressed in the laws of quantum mechanics. It is positive knowledge, deep insight into the nature of the atomic world, which constitutes the basis of this strange network of rules, formulated as rules limiting descriptions, but expressing implicitly rules holding for all physical occurrences. Beneath the disguise of a theory of physical knowledge we discern the outlines of a physical world different from what centuries of scientific research had dreamed it to be, but nevertheless demanding recognition as the world of reality.