Evaluate the following

1. $\sin(30°) =$
2. $\sin(\pi/3) =$
3. $\cos(60°) =$
4. $\cos(5\pi/6) =$
5. $\sin(135°) =$

Express the following as simple trigonometric functions. The first is an example.

1. $\sin(90° + \phi) = \cos\phi$
2. $\sin(90° - \phi) =$
3. $\cos(\beta + 90°) =$
4. $\sin(A - 180°) =$
5. $\cos(270° - \alpha) =$

Consider a cube of side a.
(a) What is the radius of a circumscribed sphere?
(b) What is the radius of an inscribed sphere?
(c) Calculate the ratio of these two radii.

Two boats leave from opposite banks of a river at the same time and travel at constant but different speeds. They pass each other 700 yards from one bank and continue to the other side of the river, where they turn around. On their return trip the boats pass again—this time 400 yards from the opposite bank. How wide is the river?

Solutions Let $f = 700$ yds and $g = 400$ yds, and let Boat 1 have speed v_1 and Boat 2 have speed v_2. Finally, let Δt_1 be the time it takes both boats to leave from the banks and meet the first time, and Δt_2 be the time *between* the two meetings. Finally, at the first meeting, Boat 2 is 700 yds from its initial position (shore), and d is the width of the river.

Using the constant speed formula from the kinematic equations I get, for the first time interval

$$d - f = v_1 \Delta t_1 \tag{1}$$
$$f = v_2 \Delta t_1 \tag{2}$$

and for the second time interval

$$f + (d - g) = v_1 \Delta t_2 \tag{3}$$
$$g + (d - f) = v_2 \Delta t_2 \tag{4}$$

There are four equations and 5 unknowns: v_1, v_2, Δt_1, Δt_2, and d. Since this is a nonlinear system, it turns out we can only determine 3 values: the width d and the two ratios v_1/v_2 and $\Delta t_2/\Delta t_1$.

First, adding (1)+(2) and (3)+(4) gives the following two equations

$$d = (v_1 + v_2)\Delta t_1 \tag{5}$$
$$2d = (v_1 + v_2)\Delta t_1 \tag{6}$$

and the ratio of these two equations gives

$$\frac{\Delta t_2}{\Delta t_1} = 2. \tag{7}$$

This is an interesting result in its own right. The time interval between the two meetings is twice as long as the first time interval, *regardless of the points chosen*! This is obvious if the boats travel at the same speed and both passing points are chosen to be in the exact middle of the river. But it is not so obvious that it works in other cases.

Second, we can eliminate the speeds by taking the ratios (3)/(1) and (4)/(2)

$$\frac{f + d - g}{d - f} = 2 \tag{8}$$
$$\frac{g + d - f}{f} = 2 \tag{9}$$

and these two equations are not linearly independent as they both result in the same value for d

$$\boxed{d = 3f - g}. \tag{10}$$

For the current case, this gives $d = 1700$ yds.

Third, the time intervals can be eliminated with the ratios (1)/(2) and (3)/(4)

$$\frac{d - f}{f} = \frac{v_1}{v_2} \tag{11}$$
$$\frac{f + d - g}{g + d - f} = \frac{v_1}{v_2} \tag{12}$$

and again these two equations result in the same value for the ratio of the two speeds

$$\frac{v_1}{v_2} = 2 - \frac{f}{g}. \tag{13}$$

For the current case, this gives $v_1/v_2 = 10/7$.

If you go on a bike ride and ride for half the time at 10 mph, and half the time at 20 mph, what is your average speed?

If you ride a bike up a hill at 10 mph, and then down the same hill at 20 mph, what is your average speed?

Consider a river flowing with speed v, and a swimmer able to swim at speed c relative to the water. (a) Calculate the time t_u it takes the swimmer to swim a distance d upstream and back, where d is the distance measured relative to the stationary river bank.

(b) Calculate the time t_a it takes the swimmer to swim a distance d directly across the river and back (perpendicular to the river bank).

Solutions The Galilean transformation is needed here, which states that

$$\vec{v}_{SG} = \vec{v}_{SW} + \vec{v}_{WG},$$

or in words, "the velocity of the Swimmer relative to the Ground is equal to the velocity of the Swimmer relative to the Water plus the velocity of the Water relative to the Ground." Note that this is a *vector* equation, so the magnitudes don't necessarily add. In this problem, let's let $\vec{v}_{WG} = -v\hat{y}$ and $|\vec{v}_{SW}| = c$.

(a) While swimming upstream, the swimmer's speed relative to the ground is reduced $\vec{v}_{SG} = (+c - v)\hat{y}$ so that

$$\Delta t_u = \frac{d}{c - v}.$$

Similarly, swimming downstream the swimmer goes faster $\vec{v}_{SG} = (-c - v)\hat{y}$ and $\Delta t_d = d/(c + v)$. The total time taken is

$$\Delta t = \Delta t_u + \Delta t_d = \frac{d}{c - v} + \frac{d}{c + v} = \frac{2cd}{c^2 - v^2}.$$

(b) If the swimmer wants to move *directly* across the river, they must angle slightly upstream so they don't drift downstream. In this case, c is the hypotenuse of the right triangle, v is one side, and therefore $|\vec{v}_{SG}| = \sqrt{c^2 - v^2}$ is the speed of the swimmer relative to the ground. The time taken to swim across the river is the same as that to swim back (same speed), so that the total time taken is

$$\Delta t = \frac{2d}{\sqrt{c^2 - v^2}}.$$

(c) The ratio shows that it is quicker to swim across and back

$$\frac{\Delta t_{up}}{\Delta t_{across}} = \frac{c}{c^2 - v^2}\sqrt{c^2 - v^2} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \geq 1.$$

This, in fact, is exactly the analysis needed to interpret the Michelson-Morley experiment. The "swimmer" in that case is light, and the "river" is the ether. Michelson and Morley measured the two travel times and tried to detect a difference, which would have allowed them to determine the speed of the Earth relative to the ether. However, since their result was that the two times were identical, FitzGerald and Lorentz proposed that objects (i.e., their measuring apparatus) *contracted* in length by a factor γ in the direction of motion. This *ad hoc* proposal would result in the two travel times being identical. Of course, there *is* a "Lorentz contraction," but for reasons having to do with observers in different reference frames (i.e., special relativity), rather than a physical contraction of objects.

The projectile motion equations are

$$y = v_{0y}t - \frac{1}{2}gt^2 \qquad\qquad x = v_{0x}t$$

$$v_y = v_{0y} - gt \qquad\qquad v_y^2 = v_{0y}^2 - 2gy$$

where

$$v_{0y} = v_0 \sin\theta_0 \qquad\qquad v_{0x} = v_0 \cos\theta_0$$

The shape of the trajectory, expressed as $y(x)$, is found by eliminating t from the x and y equations

$$y(x) = \tan\theta_0\, x - \frac{g}{2v_{0x}^2}x^2$$

(a) Sketch this function

Find (in terms of the initial conditions v_0 and θ_0)
(b) the maximum height (height at the apex), y_{max},
(c) the time to reach the apex, t_{apex}, and
(d) the horizontal distance traveled (range) on level ground, x_{max}, or R.

Answers:

$$\begin{aligned} y_{max} &= v_0^2 \sin^2\theta_0/2g \\ t_{apex} &= v_0 \sin\theta_0/g \\ R = x_{max} &= v_0^2 \sin 2\theta_0/g = v_0^2(2\sin\theta_0\cos\theta_0)/g \end{aligned}$$

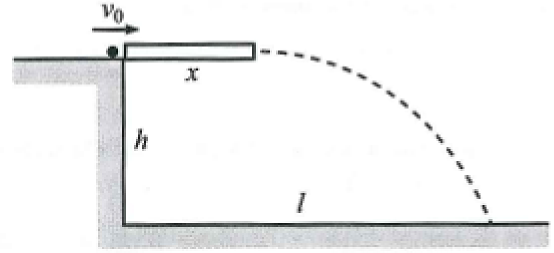
Problems for me to write on the board.

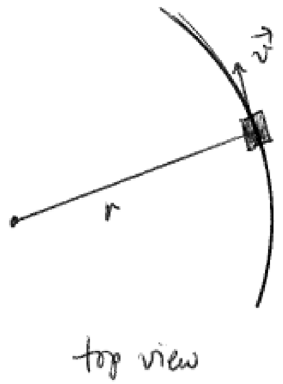
1. Given v_0, find θ_0 that maximizes R. ($\theta_0 = \pi/2$)
2. Given θ_0, find v_0 that maximizes R. ($v_0 \to \infty$)
3. At what initial angle θ_0 does the range equal the maximum height? ($\tan\theta_0 = 4$, $\theta_0 \approx 76.0°$)
4. If the initial height is h above level ground, find the range.

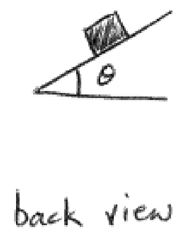
$$\left(\frac{g}{2v_{0x}^2}x^2\right) - \tan\theta_0 x - h = 0$$

$$\begin{aligned} x_{max} &= \frac{v_0^2\cos\theta_0}{g}\left(\sin\theta_0 + \sqrt{\sin^2\theta_0 + \frac{2gh}{v_0^2}}\right) \\ &= \frac{v_0^2\cos\theta_0\sin\theta_0}{g}\left(1 + \sqrt{1 + \frac{2gh}{v_0^2\sin^2\theta_0}}\right) \end{aligned}$$

5. Given v_0, find θ_0 that maximizes R.
6. A projectile is fired horizontally with speed v_0 from the top of a cliff of height h. It immediately enters a fixed tube with length x. There is friction between the projectile and the tube, the effect of which is to make the projectile decelerate with constant acceleration $-a$ ($a > 0$). After the projectile leaves the tube, it undergoes normal projectile motion down to the ground.
   (a) Find the total horizontal distance ℓ in terms of quantities given.
   (b) What value of x yields the maximum value of ℓ?







1. Draw, on each picture above, a vector representing the car's acceleration.
2. Using the perspective on the right, draw a free-body diagram for the forces on the car. Assume the surface is frictionless.
3. Write the components of N2, and solve for the speed of the car. Use values for the Daytona Speedway: $r = 1000$ feet, $\theta = 31°$.

4. Repeat the calculation assuming a static friction coefficient μ_s for

   (a) when the car is going "too fast"

   (b) when the car is going "too slow"

Viscous drag (e.g., air resistance)

For low speeds, in the "low Reynolds number regime," the drag force on an object is proportional to the velocity

$$\vec{F}_{drag} = -b\vec{v}$$

where b depends on the viscous fluid and the moving object. This is called *Stokes' law.*

1. Consider a particle of mass m in free fall.

    (a) Draw a free-body diagram and include air resistance, using Stokes' law.

    (b) Write N2 in the vertical direction, using $a = dv/dt$.

    (c) Determine a formula for the *terminal velocity*, i.e., when the acceleration is zero.

2. Consider a particle of mass m that is immersed in a fluid and starts at $t = 0$ moving with speed v_0.

    (a) Draw a free-body diagram and include viscous drag, but ignore gravity.

    (b) Write N2, using $a = dv/dt$.

    (c) Solve the full differential equation for $v(t)$.

    (d) Integrate the velocity to obtain the position $x(t)$ as a function of time.

For high speeds, when the flow becomes turbulent, the drag force is proportional to v^2

$$F_{drag} = \frac{1}{2} C_d \rho A v^2$$

where C_d is called the *drag coefficient*.

Elastic collisions in 1D

Two objects, m_1 and m_2, move in one dimension and collide.

BEFORE the collision, their velocities are v_{1i} and v_{2i}, respectively, where "i" stands for "initial."

AFTER the collision, their velocities are v_{1f} and v_{2f}, respectively, where "f" stands for "final."

Conservation of momentum (in the x direction) gives

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \tag{1}$$

Conservation of kinetic energy gives

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \tag{2}$$

Assuming that all quantities are known except for v_{1f} and v_{2f}, solve for these two unknowns in the following way:

1. Multiply Eq. (2) by 2.
2. In BOTH equations, move all terms referring to m_1 to the left side of the equation and all terms referring to m_2 to the right side of the equation.
3. Factor out m_1 and m_2 from both sides of both equations.
4. Now, divide Eq. (2) by Eq. (1). That is, set the ratio of the left hand sides equal to the ratio of the right hand sides.
5. Finally, use your knowledge of binomials to simplify this equation, and ...
6. ... group all "final" terms on one side of the equation, and "initial" terms on the other side.
7. **Five Isaac Newton points** for: Solve Eqs. (1) and (2) for v_{1f} and v_{2f} in gory detail by utilizing the quadratic equation. Write it up neatly and turn it in. Be sure to express the unknown quantities, v_{1f} and v_{2f}, completely in terms of the known quantities (all other variables). Explicitly show the two different solutions.

Result:

$$(v_{1f} - v_{2f}) = -(v_{1i} - v_{2i})$$

Newton's modification:

$$(v_{1f} - v_{2f}) = -e(v_{1i} - v_{2i})$$

where e is the *coefficient of restitution*.

Elastic collisions in 1D Conservation of momentum and energy gives

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \tag{1}$$

$$\frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2 \tag{2}$$

which are two equations for our two unknowns, v_{1f} and v_{2f}. Solving this *quadratic* set of equations gives two solutions. One of the solutions, the "trivial" solution, is

$$v_{1f} = v_{1i} \tag{3a}$$

$$v_{2f} = v_{2i} \tag{3b}$$

which corresponds to the unphysical situation of the particles passing through each other. Newton was able to cleverly combine Eqs. (1) and (2) to eliminate the trivial solution and obtain the *linear* equation

$$(v_{1f} - v_{2f}) = -(v_{1i} - v_{2i}). \tag{4}$$

Now, solving Eqs. (1) and (4) gives the following result

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}, \tag{5a}$$

$$v_{2f} = \frac{m_2 - m_1}{m_1 + m_2} v_{2i} + \frac{2m_1}{m_1 + m_2} v_{1i}. \tag{5b}$$

Note that (1) and (4) are linear equations, and can be solved by the method of Problem 2 in "Classical Mechanics Problems." Note also that Eqs. (5) are symmetric, and (5b) can be obtained from (5a) by simply making the replacements $1 \leftrightarrow 2$. Hence, our two sets of solutions to Eqs. (1) and (2) are given by Eqs. (3) and (5).

Your task This same result can be obtain by, for example, solving Eq. (1) for v_{1f}, inserting that result into Eq. (2), solving the resulting quadratic equation for v_{2f}, and re-inserting that result into Eq. (1) and obtaining two solutions for v_{1f}. You should be able to show that these two solutions are identical to Eqs. (3) and (5).

Solution Solving (1) for v_{1f}

$$v_{1f} = \frac{m_1 v_{1i} + m_2 v_{2i} - m_2 v_{2f}}{m_1} \tag{6}$$

$$= \left(v_{1i} + \frac{m_2}{m_1} v_{2i} \right) - \frac{m_2}{m_1} v_{2f} \tag{7}$$

where I've separated out the other unknown, v_{2f}. Squaring and inserting into (2) results in a quadratic equation for v_{2f}

$$a v_{2f}^2 + b v_{2f} + c = 0, \tag{8}$$

where

$$a = \frac{m_2}{m_1} M \tag{9}$$

$$b = -2\frac{m_2}{m_1}(m_1 v_{1i} + m_2 v_{2i}) \tag{10}$$

$$c = \frac{m_2}{m_1} v_{2i} \left\{2m_1 v_{1i} + (m_2 - m_1) v_{2i}\right\} \tag{11}$$

The discriminant of this quadratic equation is

$$b^2 - 4ac = \left[-2\frac{m_2}{m_1}(m_1 v_{1i} + m_2 v_{2i})\right]^2 - 4\left[\frac{m_2}{m_1} M\right]\left[\frac{m_2}{m_1} v_{2i} \left\{2m_1 v_{1i} + (m_2 - m_1) v_{2i}\right\}\right] \tag{12a}$$

$$= \left(2\frac{m_2}{m_1}\right)^2 \left[(m_1 v_{1i} + m_2 v_{2i})^2 - M v_{2i} \left\{2m_1 v_{1i} + (m_2 - m_1) v_{2i}\right\}\right] \tag{12b}$$

where I've factored out $4m_2^2/m_1^2$. Now, if the terms in the square brackets [ ] are expanded, many of them will cancel, and you'll be left with

$$b^2 - 4ac = \left(2\frac{m_2}{m_1}\right)^2 \left[m_1(v_{1i} - v_{2i})\right]^2. \tag{13}$$

The standard solution to the quadratic equation then gives a result that is easily interpreted

$$v_{2f} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{14a}$$

$$= \frac{m_1}{2m_2 M}\left[2\frac{m_2}{m_1}(m_1 v_{1i} + m_2 v_{2i}) \pm 2\frac{m_2}{m_1} m_1 (v_{1i} - v_{2i})\right] \tag{14b}$$

$$= \frac{(m_1 v_{1i} + m_2 v_{2i}) \pm (m_1 v_{1i} - m_1 v_{2i})}{M} \tag{14c}$$

Here, it is clear that the $+$ sign results in Eq. (5b), and the $-$ sign results in the trivial solution, $v_{2f} = v_{2i}$. Plugging these two results into Eq. (7) will result in Eq. (5a) and the trivial solution for v_{1f}, as expected.

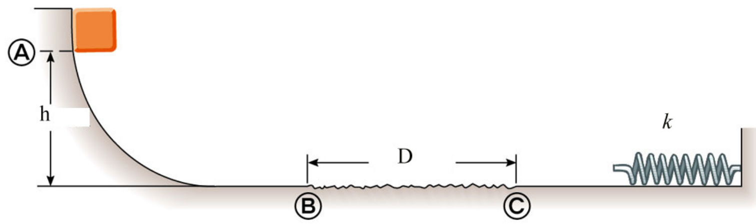
Limit check You should always check mathematical answers to make sure they agree with what you expect physically in certain simple cases. One such simple situation is to let $v_{1i} = V$ and $v_{2i} = 0$, particle 1 moves to the right and particle two is stationary. Then Eqs. (5) become

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} V, \tag{15a}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} V. \tag{15b}$$

- case i: $m_1 = m_2$. This is the case of two billiard balls colliding, where $v_{1f} = 0$ and $v_{2f} = V$. The balls simply exchange velocities, as expected.
- case ii: $m_1 \ll m_2$. Here, the stationary ball is extremely massive, and acts like a wall. The final velocities are $v_{1f} \approx -V$ and $v_{2f} \approx 0$. The massive ball is unaffected and the light ball bounces back with the same speed.
- case iii: $m_1 \gg m_2$. Here, the moving ball is extremely massive, and simply whacks the light, stationary ball "out of the way." The final velocities are $v_{1f} \approx V$ and $v_{2f} \approx 2V$. The massive ball again is unaffected and the light ball is flung forward with twice the speed.

A block of mass $m = 100$ g is released from rest at point **A**, slides down a height $h = 2$ m to the spring (with force constant $k = 25$ N/m). The only portion of the track that is not frictionless is the rough surface of length $D = 2$ m between **B** and **C**, which has a coefficient of kinetic friction $\mu = 0.3$.



(a) Calculate the speed of the block at point **B**.

(b) Calculate the speed of the block at point **C**.

(c) Calculate the maximum distance that the spring is compressed.

(d) The block now slides to the left and up the hill. Calculate the maximum height of the block on the hill.

(e) Where is the block when it finally stops?

(f) Assume that μ is unknown, and that h, D, m, and k are known but arbitrary. Determine a formula for μ if you observe that the block does not fully cross the rough patch, but only travels a distance ℓ.

(g) Determine a formula for μ if you observe that the block crosses the rough patch n times and then comes to rest a distance ℓ into the rough patch during its $(n+1)$th crossing.

ans: (a) 6.26 m/s (b) 5.24 m/s (c) 0.331 m (d) 0.8 m (e) 3 times plus 0.667 m (f) $\mu = h/\ell$ (g) $\mu = h/(\ell + nD)$

(h) Go back and calculate the mechanical energy of the block at points A, B, C, full compression, C, B, maximum height, etc.

Moment of inertia

To calculate the moment of inertia I for a rigid object that consists of discrete particles

(a) choose your axis

(b) determine the *perpendicular* distance of the ith particle from the axis, r_i

(c) evaluate the sum $I \equiv \sum_i m_i r_i^2$

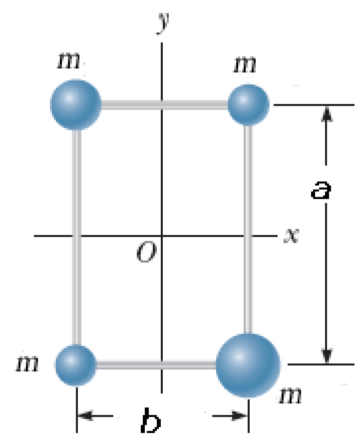
1. Consider the rigid object shown in the figure: 4 identical particles of mass m in the shape of a rectangle of sides a and b. Calculate the moment of inertia

   (a) about the x axis

   (b) about the y axis

   (c) about the z axis

   Express your answers in terms of the total mass $M = 4m$.



For a continuous object, the sum in the moment of inertia definition becomes an integral

$$I = \int r^2 \, dm$$

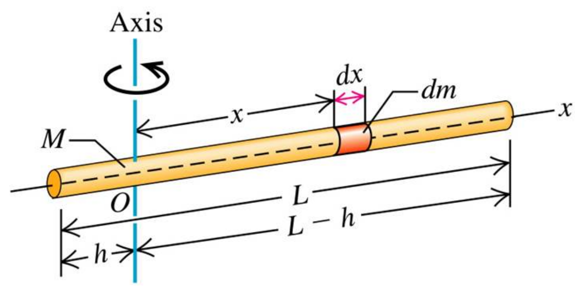
where $dm = \lambda \, dx$ or $dm = \sigma \, dA$ or $dm = \rho \, dV$, depending on the dimensionality of the object.

2. Consider a uniform rod of length L and mass M with a mass-per-unit-length $\lambda = M/L$. Calculate the moment of inertia of the rod about an axis perpendicular to the rod that

   (a) passes through the end of the rod (i.e., $h = 0$)

   (b) passes through the center of the rod (i.e., $h = L/2$)

   (c) passes through an arbitrary point of the rod (i.e., h is arbitrary)



3. Calculate the moment of inertia of a ring of mass M and radius R about an axis

   (a) perpendicular to the plane of the ring that passes through the center of the ring

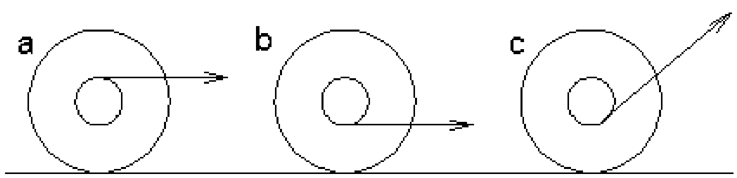
   (b) perpendicular to the axis in part (a) but that also passes through the center of the ring.

Group Project — Pulling a spool

Consider a spool of mass m with outer radius R, and inner radius r, and moment of inertia I, as shown in the figure. If a string that is wrapped around the inner radius is pulled with an external force F, then answer the following questions.

For parts (a) and (b), assume that the static friction force f is large enough (or, equivalently that the external force F is small enough) for the spool to roll without slipping. Also, express your answer in terms of the known quantities: m, R, r, I, and F. Recall that the magnitude of f is unknown.

For part (c) slightly different assumptions must be made.



(a) Calculate the acceleration of the spool.
(b) Calculate the acceleration of the spool.
(c) Determine the critical angle for which the spool will slip without rolling!

Group Project — Gravity force and orbits

Solve the following problems - in order. It helps immensely to draw a picture for each one.

1. Two particles of identical mass m are in circular orbits around their common center of mass, and they are acted on only by their mutual gravitational attraction. If the distance d between the two particles is constant, what must be the angular velocity ω of their orbit about their center of mass?

2. Three particles of identical mass m are in circular orbits around their center of mass, and they are acted on only by their mutual gravitational attraction. They are located at the vertices of an equilateral triangle with sides of length d. If the distance d between the particles is constant, what must be the angular velocity ω of their orbit about their center of mass?

3. Four particles of identical mass m are in circular orbits around their center of mass, and they are acted on only by their mutual gravitational attraction. They are located at the vertices of a square with sides of length d. If the distances between the particles is constant, what must be the angular velocity ω of their orbit about their center of mass?