Three weird facts of quantum mechanics, explained: Bohr, Schrödinger, Bell, and EPR

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We are all familiar, at least intuitively, with the procedure used to "do physics" in the macroscopic world. We take an object, subject it to known forces (say gravity or friction, or both), start it off with a particular position and velocity and follow its trajectory. When we measure the dynamical properties (say position or energy) of that object at a later time, we compare those measurements with our theoretical prediction using Newton's Laws. Newton's Laws directly predict what those quantities should be at that later time, so the comparison is straightforward. The microscopic laws of physics, quantum mechanics, aren't so simple.

Bohr 1913

One of the first "weird facts" of quantum mechanics came out of Niels Bohr's attempt to understand the atom in 1913. Only two years before, in 1911, Rutherford (with the help of Geiger and Marsden) had shown that atoms were composed of a tiny, massive nucleus with positive charge surrounded by electrons of negative charge. Hydrogen, with only one electron, was the simplest of these.

From classical electrodynamics it was predicted that the combination of a single electron orbiting a proton (as the hydrogen nucleus was called) in planetary fashion was not stable. The accelerating electron radiated electromagnetic waves, losing energy, and spiralled into the proton in about 10^{-11} s. Of course, as there is plenty of hydrogen around, this prediction must be wrong. Bohr therefore hypothesized that the electron could only occupy a "stationary state" in which it did *not* spiral into the proton. He chose what would later be called "energy eigenstates" in order to correctly predict the spectrum of hydrogen (i.e., specifically the Balmer formula).[1]

In modern notation, a single particle in a stationary state (in this example, an electron in a hydrogen atom) can be written

$$|\psi\rangle = |\psi_n\rangle. \tag{1}$$

That is, the state of the electron, $|\psi\rangle$, is a stationary state given by the quantum number n, $|\psi_n\rangle$, and has energy

$$E_n = \frac{-13.6 \text{ eV}}{n^2}, \tag{2}$$

[1] In addition, he had to make the crucial assumption that when the electron was far from the proton, the radiation would agree with the classical electrodynamic prediction. This was the first use of the so-called "correspondence principle."

where n can take on the discrete integer values 1, 2, ..., ∞. The Dirac "ket" notation $|\rangle$ is used to remind you that we are not really talking about a function, but simply labeling the state that the electron is in. The stationary states (which will turn out to be solutions to Schrödinger's equation) can also be written simply as $|n\rangle$. Of course, there are really four quantum numbers that completely describe the state of the electron in a hydrogen atom, n, ℓ, m_ℓ, and m_s, but for our purposes, all the weird physics can be had with only one quantum number.

The hydrogen spectrum is obtained by assuming that the electron can "jump" from one stationary state to another stationary state (with a lower energy) and to conserve energy must emit a photon with an energy equal to the energy difference between the two states

$$h\nu = \Delta E = E_m - E_n, \tag{3}$$

where the "quanta of light" assumption of Planck and Einstein has been used.[2] This predicted the hydrogen spectrum perfectly, but physicists were confused about what was *meant* by a state. And how did the electron know when to jump, and to what other state to jump to? Why were these the only states that were stationary? These are questions that we still have today, but we answer them in a probabilistic fashion. As Abraham Pais puts it

> At a moment which cannot be predicted an excited atom makes a transition to its ground state by emitting a photon. Where was the photon before that time? It was not anywhere; it was created in the act of transition.... Is there a theoretical framework for describing how particles are made and how they vanish? There is: quantum field theory. It is a language, a technique, for calculating the probabilities of creation, annihilation, scatterings of all sorts of particles: photons, electrons, positrons, protons, mesons, others ...[3]

Weird Fact #1 This, then, is our first "weird fact." In certain situations electrons (particles, in general) must occupy stationary states with a definite energy (completely contrary to the predictions of classical mechanics) and they emit or absorb photons when they transition between these states. They are not allowed to have any other energy.

Schrödinger 1926

The second weird fact of quantum mechanics arose with Erwin Schrödinger's equation (which is just Axiom 3 — see the Appendix — where the observable A is the energy E). He was able to derive Bohr's result from solving his eponymous equation

$$H|\psi_n\rangle = E_n|\psi_n\rangle, \tag{4}$$

which is an "eigenvalue equation," and Bohr's energies turned out to be just the energy eigenvalues. Since this is a differential equation, there are many possible solutions, one for each value of n in the set $n = 1, 2, 3, \ldots, \infty$. As is true for all linear differential equations, the most general solution is a linear combination of all viable solutions, and

[2] This assumption is related to the so-called "axioms" of quantum mechanics. We will use a few of them during our story, but for completeness, the full set of axioms is listed in a coherent fashion at the end.

[3] Pais, *Inward Bound*, pages 324-5.

this means that the electron in a hydrogen atom doesn't have be in just *one* state, as assumed in (1), but can be in a "superposition state"

$$|\psi\rangle = \sum_n c_n |\psi_n\rangle, \tag{5}$$

that is, many states at once. (This, of course, is counter to our everyday experience in which we never find objects in superposition states, but always find them in a single state.)

Weird fact #2 Again, contrary to the predictions of classical physics, and counter to our everyday experience, electrons (particles in general) are allowed to be in superposition states, in which they can take on the characteristics of each of the stationary states (for example, the energy) with a certain probability.

The measurement process

This weirdness manifests itself when we talk about the measurement process. If an electron is in such a superposition state as depicted by (5), the axioms of quantum mechanics state that if you measure the energy, you won't obtain just *any* value for the result of your measurement, but the only possible values will be E_n. In addition, the probability of measuring a certain energy, say E_n, is given by the coefficient squared, $|c_n|^2$. Since the probability of measuring *any* energy must be unity, there must be a restriction $\sum_n |c_n|^2 = 1$. The way it is usually put is as follows. When measuring the energy of one electron, there is no way of knowing which energy you will obtain, but after measuring the energies of many *identically prepared* electrons (this set is called an ENSEMBLE), the probability distribution of the different energies obtained should be as given above. Even though Schrödinger had developed the equation that led to this formalism, he thought that the situation was ridiculous. To show this, he came up with a thought experiment (involving the infamous cat) to highlight how strange this is. In his own words:

> One can even set up quite ridiculous cases. A cat is penned up in a steel chamber, along with the following device (which must be secured against direct interference by the cat): in a Geiger counter there is a tiny bit of radioactive substance, so small, that perhaps in the course of the hour one of the atoms decays, but also, with equal probability, perhaps none; if it happens, the counter tube discharges and through a relay releases a hammer which shatters a small flask of hydrocyanic acid. If one has left this entire system to itself for an hour, one would say that the cat still lives if meanwhile no atom has decayed.[4]

That is, imagine a cat in an opaque box. Also in this box is a vial of poison gas, and if this vial breaks, the cat will die. Also in this box is a radioactive atom, arranged such that if the atom decays, it will trigger a small hammer to break the vial, killing the cat. Now then, after one hour, the cat is in a superposition state (in this case there are only two states, i.e., $n = 1, 2$,

$$|\psi_{cat}\rangle = c_{alive}|\psi_{alive}\rangle + c_{dead}|\psi_{dead}\rangle, \tag{6}$$

[4]This quote is from a three-part paper, E. Schrödinger, "Die gegenwärtige Situation in der Quantenmechanik," *Naturwissenschaften*, **23** 807-812; 823-828; 844-849 (1935). Translated by John D. Trimmer.

where we have chosen the radioactive substance so that the values of c_{dead} and c_{alive} are each equal to $1/\sqrt{2}$, which means that the probabilities of the cat being alive or dead are each 1/2.

Now, even though Schrödinger — and many others — think this is weird, it is an observational fact that these rules predict accurately the outcomes of subatomic experiments. How these rules translate to the macroscopic world is a topic that I won't cover here, but has been the subject of much controversy over the past one hundred years.

Bell 1956

Our third weird fact comes into play when we consider two identical particles simultaneously, for example two electrons in an atom (now Helium). Following our previous rules, you might think that each electron must either occupy a stationary state, or perhaps occupy a superposition state. In general, this is true, but the presence of one electron affects the other electron — specifically, one electron modifies the potential energy experienced by the second electron, so the stationary states (and their energies) are modified from the one-electron case.

But more important, there really is only *one* wave function, but it depends on properties of *both* electrons. That is, we can write this wave function as a "product" state

$$|\psi\rangle = |\psi_n^a\rangle|\psi_m^b\rangle, \tag{7}$$

which means that particle a is in state n and particle b is in state m. However, in quantum mechanics, identical particles are truly identical. In classical physics, two white billiard balls, while they look the same, can be distinguished by looking closely at any possible scuffs or scratches. But two electrons, for example, are *indistinguishable*, and no one, not even God, can tell them apart. This means that we don't know whether particle a is in state n and particle b is in state m or vice versa. Therefore (7) is not an accurate representation of the state of the system. We must allow for the possibility that the particles are switched. This means that the wave function must be written as

$$|\psi\rangle = |\psi_n^a\rangle|\psi_m^b\rangle + |\psi_m^a\rangle|\psi_n^b\rangle, \tag{8}$$

which is called an "entangled" state. In reality, the wave function must be normalized, and we must write

$$|\psi\rangle = \alpha|\psi_n^a\rangle|\psi_m^b\rangle + \beta|\psi_m^a\rangle|\psi_n^b\rangle, \tag{9}$$

where $|\alpha|^2$ is the probability of finding particle a in state n and particle b in state m and $|\beta|^2$ is the probability of finding particle a in state m and particle b in state n. Of course, normalization requires that $|\alpha|^2 + |\beta|^2 = 1$. And since either case is equally probable, $|\alpha|^2 = |\beta|^2 = 1/2$. There are two choices: either $\alpha = 1/\sqrt{2}$ and $\beta = 1/\sqrt{2}$, which is called a symmetric wave function, or $\alpha = 1/\sqrt{2}$ and $\beta = -1/\sqrt{2}$, which is called an anti-symmetric wave function. The "spin-statistics theorem" in quantum mechanics states that identical boson wave functions must be symmetric and identical fermion wave functions must be anti-symmetric.

Weird fact #2 When the system consists of two identical (i.e., indistinguishable) particles, they are in a superposition state together, which is called an entangled state. It is not known which particle is in which state, but it is equally likely to be either.