established for line integrals in two dimensions. Proofs for the three-dimensional case are similar and will be omitted.

It will be assumed throughout this section that all regions are **connected**. This means that any two points in a region can be joined by a piecewise smooth curve which lies in the region. We shall also assume that for any point A in a plane region D, there exists a circle with center A which lies completely in D. A region of this type is called an **open region**. The next theorem gives us the fundamental result that if a vector function $\mathbf{F}$ is continuous on D, then the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ *is independent of path if and only if* $\mathbf{F}$ *is conservative.*

(18.15) Theorem

> If $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is continuous on an open connected region, then the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if and only if $\mathbf{F}(x, y) = \nabla f(x, y)$ for some function f.

Proof. Suppose the integral is independent of path. If (x_0, y_0) is a fixed point in D, let f be defined by

$$f(x, y) = \int_{(x_0, y_0)}^{(x, y)} \mathbf{F} \cdot d\mathbf{r}$$

where (x, y) is arbitrary in D. Since the integral is independent of path, f depends only on x and y, and not on the path C from (x_0, y_0) to (x, y). Choose a circle in D with center (x, y) and let (x_1, y) be a point within the circle such that $x_1 \neq x$. Let C_1 be any path from (x_0, y_0) to (x_1, y) and let C_2 be the horizontal segment from (x_1, y) to (x, y), as illustrated in (i) of Figure 18.8.

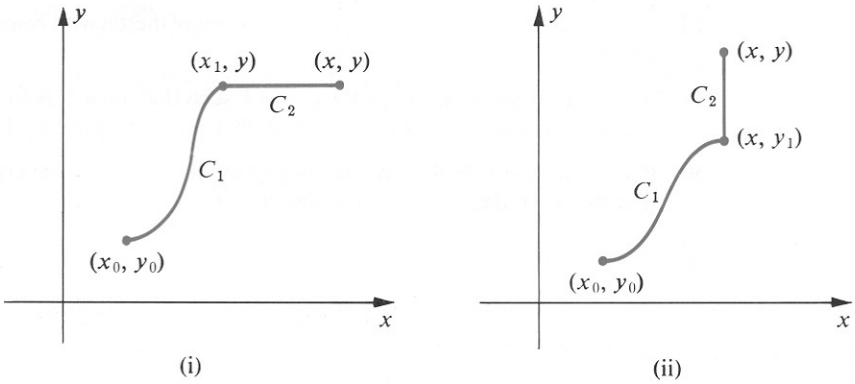

Figure 18.8

We may, therefore, write

$$f(x, y) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{(x_0, y_0)}^{(x_1, y)} \mathbf{F} \cdot d\mathbf{r} + \int_{(x_1, y)}^{(x, y)} \mathbf{F} \cdot d\mathbf{r}.$$