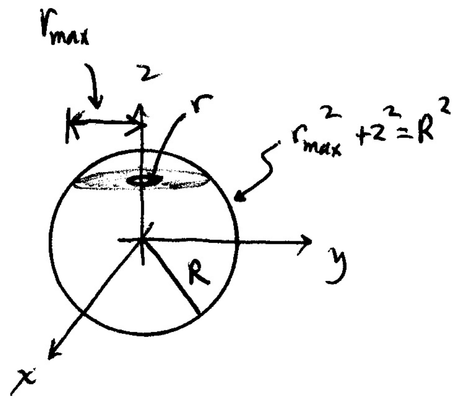
What is the moment of inertia of a uniform sphere of mass M and radius R about an axis that passes through the center of the sphere? The general equation for the moment of intertia is

$$I = \int r^2 dm,$$

where r is the perpendicular distance between the mass dm and the axis of rotation.

For the case of a sphere, we can break up the sphere into mass elements that are in the shape of rings of radius r, centered on the z axis, and parallel to the x-y plane, as shown in the figure.



The mass of this ring is $dm = \rho dV = \rho\, 2\pi r\, dr dz$. We will need to integrate over the variable r from 0 to $r_{\text{max}} = \sqrt{R^2 - z^2}$, and then integrate over z from $-R$ to R. The integral for the moment of inertia is then

$$I = \int r^2\, dm = 2\pi\rho \int\int r^3\, dr dz,$$

where the integral over r is just

$$\int_0^{r_{\text{max}}} r^3\, dr = \frac{(R^2 - z^2)^2}{4}.$$

The integral over z is then

$$I = 2\pi\rho \int_{-R}^{R} \frac{(R^2 - z^2)^2}{4} dz = \pi\rho \int_0^R (R^4 - 2R^2 z^2 + z^4)\, dz = \frac{8}{15}\pi\rho R^5.$$

Finally, noting that $\rho = \frac{M}{\frac{4}{3}\pi R^3}$, I get

$$I = \frac{2}{5} M R^2.$$