Appendix B

Blackbody Radiation

It is a highly important task to find this function. — Gustav Kirchhoff, referring to the blackbody spectrum

An object that reflects no light (electromagnetic radiation) that falls on it, but absorbs it all, is called a perfect *blackbody*. But, if it absorbs energy, it must also radiate energy if it is not to heat up indefinitely. The form of this radiation is crucial in many fields of physics and astrophysics, but measuring it and predicting it theoretically took a long time. Because the nature of the atom was not revealed until the early 20th century, the "laws" regarding thermal radiation were determined phenomenologically (i.e., from experimental observation). Gustav Kirchhoff (1824-1887) was one of the premier investigators, and his laws are now taught in introductory physics courses. One version states that the properties of the radiation depend only on the object's temperature T and not on the shape, size, or composition of the blackbody. Specifically, the following three properties were well known before 1900.

1. A blackbody emits electromagnetic radiation at all wavelengths (i.e., in all regions of the spectrum).

2. The total power emitted by a blackbody is proportional to T^4, where T is the temperature (in Kelvin) of the blackbody. (This is the Stefan-Boltzmann law.)

3. The wavelength at which a blackbody emits maximum power is inversely proportional to its temperature $\lambda_{max} \propto \frac{1}{T}$. (This is Wien's law.)

The first property means that blackbodies emit a "continuum" of radiation rather than the discrete spectral lines emitted by gaseous elements that are heated (see Appendix A).

Radiative transfer

Because objects emit different amounts of energy in different regions of the spectrum, we have to keep track of that radiation as a function of wavelength λ (or of frequency ν). The quantity $R(\lambda)d\lambda$ is defined to be the energy emitted per unit time per unit area from the surface of a blackbody, between the wavelengths λ and $\lambda + d\lambda$. $R(\lambda)$ is called the *spectral radiancy,*[1] and it has SI units W/m^3.

The "energy per unit time per unit area" is called the *energy flux.* Flux is a concept that is used in many different physics disciplines, particularly those dealing with transport of some quantity. In fluid dynamics, for example, the quantity ρv (mass density$\times$velocity) is called the mass flux, and is nothing but the mass per unit time that passes through a unit area. In laminar flow, the quantity $\rho v A$ remains constant along a streamline, where A is the cross sectional area of the streamline. This is called the continuity equation. The field of radiative transfer, while dealing with energy flux, has developed its own terminology, which we will follow.

The three experimental properties above imply three important mathematical properties of $R(\lambda)$, listed below.

1. Since the blackbody emits at all wavelengths, $R(\lambda) > 0$ for all λ. In addition, it was known that $R(\lambda) \to 0$ as both $\lambda \to 0$ and $\lambda \to \infty$.

2. The total power emitted per unit area from the blackbody's surface is just an integral of $R(\lambda)$ over all wavelengths. That is

$$\mathsf{R} \equiv \int_0^\infty R(\lambda) d\lambda = \sigma T^4, \tag{B.1}$$

where σ is the proportionality factor. This is called Stefan's Law, because it was first deduced by Jozef Stefan in 1879 from experimental observations. It is also called the Stefan-Boltzmann law because Ludwig Boltzmann derived it theoretically in 1884, and therefore $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4} is called the Stefan-Boltzmann constant. R is called the "radiancy,"[2] and has units W/m^2.

Our Sun is not a perfect blackbody, but is a very good approximation. Its surface temperature is about $T_\odot = 5780$ K, and therefore emits an energy flux $\mathsf{R} = \sigma T_\odot^4 = 6.33 \times 10^7$ W/m^2. Since the radius of the Sun is $R_\odot = 6.96 \times 10^8$ m, the radiant flux, or "luminosity," of the the Sun is $L_\odot = (4\pi R_\odot^2)\sigma T_\odot^4 = 3.85 \times 10^{26}$ W.

3. $R(\lambda)$ has exactly one maximum. That is, $dR(\lambda)/d\lambda = 0$ definines λ_{max}, which is found to be given by

$$\lambda_{max} T = b, \tag{B.2}$$

where $b = 2.898 \times 10^{-3}$ m K. This is called Wien's displacement law, which he derived in 1893. Note that b has dimensions of length$\times$temperature, so that 'm K' is 'meters$\cdot$Kelvin,' not 'milli-Kelvin.'

[1] Unfortunately, there is no standard terminology in this field. The quantity $R(\lambda)$ is sometimes written as R_λ or E_λ, and it is sometimes called radiancy, or monochromatic irradiance.

[2] Again, terminology varies, and R is sometimes called intensity or irradiance. The product $\mathsf{R}A$, where A is the surface area of the blackbody, is called the "radiant flux."

Again for the Sun, Wien's law predicts that it emits its maximum power at the wavelength $\lambda_{max} = b/T_\odot = 501$ nm, very near the center of the visible spectrum. Evolutionary biologists suggest that our eyes developed sensitivity in this spectral region simply because there is so much light available.

In 1860, Kirchhoff, speaking of $R(\lambda)$, said, "It is a highly important task to find this function." A laudable goal, but how to achieve it? Accurate measurements of spectral radiancy over a large range of wavelengths are needed, not just the visible region of the spectrum. Ångstrom was able to measure visible wavelengths to a precision of 10^{-5}, but because absolute intensities are more difficult, it was not until the early 1900s that measurements became precise enough to compare with theoretical predictions.

Cavity radiation

A simple technique to compare theoretical predictions with experimental measurements is to consider *Hohlraumstrahlung*, or cavity radiation. As Kirchhoff put it

> "Given a space enclosed by bodies of equal temperature, through which no radiation can penetrate, then every bundle of radiation within this space is constituted, with respect to quality and intensity, as if it came from a completely black body of the same temperature."

If you cut a hole in the cavity wall, there will be light emitted from that hole, and as Kirchhoff contends, that is "blackbody radiation."

It turns out that while it is straightforward experimentally to measure the spectral radiancy $R(\lambda)$ from the hole, it is much simpler to theoretically calculate the energy density of the radiation within the cavity. It can be shown (see Problem 2) that the relation between the two quantities is

$$R(\lambda) = \frac{c}{4}\, u(\lambda), \tag{B.3}$$

where $u(\lambda)d\lambda$ is the energy per unit volume between the wavelengths λ and $\lambda + d\lambda$ [the SI units of $u(\lambda)$ are obviously J/m^4], and, similar to R, the total energy density U is a sum over all wavelengths

$$U \equiv \int_0^\infty u(\lambda) d\lambda. \tag{B.4}$$

To understand the relation between R and U, another analogy with fluid dynamics is useful. As discussed above, the mass flux ρv is just the mass density ρ times the flow velocity v. Here, the energy flux R is just the energy density U times the velocity c, or $\mathsf{R} = Uc$. This works for $R(\lambda)$ and $u(\lambda)$ just as it does for R and U, because all wavelengths travel at the same speed c. However, $\mathsf{R} = Uc$ only holds when all the energy is also traveling *in the same direction*, as in laminar fluid flow, where all the mass is traveling in the same direction. Inside our cavity, electromagnetic waves are traveling in all different directions, and only those that happen to be heading out of the hole (and would reflect back into the cavity if there were no hole) contribute to $R(\lambda)$. This geometry is what is responsible for the factor of 4 in Eq. (B.3).

Early theoretical attempts to determine $u(\lambda)$

Lord Rayleigh (1900) and James Jeans (1905), using classical arguments, derived a formula for the energy density $u(\lambda)$

$$u_{RJ}(\lambda) = \frac{8\pi}{\lambda^4}\, kT. \tag{B.5}$$

This classical prediction agreed with the experimental measurements that had been made up until that time, but an obvious problem was that the integral over all wavelengths diverged, Eq. (B.4). The difficulty appeared at short wavelengths, and was therefore called the "ultraviolet catastrophe."

In 1905, with the help of Einstein, Rayleigh added an *ad-hoc*[3] exponential factor to get rid of the ultraviolet catastrophe

$$u_R(\lambda) = \frac{8\pi}{\lambda^4}\, kT\, e^{-c_2/\lambda T}. \tag{B.6}$$

This forces the integral $\int_0^\infty u(\lambda)d\lambda$ to be finite, and, as we shall see below, agrees with the correct function of Planck in the small wavelength limit if $c_2 = hc/k$.

In 1900, Max Planck derived a spectral formula by assuming that within the cavity, the electromagnetic waves and the walls could only exchange energy in discrete amounts $h\nu$.[4] He realized that this suggestion was not physical, but it was the only way that he was able to obtain a formula in agreement with experiment. The spectrum that Planck derived was

$$u_P(\lambda) = \frac{8\pi}{\lambda^4}\left(\frac{hc}{\lambda}\frac{1}{e^{hc/\lambda kT} - 1}\right), \tag{B.7}$$

which I've written in a suggestive way. The Planck function effectively replaces kT in the Rayleigh-Jeans formula with a more complicated function of λ and T, and Planck showed (Problem 5) that it agreed with the Rayleigh-Jeans formula in the long wavelength limit. Also, it did not diverge at small wavelengths, i.e., there was no ultraviolet catastrophe.

During the period 1900-1905, it was not clear which of the theoretical predictions, Eqs. (B.5)-(B.7) was correct. They all agreed with each other (and with experiments) in the limit of long wavelengths,[5] but the experiments were not precise enough to distinguish between them in the short wavelength limit. It was only after 1905, when Einstein used the same quantization ($E = h\nu$) to explain the photoelectric effect, that consensus started to back Planck's function.

Properties of the Planck function

The Planck function, Eq. (B.7), certainly satisfies the three experimental properties listed above. Figure B.1 shows $u(\lambda)$ for three values of the temperature T. It is clear that as

[3] ad-hoc, adj., made with a particular purpose, without reference to wider application.

[4] In the case of a neutral gas confined in a box, Boltzmann had already shown that during collisions with the walls the molecules exchange momentum (and energy) with the walls. But in order for this process to predict the ideal gas law, he showed that the energy exchanged can take on any value, i.e., a continuous set of values.

[5] Two sets of measurements had confirmed the blackbody spectrum in the infrared: Lummer and Pringsheim looked between 12-18 μm, and Rubens and Karlbaum looked between 30-60 μm.

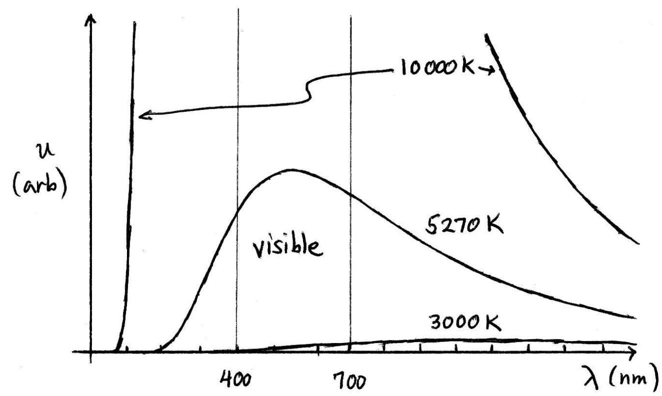


Figure B.1: Energy density $u_P(\lambda)$ given by the Planck function for three values of the temperature, 3000 K, 5270 K, and 10 000 K. These temperatures were chosen for the following reasons. A glowing metal can have a temperature on the order of 3000 K, and it appears red due to the fact that the spectral radiancy at 700 nm is greater than that at 400 nm; the Sun's surface temperature is near 5270 K — which has a peak intensity at $\lambda_{max} = 550$ nm, exactly in the center of the visible spectrum — and it appears white since the spectral radiancy is approximately flat in the visible region; and a very massive star can have a surface temperature near 10 000K, which makes it appear blue (even though the scale does not allow us to see the curve for this temperature, Wien's law tells us that its maximum must be at about 275 nm, well in the ultraviolet). Note: Rescaling the ordinate results in a plot of spectral radiancy $R(\lambda)$.

the temperature increases, the wavelength of maximum intensity decreases and the total intensity (i.e., the area under the curve) increases. Looking at the strength of the Planck function in the visible region of the spectrum, and the relative strengths in the red and blue regions, it is clear that cool blackbodies appear red (they emit more red than blue), hot blackbodies appear blue, and "medium" blackbodies (i.e., the Sun) appear white — the spectral radiancy is approximately flat across the visible spectrum.

An interesting and useful mathematical property of $u(\lambda)$, as well as $R(\lambda)$, is that of *self-similarity.* Self-similarity is commonly encountered in fractal theory where a portion of an object looks the same as the entire object. In other words, an object is self-similar if it looks the same on all scales, large and small. A *function*, on the other hand, is self-similar if you can express it as a function of only one variable. For example, I can rewrite the Planck function in the following way

$$\frac{u(\lambda)}{T^5} = \frac{8\pi hc}{(\lambda T)^5} \frac{1}{e^{hc/k(\lambda T)} - 1}. \tag{B.8}$$

Notice that the right-hand-side is a function of only the combination λT, not of λ and T separately, with all other terms being constant. This means that if I know the form of the curve for one temperature, I can determine it for another temperature in the following manner. A plot of u versus λ for one particular value of the parameter T_1 can be transformed into a plot for another value of T_2 by shrinking the abscissa axis by a factor equal to the temperature ratio T_2/T_1 and stretching the ordinate axis by a factor $(T_2/T_1)^5$.

Problems

1. What are the dimensions and SI units of "radiant flux."

2. Show that Eq. (B.3) holds. HINT: see the discussion after Eq. (B.4).

3. For the Rayleigh-Jeans energy density in Eq. (B.5), evaluate the integral $\int_a^b u_{RJ}(\lambda)d\lambda$. Which limit, $a \to 0$ or $b \to \infty$, causes the integral to diverge?

4. Rayleigh added an exponential factor $e^{-c_2/\lambda T}$ to account for the high-frequency behavior of the measured blackbody radiation. His spectral radiance was therefore

$$R(\lambda) = \frac{8\pi ckT}{4\lambda^4}\, e^{-c_2/\lambda T},$$

and this was called the "Rayleigh Law." Assuming that $c_2 = hc/k$, (a) calculate σ [i.e., evaluate the integral $\mathsf{R} = \sigma T^4 = \int R(\lambda)\, d\lambda$], and (b) calculate the constant b in Wien's law, i.e., determine the maximum of the function. How well do these agree with the similar parameters calculated from the correct Planck law?

5. Show explicitly that the Planck function, Eq. (B.7), agrees with the Rayleigh-Jeans function, Eq. (B.5), in the limit where $\lambda \to \infty$. Also show explicitly that the Planck function does not diverge in the $\lambda \to 0$ limit. That is, determine an approximation that is correct in this limit.

6. Locate the maximum of the Planck function and obtain a formula for b in terms of other fundamental constants.

7. Evaluate Eq. (B.1) using Planck's function in Eq. (B.7). Obtain a formula for the Stefan-Boltzmann constant σ in terms of other fundamental constants. This theoretical prediction of a quantity that had previously only been experimentally measured was one of the great successes of Planck's theory.

8. Derive the expression for the spectral radiancy as a function of frequency $R(\nu)$ from a knowledge of $R(\lambda)$, using the Planck function, given that $R(\nu)d\nu = -R(\lambda)d\lambda$. This last equation simply states that the energy emitted between λ and $\lambda + d\lambda$ must be the same as that emitted between ν and $\nu + d\nu$. But, since the the two variables are related by $c = \lambda\nu$, the derivative $d\lambda/d\nu$ is needed. Finally, the negative sign just ensures that both $R(\lambda)$ and $R(\nu)$ are positive.

Solution The derivative is

$$\frac{d\lambda}{d\nu} = -\frac{c}{\nu^2}$$

so that $R(\nu) = R(\lambda)c/\nu^2$. For the Planck function I obtain

$$u(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT} - 1},$$

and for the Rayleigh-Jeans function I obtain

$$u(\nu) = \frac{8\pi\nu^2}{c^3}\, kT.$$

Appendix C

The Photoelectric Effect

It was in 1905 that Einstein made the first coupling of photo effects with quantum theory by bringing forward the bold, not to say the reckless, hypothesis of an electro-magnetic light corpuscle of energy $h\nu$, which energy was transferred upon absorption to an electron. – R. A. Millikan, 1916

Heinrich Hertz (1857-1894) studied the spark discharges that occurred between two metal surfaces when they were held at different electric potentials, and in 1886 was the first to create and detect the electromagnetic waves that had been predicted by Maxwell in 1865. In addition to the waves, he noticed that charged objects would easily lose their charge when illuminated by light. Then, in 1887, in a series of experiments with spark discharges, he found that not only a large potential difference between the two surfaces (now called an anode and a cathode) was able to cause sparks, but ultraviolet light can *also* produce sparks. Figure C.1 shows a schematic of Hertz's experimental setup. He had created what we now would call a "vacuum diode." The vacuum chamber was important, because air between the two metal plates inhibits current flow unless the electric field between the plates exceeds the "breakdown" potential of air (which is about 3×10^6 V/m). In this case, the electric field ionized the air and it becomes a good conductor—this is the physical mechanism of lightning.

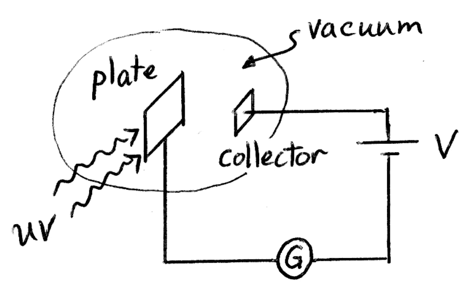


Figure C.1: The 1887 experiment by Hertz that first detected the photoelectric effect. After placing a metal plate and a collector in a vacuum chamber, he biased the plate negative with respect to the collector. If electrons can "jump" off the plate, cross the vacuum gap and reach the collector, a current will flow as measured by the galvanometer G.

If the space is evacuated, however, there are no air molecules for the electrons to collide with, and the only impediment to current flow is getting the electrons to leave the plate in the first place. Hertz realized that there were two possible methods that could induce the electrons to leave the metallic plate:

1. Heat up the plate, and the electrons would leave via *thermionic emission*.

2. Illuminate the plate, and the electrons would leave via *photo-emission*.

Both mechanisms give some of the electrons enough energy to overcome the binding energy of the metal, also known as the "work function," ϕ. That is, ϕ is just the minimum energy necessary for an electron to escape from the metal. (Typically, work functions for metals are between 1 eV and 10 eV.) That a *minimum* energy exists makes sense because removing a negatively charged electron from a neutral metal plate results in a positively charged metal plate. The resulting opposite charges attract, and the electron is pulled back toward the metal plate, *unless* an external force does enough work to overcome that attractive force. As Einstein put it

> Energy quanta penetrate into the surface layer of the body, and their energy is transformed, at least in part, into kinetic energy of the electrons. The simplest way to imagine this is that a light quantum delivers its entire energy to a single electron; we shall assume that this is what happens.[1]

Cutoff wavelength

One of the crucial experimental results, which was a key clue in determining that the underlying physical mechanism is quantum in nature, was the existence of a cutoff wavelength. If you, as the experimenter, vary the wavelength λ of the light incident on the plate, while keeping the potential bias V constant, the current I measured by the galvanometer would also vary. However, if the wavelength was greater than some maximum wavelength, usually called the "cutoff" wavelength, $\lambda > \lambda_c$, there would be no current, *regardless* of the intensity of the incident light. (See Fig. C.2 for a typical current trace.) In this regime, the fact that there is no current implies that the electrons are not receiving enough energy to overcome the work function, and any explanation of this must be based on a theory of the interaction of light and matter.[2]

Maxwell's wave theory of light predicted that matter obtains energy from light in a continuous manner, just like an ocean wave washing up on the shore. As the amplitude E_0 of the light wave increases, the intensity also increases ($I \propto E_0^2$), so that by increasing the light intensity an electron should be able to absorb as much energy as needed, *regardless of the light's wavelength*. Figure C.2 shows that this is not what actually happens. In 1905, Einstein realized that an adoption of Planck's quantum hypothesis not only correctly predicts the features of Fig. C.2, but many other discrepancies as well.

Here is Einstein's logic. Light of frequency ν exists in discrete packets with energy $E = h\nu = hc/\lambda$, called photons. If, during a "collision" with an electron, a photon is

[1] Einstein, 1905.

[2] Refer to Problems 5 and 6 for a simple theory of this interaction.

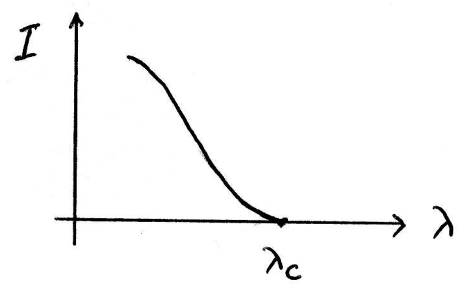


Figure C.2: The current I as measured by the galvanometer as a function of the wavelength λ of the light incident on the metal plate. If λ is small enough, this light causes photoemission of the electrons.

"annihilated," that is, it gives up all its energy to the electron, then in order for the electron to be ejected from the metal plate, the photon's energy must be greater than the work function of the metal, $E > \phi$, or

$$\lambda < \frac{hc}{\phi} \equiv \lambda_c. \tag{C.1}$$

For example, if the plate is made of nickel, whose work function is about 5 eV, then λ_c can be calculated to be about 250 nm, which means the incident light must be in the ultraviolet. Of course, in practice it is an experimental measurement of λ_c that is used to determine ϕ.

The photoelectric equation

Applying the concept of energy conservation to the interaction between the photon and electron results in the following equation

$$h\nu = \phi + K_{max}, \tag{C.2}$$

where $h\nu$ is the total energy before the interaction, since the electron is assumed to be at rest, and the right hand side is the total energy of the electron after the interaction (the photon no longer exists). The quantity K_{max} is the maximum kinetic energy of the electron after it has left the metal's surface.[3]

It was found experimentally in 1902 by Philipp Lenard [Nobel Prize, Physics, 1905] that K_{max} was independent of the intensity of the light, and also that K_{max} increased with the frequency of the light ν, both of which are predicted by Einstein's photoelectric equation (C.2). In fact, a straightforward determination of h can be made by measuring the slope of K_{max} versus ν. This is exactly what Robert Millikan did in 1916 to obtain the most precise value for h at that time. He obtained

$$h = (6.56 \pm 0.03) \times 10^{-34} \text{ J s}.$$

[3]Equation (C.2) assumes, as Einstein did, that the "light quantum delivers its entire energy to a single electron."

All this is fine, but how is one to determine K_{max}? It turns out that there is a very simple method. For a given frequency of incident light, reverse the polarity of the battery in Fig. C.1 until it is strong enough to *stop* the current. This value of V is called the "stopping potential," V_s. When the battery's polarity is reversed, the electric field now points *from* the plate *to* the collector, which serves to repel the electrons from the collector. At this critical value of the potential, the electric field is just barely large enough to repel the most energetic electrons, those with energy K_{max}, which means that $K_{max} = eV_s$. The experimentally measured quantities V_s and ν are therefore related by

$$V_s = \left(\frac{h}{e}\right)\nu - \frac{\phi}{e}, \tag{C.3}$$

which means that in actuality the experimenter measures a slope of h/e, rather than h directly.

Einstein's revolution

The photoelectric effect, and the various interpretations of Einstein's explanation, is useful to illustrate some issues in the philosophy of science and some of the consequences of Einstein's quote on page xv. For example, it is usually stated that the experimental facts of the photoelectric effect, and Einstein's explanation, unambiguously suggest that light comes in discrete clumps, or photons. In fact, the situation is not that clear, as we shall see below.

Einstein's 1905 paper, in which he explained the photoelectric effect, was primarily a study of the thermodynamics of radiation, and in particular how that applied to blackbody radiation. He limited his analysis to the so-called "Wien regime," which can be expressed as $h\nu \gg kT$. This is the regime where the ultraviolet catastrophe (see App. B) rears its ugly head. That is, in this regime, "the classical theory becomes an unreliable predictor for the quantum results."[4] After a study of Planck's explanation of blackbody radiation, Einstein was prompted to make the "light-quantum hypothesis:"

> Monochromatic radiation...behaves in thermodynamic respect as if it consists of mutually independent energy quanta of magnitude $R\beta\nu/N$.[5]

You can think of this hypothesis (not a theorem) as "just a curious property of pure radiation in thermal equilibrium, without any physical consequence,"[6] but Einstein next made a statement about *physical reality*, called the "heuristic principle:"

> If...monochromatic radiation...behaves as a discrete medium consisting of energy quanta of magnitude $R\beta\nu/N$, then this suggests [that] the laws of the

[4] Emch and Liu, *The logic of thermostatistical physics*, Springer 2001, page 363.
[5] Einstein, 1905. In Einstein's notation $R\beta/N = h$.
[6] Pais, *Subtle is the Lord*, page 377.

> generation and conversion of light are also constituted as if light were to consist of energy quanta of this kind.[7]

That is, the "light-quantum hypothesis" describes a property of radiation, nothing more or less, but the "heuristic principle," on the other hand, makes the stronger claim that this property can be extended to the interaction of light and matter. Einstein is now describing the underlying reason for a physical *process*, which, if not falsified, is the first step on the road to becoming a physical theory.

In 1905, there was no quantum theory for electrons — the Schrodinger equation did not make an entrance until 1926 — and so Einstein, as might be expected, treated the electrons classically. Only the light was assumed to be quantized, and this was enough to explain all the strange experimental observations. However, in 1927 after it became possible to describe the electrons using a quantum theory, Gregor Wentzel was able to explain the photoelectric effect *without photons*! Either the electron *or* the light must be quantized, but both is not necessary. We can conclude that the photoelectric effect does *not* prove the existence of photons, but is somewhat more ambiguous. In addition, the Compton effect (Appendix F) is also an experiment that claims to prove the existence of photons. However, his explanation is from 1923, again before the advent of the Schrodinger equation. In 1927, Schrodinger himself explained the Compton effect without photons, although he had to quantize the electron, just as Wentzel did.

So we are left with a conundrum: are photons real or not? This is not the proper question to ask, however. Two better questions are, "Do photons explain nature?" and "Is the concept of a photon required to explain experimental observations?" The answer to the first question is yes, but, considering only the photoelectric effect and the Compton effect, the answer to the second question is no. Are there any other observations that can decide the issue? In fact, there are. When an atom radiates light, it recoils. If we viewed the electromagnetic radiation classically, atoms would radiate a spherical wave in all directions (if the atom were spherically symmetric), and conservation of momentum would dictate that the atom would not move. However, since atoms do recoil, this implies that the radiation is emitted in a particular direction, and in fact, is a photon.

The moral of this story is that while there might be one experiment that is conceptually straightforward and that experiment comes to be known as the "proof" of a concept, it is usually several experiments that result in a "preponderance of the evidence." That is, one experiment does not usually remove "reasonable doubt."

Collateral Reading

The following articles and sections of books give a brief introduction to problems of epistemology, especially as it applies to science. That is, as scientists we want to make statements with certainty, or, barring that, at least know the degree of certainty that holds for each particular statement.

- Sam Inglis, review of *Karl Popper: Philosophy and Problems*, Am. J. Phys., **65** 162-164 (1997). (ERAU: `Reynolds' office`)

[7]Einstein, 1905.

- Bertrand Russell, *The Problems of Philosophy*, Chapter 1, 1912. (ERAU: `online`)
- Hans Reichenbach, *Philosophic foundations of quantum mechanics*, Sections §4-6, University of California Press, 1944 (Dover, 1998). (ERAU: `Reynolds' office`)

Problems

1. In his measurement of h, Millikan used sodium metal for the material of his metal plate, and was able to determine that the minimum (cutoff) frequency was 0.439×10^{15} Hz. What does this imply for the work function for sodium? How does Millikan's value compare with the presently accepted value?

   Solution Multiplication gives $\phi = h\nu = (6.561 \times 10^{-34}\ \text{J s})(0.439 \times 10^{15}\ \text{Hz}) = 2.88 \times 10^{-19}\ \text{J} = 1.798$ eV. The current values for sodium range from 1.82 eV to 2.75 eV depending on the surface cleanliness. This variable (surface cleanliness) was what Millikan spent several years trying to improve. A recent value from CRC is 2.36 eV.

Appendix D

Rutherford Scattering

All science is either physics or stamp collecting. — Ernest Rutherford

After studying radioactivity for several years, and winning a Nobel Prize (in Chemistry!) for his efforts (see Section 3.7), in 1911 Ernest Rutherford attacked the question of the composition of matter from a different perspective. Along with two students, Hans Geiger and Ernest Marsden, Rutherford directed the α rays emitted by "radium emanation," ^{222}Rn, at several thin solid targets, primarily gold.[1] (He had by that time definitively determined that α rays were nothing more than helium nuclei.) At this time, the prevailing model of the atom was J. J. Thomson's "plum pudding" model, in which he envisioned a smeared out positive charge with electrons embedded like plums in a pudding. One way to test this model was to fire a charged particle at an atom and then measure its trajectory. This would give information about the location of the electric charges. In fact, this is the primary method that has been used over the past 100 years to investigate the structure of subatomic particles.

Rutherford's results showed that the atom consisted of a small, massive "nucleus" that was positively charged, surrounded by several light, negatively charged electrons. The incoming α particle was deflected only by the nucleus and not by the light electrons, so Rutherford developed a theory of scattering to analyze his results. This theory is sufficiently important that I will derive its general form, and then apply it to two specific situations.

[1] Radon-222 is the daughter of ^{226}Ra, and it α decays to radium A (^{218}Po) which then α decays to ^{214}Pb which then β decays to radium C (^{214}Bi). This is part of the $4n+2$ natural decay series (see Section 3.8.1) starting with ^{238}U. All of these isotopes were present and emitting α particles, each with a characteristic energy. As Geiger and Marsden stated in "On a Diffuse Reflection of the α-Particles," *Proc. Roy. Soc.*, **82**, 495-500 (1909), "The tube contained an amount of emanation equivalent to about 20 milligrammes $RaBr_2$ at a pressure of a few centimetres. The number of α-particles expelled per second through the window was, therefore, very great, and, on account of the small pressure inside the tube, the different ranges of the α-particles from the three products (i.e. emanation, RaA, and RaC) were sharply defined."

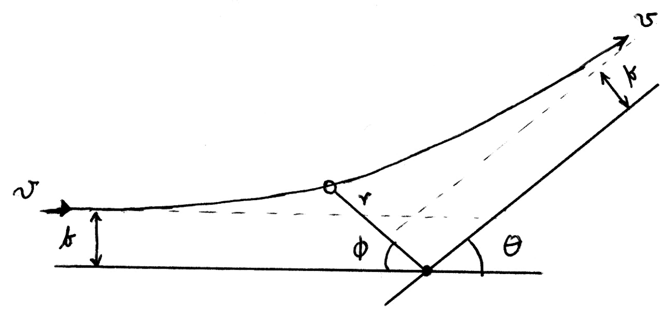


Figure D.1: Scattering geometry for a fixed, repulsive scattering center. The scattered particle initially has a speed v, and if no force were present, its straight line trajectory would take it to within a distance b — the impact parameter — of the scattering center. By symmetry, it will have a final speed v, but in a direction θ relative to its initial direction.

Scattering by a central force

The standard scattering problem is as follows: An object approaches a "scattering center" with speed v, and if it felt no force it would miss the scattering center by a distance b, known as the "impact parameter." See Fig. D.1. If the force exerted on the object is in the radial direction, and depends only on the radial distance r, then the resulting trajectory will be symmetric, and the object will head away from the center asymptotically approaching a line that is also a distance b from the center. At any given instant, the object will be located at $(r,\ \phi)$ relative to the center, and the scattering angle θ is the direction that it is heading (relative to its initial direction) when it is far away from the center.

For any given force that the scattering center exerts on the object, the main theoretical prediction is the function $\theta(b)$. That is, how does the scattering angle θ depend on the impact parameter b? For repulsive forces, we can predict some general features of $\theta(b)$. First, if $b = 0$, then the object hits the (repulsive) center head on and simply "bounces" back, resulting in $\theta = \pi$. As b increases θ must decrease, until for large b, θ must be small. In the limit that $b \to \infty$, it must be the case that $\theta \to 0$, as long as the scattering force gets weaker with distance. See Fig. D.2, which shows the general case of $\theta(b)$. It is correct for small b and for large b for all repulsive forces, but detailed shape depends on the actual force law.

Hard sphere (billiard-ball) collisions

One of the simplest types of collisions to analyze is that of two solid spheres of radius R, and the only force that exists between them is a repulsive, elastic contact force when they touch. In this case, $b = 0$ results in $\theta = \pi$, as we predicted above. However, if b is greater than $2R$, then the spheres miss each other completely, and there is no scattering, which means $\theta = 0$. Using the law of reflection (the angle of incidence equals the angle of

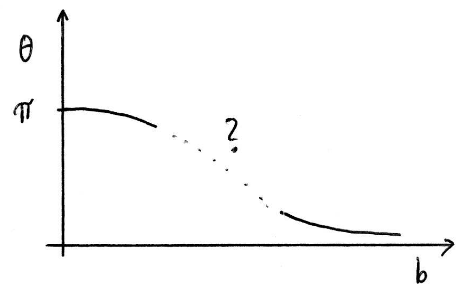


Figure D.2: Typical plot of scattering angle θ versus impact parameter b for a repulsive force. A direct hit ($b = 0$) must cause direct backscattering ($\theta = \pi$), and as b increases, θ must decrease toward zero. The exact form of the decrease depends on the form of the repulsive force, so that a measurement of $\theta(b)$ can be inverted to infer $F_r(r)$.

reflection), you can show (see Problem 1) that b and θ are related by

$$b = 2R\cos\left(\frac{\theta}{2}\right). \tag{D.1}$$

To obtain θ as a function of b, you simply need to invert the formula above

$$\theta = \begin{cases} 2\arccos(b/2R) & b < 2R \\ 0 & b \geq 2R \end{cases} \tag{D.2}$$

Rutherford scattering

Ernest Rutherford, of course, was interested in the case where the force law was a repulsive Coulomb force, which was the case in his experiment of α particles scattering off gold nuclei. The gold nucleus acted as the scattering center — it was very massive and did not move very much during the "collision" — and the α particles were the objects being scattered. If the repulsive force is

$$F = \frac{1}{4\pi\epsilon_0}\frac{(ze)(Ze)}{r^2}, \tag{D.3}$$

where Ze is the positive charge of the scattering center, and ze is the positive charge of the object being scattered, then you can show that

$$b = \frac{zZ}{2K}\frac{e^2}{4\pi\epsilon_0}\cot\left(\frac{\theta}{2}\right), \tag{D.4}$$

where K is the initial kinetic energy (when it is very far away) of the α particle. Notice that the scattering angle depends on the speed of the object, which was not true in the billiard-ball case. There, since there was no force except when the spheres made contact, the speed did not matter at all, only the angle of the collision. Here, if the α particle moves quickly, it spends less time in the region where the repulsive electric force is strong, and therefore the scattering angle is small.

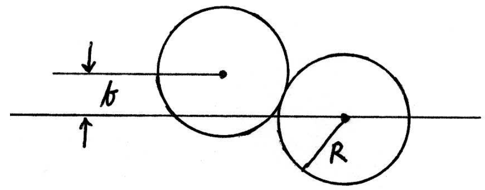


Figure D.3: Scattering geometry for two solid spheres of radius R.

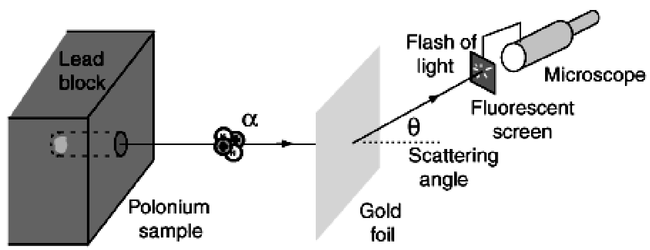


Figure D.4: Schematic of the Geiger-Marsden experiment. The scattering angle θ was measured by noting a flash on the fluorescent screen in a darkened room. From Hyperphysics.

The Rutherford scattering formula Unfortunately, Eq. (D.4) is not in the proper form for comparison with experimental results. Why not? Well, since the detector is typically located at an angle θ from the initial projectile direction, or at least the number of particles that are deflected by an angle θ is measured, we wish to predict the probability for an α particle to be scattered into any angle between θ and $\theta + d\theta$. In addition, there are many α particles and many nuclei in the thin foil target, which means the density of the gold nuclei and the thickness of the foil must be taken into account. If all these factors are included, the probability above, that for scattering into any angle between θ and $\theta + d\theta$, is given by $N(\theta)d\theta$. For the Coulomb force, Rutherford showed that

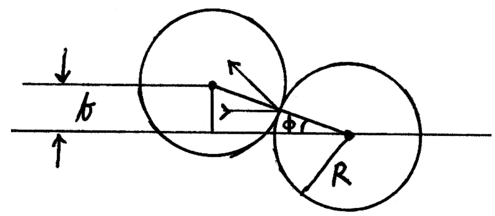
$$N(\theta) = \frac{nt}{4r^2}\left(\frac{zZ}{2K}\right)^2\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \sin^{-4}\left(\frac{\theta}{2}\right), \tag{D.5}$$

where n is the number density of the scatterers, t is the foil thickness, and r is the distance of the detector from the point where the beam hits the foil. Geiger and Marsden were able to reproduce the dependence of N on θ, Z, t, and K. All of the measurements matched the predictions, which led to the acceptance of a "nuclear" atom.

Problems

1. Derive the scattering formula, Eq. (D.1), for two solid spheres.

Solution The right triangle in the figure has a hypotenuse of $2R$, and hence $\sin\phi = b/2R$. The scattering angle θ (see Fig. D.1) is given by $\theta + 2\phi = \pi$, where the factor of 2 comes from the fact that the angle of incidence equals the angle of reflection. Solving for b gives



$$b = 2R\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = 2R\cos\left(\frac{\theta}{2}\right).$$

2. Sketch the function $\theta(b)$ for billiard-ball collisions. One method is to sketch $b(\theta)$ from Eq. (D.1), and then invert the sketch (flip it mirror-like around the line $b = \theta$).

3. Sketch the function $\theta(b)$ for Coulomb collisions (i.e., Rutherford scattering).

4. If an α particle heads directly toward a gold nucleus ($b = 0$), how much kinetic energy must it have so that its distance of closest approach (defined to be where its kinetic energy is zero) is equal to the nuclear radius?

Solution Gold has $A \sim 197$ so that $R = R_0 A^{1/3} = 1.2\text{ fm} \times 5.8 \sim 7\text{ fm}$. The potential energy between the α particle ($z = 2$) and the gold nucleus ($Z = 79$) when they are 7 fm apart is

$$|U| = \frac{e^2}{4\pi\epsilon_0}\frac{(2)(79)}{7\text{ fm})} \approx 32\text{ MeV}.$$

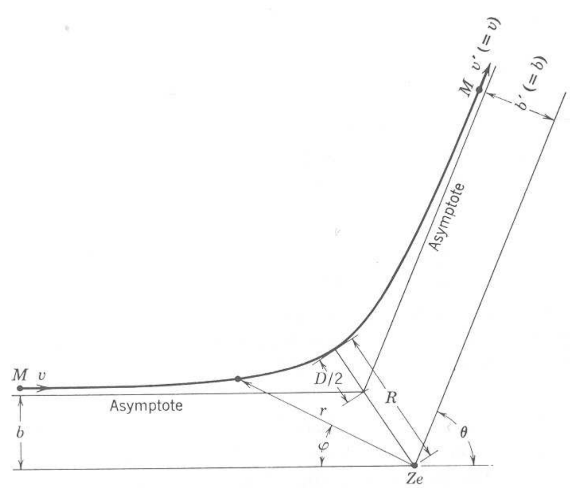
This is just what K must be when they are very far apart. However, this is much larger than the ~ 5 MeV α particles that are emitted radioactively, so that Rutherford's nuclei did not get close enough to experience the strong force.

5. Rutherford scattering. In polar coordinates, with a scattering nucleus (of charge $+Ze$) fixed at the origin, the equation of the trajectory of the α particle (of charge $+ze$) can be shown to be

$$\frac{1}{r} = \frac{1}{b}\sin\varphi + \frac{D}{2b^2}(\cos\varphi - 1),$$

where b is the "impact parameter," and D is the "distance of closest approach" in a head on collision ($b = 0$), which is given by

$$D = \frac{1}{4\pi\epsilon_0}\frac{zZe^2}{Mv^2/2}.$$



In a head-on collision, the α particle will stop and turn around at this location distance from the nucleus. (a) Show that D is the distance at which the potential energy of the α particle is equal to its initial kinetic energy ($Mv^2/2$). (b) Show that the trajectory equation is a hyperbola. (In the figure, θ is the scattering angle.)

Solution (a) The total energy of the α particle is $E = K + U$. When it is very far from the gold nucleus, the potential energy is zero, so that $E = K = \frac{1}{2}mv^2$. At its distance of closest approach, the "turning point," the kinetic energy is zero so that $E = U = +zZe^2/4\pi\epsilon_0 r$. The + sign indicates that the Coulomb force is repulsive. Since E is the same in both cases, setting $K = U$ and $r = D$ gives the formula for D. (b) This one is hard, and if anyone made a reasonable attempt, give them some credit.

6. In Geiger and Marsden's experiment, α particles impinged on a gold foil. Consider one α particle heading directly toward one gold nucleus (^{197}Au of course). How much initial kinetic energy must the α particle have (when it is very far from the nucleus) in order to have its distance of closest approach be equal to the radius of the nucleus?

Solution Here, $z = 2$ and $Z = 79$, so that the kinetic energy can be written

$$\frac{1}{2}mv^2 = \frac{zZe^2}{4\pi\epsilon_0 D} = \frac{(2)(79)(1.4399 \text{ eV nm})}{(1.2 \text{ fm})(197)^{1/3}} = 32.6 \text{ MeV}.$$

This is quite a bit of energy. Recall that Rutherford was using α particles from radioactive decay, and they tended to have energies around 3-5 MeV. So his projectiles never got close to the gold nuclei, which means that he "saw" strictly the $1/r$ potential and no hint of structure in the nucleus.

7. In a collision between hard spheres, there is no scattering if b is larger than a maximum value. This means that you must "aim well" in order to see an effect. Not true for the $1/r^2$ Coulomb force: *any* impact parameter will cause scattering. It is instructive to investigate the "Born approximation," where we take the limit of large impact parameter (and thus a small scattering angle). Determine the relationship between b and θ in this limit.

Solution Expanding the cotangent in Eq. (D.4) for small θ gives

$$b \approx \frac{zZ}{2K}\frac{e^2}{4\pi\epsilon_0}\left(\frac{2}{\theta}\right),$$

which shows that b and θ are inversely proportional. More interesting is that, in this limit, the quantity $b\theta K$ is constant.

Appendix E

The Stern-Gerlach Experiment

...quantum-mechanical states are to be represented by vectors in abstract complex vector space. —J.J. Sakurai[1]

In 1922, Otto Stern and Walther Gerlach measured the magnetic moment of the silver atom using a technique that has come to be known as the "molecular beam method." Due to the electron configuration of silver ($Z = 47$), it was essentially a measurement of the magnetic moment of the electron (see Section 2.4 and page 76). It is also a demonstration of the simplest system that is inherently quantum mechanical, and it is instructive to realize just how inadequate our macroscopic intuition really is.

Torques and Forces

Recall from elementary electromagnetism that electric dipoles and magnetic dipoles experience forces and torques due to electric fields and magnetic fields, respectively. For magnetic dipoles, if the magnetic field $\vec{B}$ is uniform, then the force on the dipole is zero, but the torque on the dipole is equal to $\vec{\tau} = \vec{\mu} \times \vec{B}$, where $\vec{\mu}$ is the magnetic dipole moment. This just says that field tries to align the moment with the field vector. More important for the Stern-Gerlach experiment is the fact that if the field is nonuniform, then the dipole feels a net force that is due to the gradient in the field. Specifically, a magnetic dipole feels a force

$$\vec{F} = \nabla \left(\vec{\mu} \cdot \vec{B} \right), \tag{E.1}$$

which, if the field points primarily in the z direction, and its magnitude also varies in the z direction, becomes approximately

$$F_z \approx \mu_z \frac{\partial B_z}{\partial z}. \tag{E.2}$$

This was exactly the case in the Stern-Gerlach experiment. Otto Stern had the idea for this experiment in 1921 in order to see if he could detect the "space quantization" of the atom. In Bohr's atomic model, the angular momentum perpendicular to the plane of the electron orbit was quantized, $L_z = n\hbar$, which meant that the magnetic moment due to the

[1]Sakurai, *Modern Quantum Mechanics*, page 10.

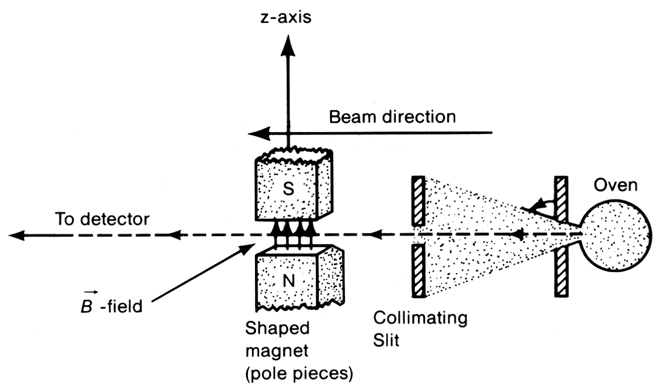


FIGURE 1.1. The Stern-Gerlach experiment.

Figure E.1: Schematic of the Stern-Gerlach experiment. Silver atoms were heated in an oven, allowed to escape and sent in a particular direction via a collimated slit, passed through a nonuniform magnetic field, and finally impinged on a photographic plate. Figure 1.1 from Sakurai, *Modern Quantum Mechanics*.

orbital motion was also quantized. It can be expressed (Problem 2) in units of the Bohr magneton

$$\mu_z = -n\mu_B, \tag{E.3}$$

exactly as in Eq. (2.9). From observations of the Zeeman effect, where the spectral lines of atoms that have been place in a magnetic field are split into two, three, or more components, it was postulated that the magnetic moment vector of an atom was forced to be either parallel or anti-parallel to the external magnetic field. Since these two states have different energies (the torque equation above implies that there is a potential energy due to the interaction that is $U = -\vec{\mu} \cdot \vec{B}$), this would explain the splitting of the spectral lines. Stern proposed to verify this by means of Eq. (E.2).

As depicted in Fig. E.1, silver atoms would be heated in an oven, allowed to "effuse"[2] through a hole in the oven and then a collimating slit. They then would pass through a nonuniform magnetic field which would exert a force on the atoms via Eq. (E.2) and therefore spread out the beam. Classically, the magnetic moment vectors of the atoms point in random directions, and therefore the z components would take on a continuous range of values, which means that the initially narrow beam would be spread out. However, if the quantum intuition of physicists like Niels Bohr was correct, then only certain discrete values of μ_z would be allowed and the beam would split into two or more discrete beams, resulting in discrete lines on the detecting photograph. This, in fact, was Stern's motivation: to "decide unequivocally between the quantum theoretical and classical views."[3] It took a year to complete the experiment because the deflection of the beam was small, and the entire apparatus had to be aligned to a tolerance of 0.01 mm or the result would be inconclusive.

Since silver has 47 electrons, the first 46 form a spherical cloud — and each pair of them

[2]effusion, n., the flow of gas through an aperture whose diameter is small as compared with the distance between the molecules of the gas.

[3]Quoted in Friedrich and Herschbach, *Physics Today*, 2003.

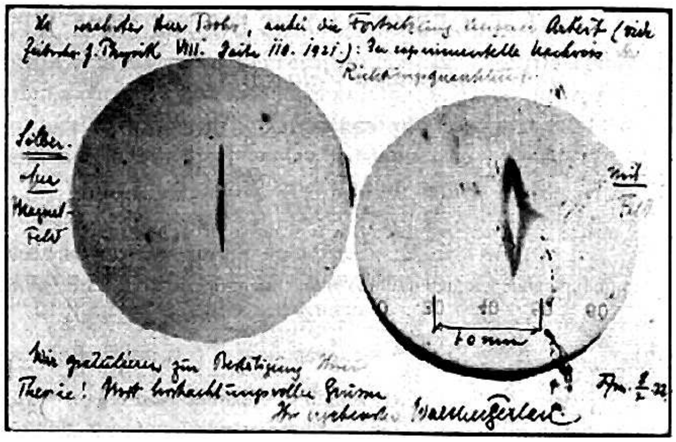


Figure E.2: Postcard from Gerlach to Bohr. The message reads "Attached [is] the experimental proof of directional quantization. We congratulate [you] on the confirmation of your theory." Note the scale denoting 1.0 mm at the bottom right. Figure 4 from Friedrich and Herschbach, *Physics Today*, 2003.

has their spins anti-aligned — while the 47th electron, the only one in the $5s$ subshell, is the only one to contribute to the magnetic moment of the atom (see Problem 67 for a proof that the nucleus contributes only negligibly to the atom's magnetic moment). Of course, Stern and Gerlach did not know about spin (it wasn't proposed by George Uhlenbeck and Samuel Goudsmit until 1925), but they assumed the Bohr model, which stated that the unpaired electron would have a nonzero orbital angular momentum, and hence a nonzero magnetic dipole moment, as in Eq. (E.3). Of course, as we now know, the 47th electron has zero orbital angular momentum, so that the magnetic moment of the atom is solely due to its spin, and can take on the values

$$\mu_z = -g m_s \mu_B, \tag{E.4}$$

as given in Eq. (2.10). When they found that the beam was split into two beams, and that the strength of the splitting implied that silver had a magnetic moment equal to μ_B to within 10%, Gerlach sent a postcard to Bohr in congratulations (see Fig. E.2). They thought that they had confirmed Eq. (E.3) with $n = 1$. But in fact they had confirmed Eq. (E.4), with $g \approx 2$ and $m_s = \pm\frac{1}{2}$. It wasn't until 1927, after spin was discovered and after Schrodinger modeled the hydrogen atom, that it was recognized that they had actually measured the spin of the electron. In 1922, Stern and Gerlach were completely in the dark about the true nature of their result.

Sequential Stern-Gerlach experiments

What is special about the z axis? Could we turn the magnets in Fig. E.1 horizontal and measure μ_x? Of course, but then, quantum mechanics tells us, we would have no knowledge of the z component. Recall from Section 2.3 that it is possible to know simultaneously only the magnitude and *one* component of the spin vector — the other two components, as given

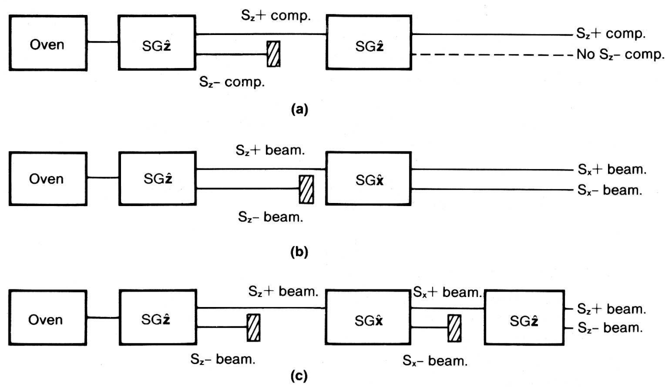


Figure E.3: Schematic of three different possibilities for Stern-Gerlach type experiments that are run in sequence, i.e., one after the other. Figure 1.3 from Sakurai, *Modern Quantum Mechanics*.

by the angle ϕ, are completely "unknowable." Since the magnetic moment is proportional to spin, this restriction applies also to it. To illustrate this restriction, consider Fig. E.3(a). We place two identical deflecting magnetic fields that are both oriented in the z direction, so they each effectively measure the z component of the magnetic moment (or spin). After the first "SGz" apparatus, which splits the beam in two, if we block the beam that was deflected downward, and let the upward-deflected beam go through *another* SGz apparatus, then there should be no downward deflected beam. This is because all the atoms entering the second apparatus must have their magnetic moments pointing in the $+z$ direction, since they had just been measured by the first SGz apparatus.

Figure E.3(b) shows a similar setup, but with an SGx apparatus coming second. Again, if we block the "spin down" component and let the spin up component through, the result is that 50% of the beam is deflected right (which means it has $S_x = +\frac{1}{2}\hbar$) and 50% of the beam is deflected left (which means it has $S_x = -\frac{1}{2}\hbar$). Now, it is tempting to conclude that we have just measured *two* components of the spin vector, something we thought was impossible. That is, a reasonable interpretation seems to be that half of the atoms in the $S_z = +\frac{1}{2}\hbar$ beam leaving the first, SGz, apparatus have both $S_z = +\frac{1}{2}\hbar$ and $S_x = +\frac{1}{2}\hbar$, and the other half have both $S_z = +\frac{1}{2}\hbar$ and $S_x = -\frac{1}{2}\hbar$.

Figure E.3(c) will show that our "reasonable interpretation" above is wrong. If we now block the $S_x = -\frac{1}{2}\hbar$ beam after the second, SGx, apparatus, and then run the remaining $S_x = +\frac{1}{2}\hbar$ beam through a third, SGz, apparatus, we find that even though we started with atoms that had only $S_z = +\frac{1}{2}\hbar$, we now have both components. This effectively proves that we cannot measure two components of the spin vector simultaneously. Specifically, when the atoms passed through the SGx apparatus that measured their S_x, it destroyed any prior knowledge about the z component. This result is not due to any experimental inaccuracy or error, but is simply a microscopic limitation, as expressed by the Heisenberg uncertainty principle. It turns out that this situation is almost identical with the *classical* case of light passing through sequential polarizing filters. It means that one method of representing the atom's spin is to use a wave equation (since the results of the experiment with light is due to the phenomenon of superposition), and that's exactly what quantum mechanical equations are, wave equations. The Schrodinger equation is a nonrelativistic wave equation, and the Dirac and Klein-Gordon equations are relativistic wave equations.

Collateral Reading

- Bretislav Friedrich and Dudley Herschbach, "Stern and Gerlach: How a Bad Cigar Helped Reorient Atomic Physics," *Physics Today*, **56**(12) 53-59 (December 2003).
- J. J. Sakurai, *Modern Quantum Mechanics*, Benjamin-Cummings, 1985. Chapter 1.

Problems

1. What is meant by the term "space quantization?" Is space really quantized?
2. Derive Eq. (E.3). Recall that the magnetic moment of a current loop has a magnitude $\mu = IA$.
3. Estimate the separation distance of the images observed on the screen of Stern and Gerlach's experiment. Their source of atoms was an oven of temperature 1000 °C, their deflecting magnet was 3.5 cm long, and the magnetic field gradient was 10 T/cm. Make any other assumptions that you need (but be sure to state them).

   Answer The splitting was 0.2 mm.

Appendix F

The Compton Effect

In 1923, Arthur Holly Compton [Nobel Prize, Physics, 1927] performed a simple experiment concerning the scattering of X-rays. He used the K_α X-rays from molybdenum (see Section 4.4 for a description) and scattered them off a carbon target. X-rays had already been scattered from crystal targets by the Braggs (father William Henry and son William Lawrence) in 1913, and they had found that the wavelength of the scattered X-rays was identical to the incident X-rays, but the *intensity* varied with scattering angle. The angles that exhibited the highest intensity were explained by constructive interference of the electromagnetic wave with itself as it was scattered by parallel planes in the crystal, and those that exhibited low intensity were the result of destructive interference. In these solids, however, the electrons were strongly bound to the atoms so that they did not interact individually with the incident X-rays.

Compton used carbon (in the form of graphite) as his target. In this case, the high electrical conductivity means that there are plenty of effectively "free" electrons willing and able to collide with the X-ray photons allowing them to exhibit their particle nature. In observing the scattered X-rays at different angles, Compton found that the wavelength *increased*, which means that they must have lost energy during the interaction with the graphite. The only way that Compton was able to explain this effect is by invoking the quantum nature of light and treating the interaction as a two-body collision between photon and electron, although using relativistic dynamics. This experiment, along with Planck's explanation of the blackbody spectrum and Einstein's explanation of the photoelectric effect, finally convinced most physicists that photons were "real." In fact, it wasn't until after this experiment, in 1926, that the term "photon" was first used.

Two-body collisions

While the analysis is somewhat complicated algebraically, it is a straightforward application of the laws of conservation of energy and momentum. Consider the geometry as shown in Fig. F.1, where a stationary electron of mass m_e is hit by a photon of energy E and momentum $\vec{p} = p\hat{x}$. After the collision, the photon now has a different energy E'

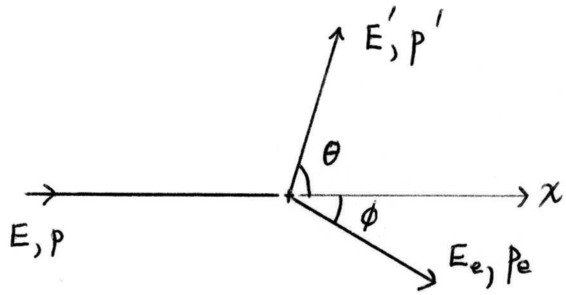


Figure F.1: Scattering geometry for an incoming particle of energy E and momentum p, moving in the x direction, and impacting a stationary particle at the origin. After the collision, the incoming particle moves off with energy E' and momentum p' at an angle $+\theta$ with respect to the positive x axis, and the stationary particle moves off with energy E_e and momentum p_e at an angle $-\phi$ with respect to the positive x axis.

with a component of momentum in the $\hat{y}$ direction, and the electron increases its energy to E_e and has a component of momentum in the $-\hat{y}$ direction. The three conservation equations become (energy, x-momentum, and y-momentum)

$$E + m_e c^2 = E' + E_e \tag{F.1a}$$
$$p = p' \cos\theta + p_e \cos\phi \tag{F.1b}$$
$$0 = p' \sin\theta - p_e \sin\phi \tag{F.1c}$$

where the scattering angles θ and ϕ are both taken to be positive. In principle, a knowledge of the initial conditions — the photon's initial wavelength λ, from which both its energy and momentum can be calculated — allows us to calculate the final conditions — the direction and momentum of both the photon and the electron. However, there are four unknowns, p_e, p', θ, and ϕ, but there are only three equations. Hence the system is underdetermined, and the best we can do is to either (a) fix one of the unknowns and solve for the other three in terms of the first, or (b) eliminate two of the unknowns to obtain a relationship between the other two unknowns. Since there is no general experimental method to fix one of the unknowns, the second approach is the one that allows a comparison with experiment. To illustrate the procedure, I'll first analyze a classical, nonrelativistic collision between two point particles that you have seen before in elementary mechanics. Then I'll look at the Compton scattering experiment, where relativistic dynamics are required, but the physical principles are identical.

Billard ball collision. Let's assume that Fig. F.1 and Eqs. (F.1) applies to two billiard balls: a cue ball (instead of a photon), and an eight-ball (instead of an electron). In the nonrelativistic case, the energy of each particle is just the kinetic energy, given by $K = mv^2/2$ plus the rest energy, and the momentum is the linear momentum $\vec{p} = m\vec{v}$. Assuming that the masses of the cue ball and eight-ball, m and m_e respectively, do not change, the final unknowns are just the two speeds, v' and v_e, and the two angles, θ and ϕ. Our goal is to determine these quantities in terms of the one initial quantity: the initial kinetic energy K of the cue ball. The technique is identical to that used in Problem 60.

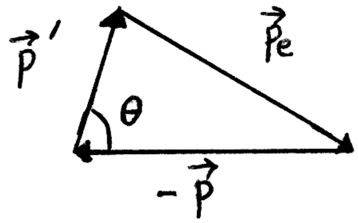


Figure F.2: Graphical representation of conservation of momentum for the Compton scattering geometry. The difference in the momenta before and after the collision must be zero, which means that a vector sum of all the momentum vectors must return to the original starting location.

The first step is to combine Eqs. (F.1b) and (F.1c) by squaring both sides and adding the two equations. Using elementary trigonometric relations, I get

$$p^2 - 2pp' \cos\theta + p'^2 = p_e^2. \tag{F.2}$$

This fundamental relation can be obtained very easily with a graphical analysis, as shown in Fig. F.2. Since the momentum *vector* is conserved, a triangle can be drawn whose three sides are the three momenta in the problem. A trivial application of the law of cosines results in Eq. (F.2). Notice that we have effectively eliminated the angle ϕ.

The second step is to express the energy of each particle in terms of its momentum. Since this problem is nonrelativistic, we have

$$E = mc^2 + K \tag{F.3a}$$

$$E' = mc^2 + \frac{p'^2}{2m} \tag{F.3b}$$

$$E_e = m_e c^2 + \frac{p_e^2}{2m_e}, \tag{F.3c}$$

where K is the initial kinetic energy of the cue ball.[1] Conservation of energy, Eq. (F.1a), becomes

$$\begin{aligned} K &= \frac{p'^2}{2m} + \frac{p_e^2}{2m_e} \\ &= \frac{p'^2}{2m} + \frac{1}{2m_e}\left(p^2 - 2pp'\cos\theta + p'^2\right), \end{aligned} \tag{F.4}$$

where I have eliminated p_e using Eq. (F.2). This is our final result, and it is a relation between p' and θ. That is, given the fact that the cue ball deflects at a certain angle, then Eq. (F.4) tells us what its final momentum and energy *must be.*

Of course, as you may know from playing billiards, the cue ball can deflect at any angle depending on how it impacts the eight-ball. However, if you were to measure its final momentum, you would find that p' and θ are always related by Eq. (F.4) — see Problems 1 and 2.

[1] This initial energy is a known quantity, so we are interested in solving for the other quantities in terms of K. If desired, we could express it as $K = p^2/2m$ and take p and m as our known quantities.

Photon-electron collision. How do we analyze the case of a X-ray interacting with an electron? At first glance, it appears to be a completely different situation, but in fact is remarkably similar. Energy and momentum are still conserved, so Eqs. (F.1) and (F.2) still apply. The only difference is that we must allow for the fact that the electron might be moving relativistically after the collision, and of course the photon is always moving ultra-relativistically, so we must replace Eqs. (F.3) with

$$E = pc \tag{F.5a}$$
$$E' = p'c \tag{F.5b}$$
$$E_e = \sqrt{(m_e c^2)^2 + p_e^2 c^2}. \tag{F.5c}$$

The conservation of energy equation, Eq. (F.1a), becomes slightly more complicated

$$(pc + m_e c^2 - p'c)^2 = (p^2 - 2pp' \cos\theta + p'^2)c^2 + (m_e c^2)^2. \tag{F.6}$$

Again, I have eliminated p_e using Eq. (F.2), so this is our final result, a relation between p' and θ.

There are two key differences between this case and the billiard-ball collision. First, the initial condition is given by the photon momentum p rather than the cue ball kinetic energy K. Second, we eliminated ϕ (rather than θ) because the electron remains in the target so that only θ is measureable. With billiard balls, in principle we could measure either θ or ϕ.

Compton wanted to compare this theoretical prediction, Eq. (F.6), with experimentally observable quantities. It turns out that photon momentum is not simple to measure, but photon wavelength is, so using the de Broglie relation $p = h/\lambda$ and $p' = h/\lambda'$, where λ and λ' are the photon wavelength before and after the collision, results in the famous Compton scattering formula

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta). \tag{F.7}$$

The quantity

$$\lambda_C \equiv \frac{h}{m_e c}, \tag{F.8}$$

is called the *Compton wavelength* of the electron. It is *not* the de Broglie wavelength of the electron matter wave, but rather indicates the magnitude of the wavelength *shift* of the photon when it collides with an electron. Equation (F.7) predicts that the wavelength of the scattered light is longer than the wavelength of the incident light — how much longer depends on the angle of deflection. Figure F.3 shows some of Compton's original data clearly showing this effect.

This process is fundamental to the operation of NASA's Compton Gamma Ray Observatory, which was in Earth orbit from 1991 through 2000. The NASA web site states, "The Observatory was named in honor of Dr. Arthur Holly Compton, who won the Nobel prize in physics for work on scattering of high-energy photons by electrons — a process which is central to the gamma-ray detection techniques of all four instruments [on the CGRO]."

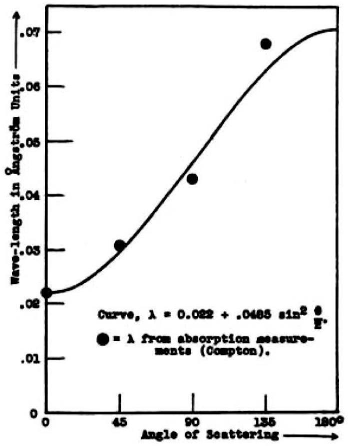


Figure F.3: Wavelength of the scattered γ-rays as a function of angle of deflection θ. The four dots are experimental measurements, and the curve is Eq. (F.7). Figure 5 from Compton, 1923.

Compton scattering vs. Bragg scattering

In principle, when X-rays are incident on a solid, both types of scattering will occur. The wave properties of light allow it to interfere with itself and show preferential scattering directions, while the particle properties allow it to interact with single electrons.

Problems

1. A cue ball of mass m with kinetic energy K collides elastically with an eight-ball of mass m_e at rest. Derive the relationship between the deflection angle of the cue ball (θ) and its kinetic energy *after* the collision ($p'^2/2m$). That is, express θ as a function of K' and K.

   Solution Manipulating Eq. (F.4) by expressing the momenta in terms of the kinetic energy ($p = \sqrt{2mK}$) gives

$$\cos\theta = \frac{(m+m_e)K' + (m-m_e)K}{m\sqrt{KK'}}.$$

2. For the situation described in Problem 1, if $m < m_e$ then it is possible for m to bounce directly backwards (i.e., with a deflection angle of $\theta = \pi$). However, if $m > m_e$, then there is a maximum deflection angle. Find this angle.

3. Derive Eq. (F.7).

   Solution Expanding the square on the left-hand-side of Eq. (F.6) and canceling terms gives

$$m_e c(p - p') = pp'(1 - \cos\theta).$$

Expressing the momenta in terms of the wavelength gives

$$\frac{p - p'}{pp'} = \frac{1}{h}\lambda\lambda'\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = \frac{\lambda' - \lambda}{h}.$$

Simplification results in Eq. (F.7).

4. If the incident photon has energy E (or $h\nu$), what is the maximum possible kinetic energy imparted to the scattered electron? At what angle ϕ does this electron scatter?

Solution The electron will have maximum kinetic energy is the photon loses a maximum amount of energy. This occurs when the photon is backscattered, i.e., $\theta = \pi$. For this angle, the Compton scattering formula can be written

$$\frac{1}{E'} - \frac{1}{E} = \frac{2}{m_e c^2}.$$

Solving this for E' and then forming the quantity $E - E'$, I get

$$E - E' = \frac{2E^2}{m_e c^2 + 2E}.$$

Since this is the energy lost by the photon, it is also the energy gained by the electron.

5. Obtain a formula that gives the electron's kinetic energy as a function of the photon's scattering angle θ.

Solution Following the method of solution for the previous problem, but retaining an arbitrary angle θ, the Compton scattering formula is

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{m_e c^2}\left(1 - \cos\theta\right),$$

and forming $E - E'$ again (which is just the electron kinetic energy)

$$K_e = E - E' = (m_e c^2)\frac{E^2(1 - \cos\theta)}{E(1 - \cos\theta) + m_e c^2}.$$

Since the quantity $(1 - \cos\theta)$ is positive definite, the electron kinetic energy is also positive definite. Also, you can show that K_e is zero when $\theta = 0$, and that it is a maximum when $\theta = \pi$.

6. From Figure F.3, can you calculate the Compton wavelength λ_C? The units of the y axis are Ångstroms.

Appendix G

Cosmic Rays and Muons

> *Coming out of space and incident on the high atmosphere, there is a thin rain of charged particles known as the primary cosmic radiation.* — Cecil Powell [Nobel Prize, Physics, 1950]

The charged particles that make up the "primary" cosmic rays are protons, α particles, heavier nuclei, and electrons, and they impact the Earth from all directions and with various energies. Most of these are protons (about 80%), second in abundance are α particles (about 14%), while electrons make up less than 1%. When they impact nuclei in the atmosphere — mostly oxygen and nitrogen nuclei — their energies are such that they create "showers" of hadrons, mostly pions, along with some kaons, and anti-protons, and anti-neutrons. These then decay into photons, electrons, positrons, neutrinos, and muons (which themselves decay into electrons and neutrinos). These are all called "secondary" cosmic rays.

Where do the primary cosmic rays come from? Some come from the sun (mostly due to solar flares), most come from galactic supernovae, and a few with the highest energy are suspected to originate from outside the Milky Way. You might suspect the solar wind—a neutral plasma that consists of low energy protons, electrons, and helium nuclei—as a source of cosmic rays. Due to their low energies, however, these particles are stopped from reaching the atmosphere by the Earth's magnetic field, except in the polar regions. While they have enough energy to cause aurora, they do not cause showers of secondary subatomic particles.

How many are there? About 1 charged particle per second per cm^2 impacts the Earth.[1] This is a far cry from the 6×10^{10} neutrinos $\text{s}^{-1}\ \text{cm}^{-2}$ that come from the Sun.

What are their energies? The typical kinetic energy of these particles is about 10 MeV to 100 MeV, although there are some at higher energies. Figure G.1 shows the distribution of the measured energy per particle. In fact, the cosmic ray with the highest energy has been measured at 48 J! These ultra-high energy cosmic rays are suspected to be extra-galactic, as there is no plausible mechanism of acceleration to these energies by

[1] Henley and Garcia, *Subatomic Physics*, page 597.

a supernova, for example. Again, compare these energies to those of solar neutrinos that have only 0.26 MeV.

What happens to the secondary cosmic rays? The pions decay via the following modes

$$\pi^0 \rightarrow 2\gamma \qquad (G.1)$$
$$\pi^\pm \rightarrow \mu^\pm + \nu, \qquad (G.2)$$

where the neutral pions decay electromagnetically with an average lifetime of 8.4×10^{-17} s, and the photons subsequently create electron-positron pairs. Most of the energy of the original cosmic ray follows this path. Some of the energy goes into charged pions, which decay into muons with an average lifetime of 2.6×10^{-8} s. This long lifetime indicates that the decay is due to the weak interaction, and is therefore relatively unlikely. The muons then decay into electrons (or positrons) and neutrinos

$$\mu^\pm \rightarrow e^\pm + 2\nu, \qquad (G.3)$$

and their average lifetime is 2.2 μs, also a weak interaction.[2]

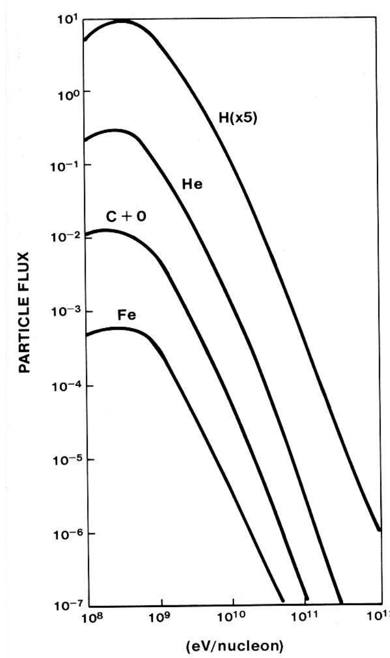


Figure G.1: The energy spectrum of the different nuclei that make up cosmic rays. Carbon and oxygen are lumped together. From Friedlander, *Cosmic Rays*, Figure 6.4.

What happens to these secondary cosmic rays as they pass through the atmosphere? First of all, in addition to possible decay, the charged particles cause ionization of the atmospheric molecules and therefore lose energy. For example, a typical muon loses about 2 GeV of kinetic energy before it hits the ground (if it hasn't decayed yet), and by the time they do reach the ground, the average muon energy is about 4 GeV. Secondly, the showers spread out laterally from the direction of the primary cosmic ray. The main hadronic core (pions, etc.) covers a few meters by the time it hits the ground, and the electromagnetic particles (electrons, positrons, photons) have spread further, about 100 m. Finally, the muons have spread the furthest, almost 1 km.

Muons as clocks

This spreading means that muons are continually bombarding the Earth's surface and, since it is not clear what direction they came from, statistical methods must be used to interpret the muon flux. That is, the muons are all "born" at different altitudes, they travel downward with different speeds, and they "live" for different intervals of time. Therefore, you might expect that the muon flux would increase with increasing altitude, at least initially, reach a maximum at some altitude, and then finally decrease. This is precisely

[2]Recall that the weak force is responsible for changing one family of quarks or leptons into another.

what is observed, but the exact shape of this curve is a convolution of a source function and a decay function, and therefore requires lots of modeling to interpret.

However, for our purpose — special relativity — we want to use the muons as a clock. In Chapter 5 we assume that our muons are all created at the same altitude, and all live for the same amount of time, 2.2 μs. You might think that we are not justified in doing this, because of the statistical spread of muon lifetimes, but that turns out not to be true. Scott and Burke state the case:

> It may seem at first glance that a real particle that is formed and later decays does not constitute an accurate clock, because of the uncertain nature of the decay process. Given a number of particles, some will decay at times less than the mean life, some will decay at times greater than the mean life, and in general it is impossible to predict exactly when any given particle will decay. However, it is possible to determine the *mean* lifetime of a number of particles to any desired accuracy simply by observing a sufficient number of such particles, and in this sense, decaying particles are just as good clocks as vibrating molecules. Indeed, for a vibrating molecule it is necessary to observe it for a large number of cycles in order to determine its frequency precisely; this is analogous to observing a large number of decays in an exponentially decaying system.[3]

Collateral Reading

- "The early history of cosmic ray research." by Q. Xu and L. M. Brown, *Am. J. Phys.*, **55** 23-33 (1987).
- Michael W. Friedlander, *Cosmic Rays*, Harvard University Press, 1989. (ERAU: `QC 485 .F75 1989`)

Problems

1. Calculate the energy in MeV of a 48-J proton. Also calculate γ and β for the same proton.
2. (a) Calculate the reaction energy for a pion decaying into a muon and a neutrino. (b) Using the conservation of momentum, calculate how much energy the muon has. HINT: You can assume the muon is non-relativistic (check this), but you must take relativistic effects into account for the neutrino. One approximation is to take the highly relativistic limit for the neutrino, where the relationship between its energy and momentum is $E_\nu = p_\nu c$. As usual, ignore the neutrino mass.
3. Why can't a π^0 decay into a μ^- and a μ^+?
4. If a muon μ^- is "born" due to a pion decay $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$ at an altitude of 20 km, how fast must it be traveling to reach the ground before it decays 2.2 μs later? Express your answer in the form $\beta = 1 - \epsilon$, and calculate ϵ.

[3] Scott and Burke, *Special Relativity Primer*, page 5.