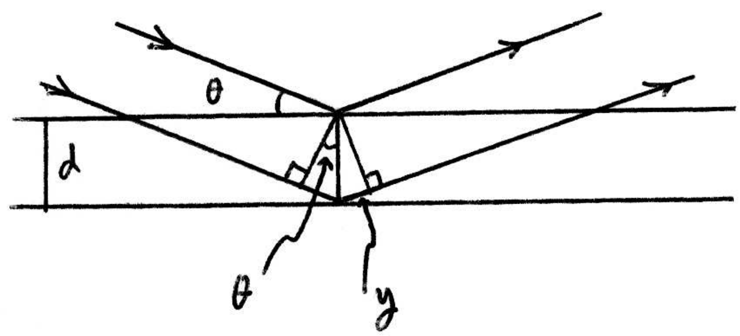
Derivation of *Bragg's Law*, also known as the Bragg scattering condition

There are (at least) three ways to derive the Bragg scattering condition, applicable to electromagnetic waves, usually X-rays, scattering from a crystalline solid. This "law" was obtained by William Lawrence Bragg in 1912 (published in 1913[1]) while trying to decipher the X-ray scattering results of Max von Laue. He and his father, William Henry Bragg, shared the 1915 Nobel Prize in Physics "for their services in the analysis of crystal structure by means of X-rays."

Incident plane waves are scattered by the atomic nuclei (called scattering centers) in a crystal in a manner that resembles reflection. (This atomic scattering is called Rayleigh scattering, and explains why the sky is blue.) Each atom acts as a point source so that all the atoms in a single plane together radiate coherently to produce reflection, similar to Huygens's construction of reflection from a plane mirror. This means that two parallel planes act very similarly to a thin film, except that the index of refraction of the "medium" between the planes is unity.

- First, consider two parallel rays of light incident on, and reflecting from, two of the parallel atomic planes (called "crystallographic planes"), separated by a distance d.



Given Huygens's principle, which states that perpendicular lines connecting the rays mark points of identical phase, we can selectively draw these perpendicular lines so that $2y$ is the extra distance that the lower ray travels. Given the geometry of the small right triangle, where $\sin\theta = y/d$, we have

$$\Delta r = 2y = 2d\sin\theta.$$

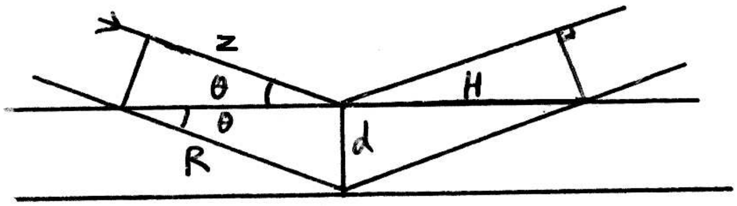
This implies that for constructive interference

$$2d\sin\theta = n\lambda$$

is required, which is just the Bragg condition.

[1] W. L. Bragg, "The Diffraction of Short Electromagnetic Waves by a Crystal," *Proc. Cambridge Philosophical Soc.*, **17**, 43-57 (1913).

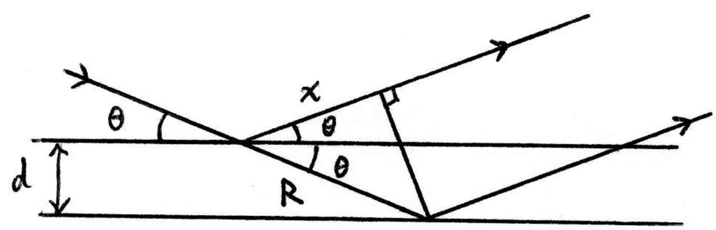
- For the second method, two different perpendicular connectors between the rays can be drawn, as shown below.



In this case the path difference is $2R - 2z$. In order to relate R and z to d and θ, the two triangles give the following trigonometric relations: $z = H\cos\theta$, $H = R\cos\theta$, and $d = R\sin\theta$. Using these three relations in order allows us to simplify the expression for the path difference

$$\begin{aligned}\Delta r &= 2(R - z) \\ &= 2(R - H\cos\theta) \\ &= 2R(1 - \cos^2\theta) \\ &= 2\,\frac{d}{\sin\theta}\,\sin^2\theta = 2d\sin\theta.\end{aligned}$$

- Third, we can assume that the two incoming rays are co-linear, and that when they reflect off their respective planes, the path difference is $R - x$, as shown in the figure.



From the large right triangle we have $x = R\cos 2\theta$, and from the smaller $d = R\sin\theta$ (this is identical to one of the relations in the second derivation). Again, using these in order, along with standard trig identities, gives

$$\begin{aligned}\Delta r &= R - x \\ &= R(1 - \cos 2\theta) = R(2\sin^2\theta) \\ &= \frac{d}{\sin\theta}\,2\sin^2\theta = 2d\sin\theta.\end{aligned}$$

As expected, we obtain the same expression for the path difference.