Newton's generalization of the binomial theorem

The **binomial theorem** states that the binomial $(a+b)$ raised to an integer power n is given by the sum

$$(a+b)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{n-k} b^k = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^{n-k} b^k = a^n + n a^{n-1} b + \cdots + n a b^{n-1} + b^n$$

where $\left(\begin{array}{c} n \\ k \end{array} \right)$ is the number of combinations of n things chosen k at a time. These are also known as the *binomial coefficients*.

Newton showed that the binomial theorem was valid even if n was *not an integer*. In this case the sum is infinite and is given by the **Newton series**, also known as *Newton's generalization of the binomial theorem*

$$(a+b)^r = \sum_{k=0}^{\infty} \left(\begin{array}{c} r \\ k \end{array} \right) a^{r-k} b^k = \sum_{k=0}^{\infty} \frac{r(r-1)\cdots(r-k+1)}{k!} a^{r-k} b^k = a^r + r a^{r-1} b + \frac{r(r-1)}{2!} a^{r-2} b^2 + \cdots$$

where $\left(\begin{array}{c} r \\ k \end{array} \right)$ is the generalized binomial coefficient. Note that since r is not an integer, there is no value of k such that $r - k + 1 = 0$, which would truncate the series—hence it is an infinite series.

If we let $a = 1$ and $b = x$, then the Newton series is identical to the the Taylor series of the function $f(x) = (1+x)^r$. Newton's generalization is very useful as an approximation scheme when $x \ll 1$. In that case, each successive term is smaller than the previous term, and $f(x)$ can be evaluated to any desired accuracy by retaining the proper number of terms. For most cases, however, only two terms are needed, and the following approximation is used ubiquitously in science and engineering

$$(1+x)^r \approx 1 + rx.$$

You should memorize this.