Hyperspheres

An "n-sphere" is the set of points in $(n+1)$-dimensional Euclidean space that are at a fixed distance a from the origin. You know that the volume enclosed by a 2-sphere (the ordinary sphere) is $4\pi a^3/3$, and the "volume" enclosed by a 1-sphere (the ordinary circle) is πa^2. Calculate the volume of the 3-sphere by evaluating the integral

$$V_3 = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} \int_{-\sqrt{a^2-x^2-y^2-z^2}}^{\sqrt{a^2-x^2-y^2-z^2}} du\ dz\ dy\ dx.$$

I suggest that you start with the simpler problems of confirming (via direct integration) the area of a circle

$$V_1 = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\ dx = \pi a^2$$

and the volume of a sphere

$$V_2 = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz\ dy\ dx = \frac{4}{3}\pi a^3.$$

HINT: In the V_2 integral, switch to polar coordinates after the first (trivial) integration over z, and in the V_3 integral, switch to spherical coordinates after the first (trivial) integration over u.

PS 226 — Diagnostic Quiz

Name ________________________________ Score: $\boxed{}$/10

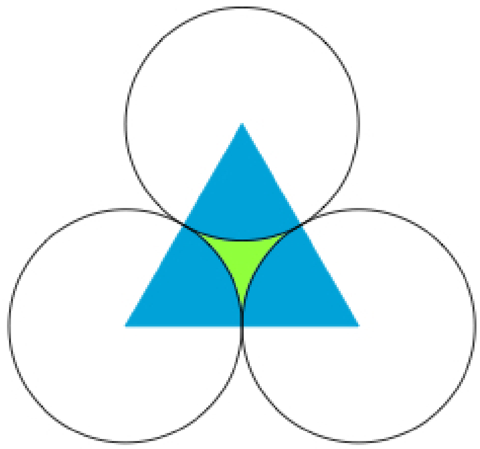
Vectors are an extremely important mathematical tool in physics. In order to gauge your current knowledge state, please write down (and explain with words and diagrams) any thing that you know about the following quantities. That is, definitions, formulas, etc. Note, $\vec{A}$ and $\vec{B}$ are both vectors.

$$\vec{A}+\vec{B}, \qquad \vec{A}-\vec{B}, \qquad \vec{A}\cdot\vec{B}, \qquad \vec{A}\times\vec{B}, \qquad \vec{A}\vec{B}, \qquad \frac{\vec{A}}{\vec{B}}$$

PS 226 — Quiz

Name ________________________________ Score: $\boxed{}$/10

Consider three identical circles of radius r. If you place all three in contact, as shown, there is a small (light green) area that they enclose. Determine that area.



PS 226 — Quiz

Name ______________________________ Score: $\boxed{}$/10

A proton is approximately a sphere with a diameter of about 1.2 fm (1 fm $= 10^{-15}$ m), and a mass of approximately 1.67×10^{-27} kg. (a) What is the density of a proton? (b) Is this more dense or less dense than water?

(a)
(b)

PS 226 — Quiz

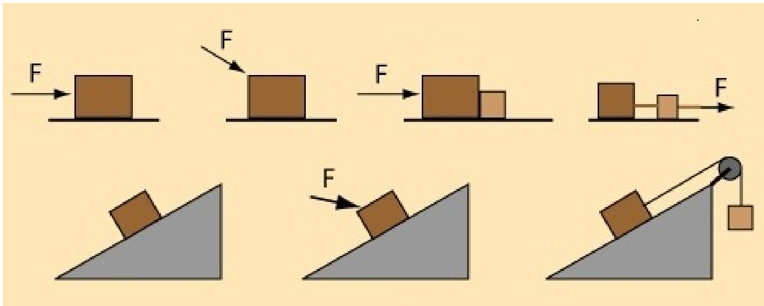
Name ______________________________ Score: $\boxed{}$/10

A tennis ball, initially traveling horizontally to the right at 41 m/s, hits a wall and bounces off with the same speed, but opposite direction. If it is in contact with the wall for 3 ms, what is the average acceleration of the tennis ball — both (a) magnitude *and* (b) direction — while it is in contact with the wall?

(a)
(b)

PS 226 — Take-home Quiz

Name ______________________________ Score: ☐/10



1. For each of the situations shown in the figure, draw correct Free Body Diagram(s) for the relevant object(s). Consider all surfaces to be frictionless, and the pulley to be massless and frictionless. Correctly label all the forces acting.

(a) The block on the horizontal surface, with a horizontal force applied.

(b) The block on the horizontal surface, with a force applied at an angle ϕ below horizontal.

(c) The two blocks on the horizontal surface, with a horizontal force applied to the left block.

(d) The two blocks on the horizontal surface, connected by a massless string, with a horizontal force applied to the right block.

(e) The block on the angled surface. The surface makes an angle of θ above horizontal.

(f) The block on the angled surface with an applied force. The surface makes an angle of θ above horizontal. The force is applied at an angle ϕ below horizontal.

(g) The block on the angled surface, connected by a massless string to a second block, which is hanging. The surface makes an angle of θ above horizontal.

PS 226 — Quiz

Name ______________________________ Score: $\boxed{}$/10

A daredevil (of mass m) rides a motorcycle on the "Wall of Death," where she rides (at speed v) in a horizontal circle (of radius r) on the inside of a cylinder. Using our proven technique (picture, free-body diagram, and application of Newton's second law), determine the minimum value of μ_s that must exist between the tires and the wall in order for the motorcycle not to fall.

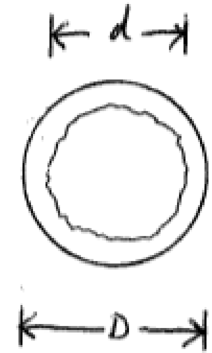
HINT: Analyze the motion at one instant.

$\mu_{s,min} =$

PS 226 — Quiz

Name ______________________________ Score: ☐/10

In Arthur C. Clarke's science fiction novel *Rendezvous with Rama*, a cylindrical spacecraft entered the solar system, spinning about its axis of symmetry. Although the outside of the cylinder had a diameter of $D = 20$ km, it was partially filled with "mountains" so that the "ground" that you walked on had a diameter of only $d = 16$ km. It was observed to spin at a rate of 0.25 rpm (revolutions per minute). Calculate the effective acceleration due to "gravity" you would experience standing on the interior.



PS 226 — Quiz

Name ________________________________ Score: ☐/10

A (nonuniform) force in the x-direction takes the form $F_x(x) = ax^2$, where $a = 5$ N/m^2. If a particle of mass $m = 5$ kg moves from $x_a = 2$ m to $x_b = 3$ m under the influence of this force, calculate how much work this force does (in Joules) on the particle during this displacement.

PS 226 — Quiz

Name ______________________________ Score: [ ] /10

The Sacramento River in California loses 2 feet of elevation as it travels from Sacramento to empty into the San Francisco Bay. (a) How much gravitational potential energy does 1 kg of water lose? (b) If there were no friction, and the river started at rest in Sacramento, how fast would it be moving when it reached the Bay?

(a)
(b)

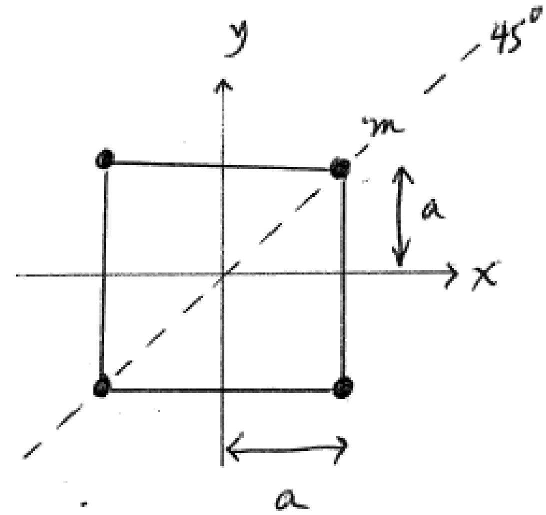
1 inch = 2.54 cm

PS 226 — Quiz

Name ______________________________ Score: $\boxed{}$/10

Consider the rigid object shown in the figure: 4 identical particles of mass m at the corners of a square of sides $2a$ (connected by . You have already calculated the moment of inertia about the x, y, and z axes.

Now I want you to calculate the moment of inertia about a diagonal axis that is in the x-y plane, but is rotated 45° from the x axis, as shown.



The solution for arbitrary rotation angle θ can be found using the rotation matrix. That is, if the coordinates of a point are (x, y) in the standard Cartesian frame, then the coordinates (x', y') in a frame rotated an angle θ counter-clockwise are

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$\vec{r'} = \mathsf{R}\,\vec{r}$$

where R is the rotation matrix.

The four mass points in the problem are at positions

$$(x, y) = (a, a), (-a, a), (-a, -a), (a, -a)$$

and the y' values (in the rotated frame) are

$$\begin{aligned} y_1' &= a(-\sin\theta + \cos\theta) \\ y_2' &= a(+\sin\theta + \cos\theta) \\ y_3' &= a(+\sin\theta - \cos\theta) \\ y_4' &= a(-\sin\theta - \cos\theta) = -a(+\sin\theta + \cos\theta) \end{aligned}$$

and the squares of these values are

$$\begin{aligned} (y_1')^2 &= a^2(1 - 2\sin\theta\cos\theta) \\ (y_2')^2 &= a^2(1 + 2\sin\theta\cos\theta) \\ (y_3')^2 &= a^2(1 - 2\sin\theta\cos\theta) \\ (y_4')^2 &= a^2(1 + 2\sin\theta\cos\theta) \end{aligned}$$

The moment of ineria about the x' axis is

$$I_{x'} = \sum_i m_i (y_i')^2 = m \sum_{i=1}^{4} (y_i')^2 = m(4a^2) = 4ma^2$$

where all the cross terms that depend on θ cancel out. Similarly, as you can check for yourself, $I_{y'} = 4ma^2$. This is a consequence of an important theorem regarding the symmetry of an object, which can be stated as follows.

Theorem If an object is symmetric about a certain axis, and that symmetry is of order 3 or higher, then all moments of inertia about axes perpendicular to that axis of symmetry are identical.

Definition An object has a symmetry of order n about an axis if when you rotate that object by $360°/n$ it maps into itself. For example, an equilateral triangle has a symmetry of order 3 about an axis perpendicular to the plane of the triangle and through the geometric center. Also, a square has a symmetry of order 4.

PS 226 — Quiz

Name ______________________________ Score: ☐/10

You are standing on the equator of the Earth and the Sun is directly overhead. Calculate the net bra citational force ON you DUE TO the Earth and the Sun. That is, calculate its magnitude and state which direction it points. A picture and force diagram is helpful.

Let your mass be $m = 100$ kg. See the table for other useful numerical values.