

Gravitation

13-1 NEWTON'S LAW OF GRAVITATION

Learning Objectives

After reading this module, you should be able to . . .

- 13.01** Apply Newton's law of gravitation to relate the gravitational force between two particles to their masses and their separation.
- 13.02** Identify that a uniform spherical shell of matter attracts a particle that is outside the shell as if all the shell's mass were concentrated as a particle at its center.

- 13.03** Draw a free-body diagram to indicate the gravitational force on a particle due to another particle or a uniform, spherical distribution of matter.

Key Ideas

- Any particle in the universe attracts any other particle with a gravitational force whose magnitude is

$$F = G \frac{m_1 m_2}{r^2} \quad (\text{Newton's law of gravitation}),$$

where m_1 and m_2 are the masses of the particles, r is their separation, and $G (= 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2)$ is the gravitational constant.

- The gravitational force between extended bodies is found by adding (integrating) the individual forces on individual particles within the bodies. However, if either of the bodies is a uniform spherical shell or a spherically symmetric solid, the net gravitational force it exerts on an *external* object may be computed as if all the mass of the shell or body were located at its center.

What Is Physics?

One of the long-standing goals of physics is to understand the gravitational force—the force that holds you to Earth, holds the Moon in orbit around Earth, and holds Earth in orbit around the Sun. It also reaches out through the whole of our Milky Way galaxy, holding together the billions and billions of stars in the Galaxy and the countless molecules and dust particles between stars. We are located somewhat near the edge of this disk-shaped collection of stars and other matter, 2.6×10^4 light-years (2.5×10^{20} m) from the galactic center, around which we slowly revolve.

The gravitational force also reaches across intergalactic space, holding together the Local Group of galaxies, which includes, in addition to the Milky Way, the Andromeda Galaxy (Fig. 13-1) at a distance of 2.3×10^6 light-years away from Earth, plus several closer dwarf galaxies, such as the Large Magellanic Cloud. The Local Group is part of the Local Supercluster of galaxies that is being drawn by the gravitational force toward an exceptionally massive region of space called the Great Attractor. This region appears to be about 3.0×10^8 light-years from Earth, on the opposite side of the Milky Way. And the gravitational force is even more far-reaching because it attempts to hold together the entire universe, which is expanding.

This force is also responsible for some of the most mysterious structures in the universe: *black holes*. When a star considerably larger than our Sun burns out, the gravitational force between all its particles can cause the star to collapse in on itself and thereby to form a black hole. The gravitational force at the surface of such a collapsed star is so strong that neither particles nor light can escape from the surface (thus the term “black hole”). Any star coming too near a black hole can be ripped apart by the strong gravitational force and pulled into the hole. Enough captures like this yields a *supermassive black hole*. Such mysterious monsters appear to be common in the universe. Indeed, such a monster lurks at the center of our Milky Way galaxy—the black hole there, called Sagittarius A*, has a mass of about 3.7×10^6 solar masses. The gravitational force near this black hole is so strong that it causes orbiting stars to whip around the black hole, completing an orbit in as little as 15.2 y.

Although the gravitational force is still not fully understood, the starting point in our understanding of it lies in the *law of gravitation* of Isaac Newton.

Newton's Law of Gravitation

Before we get to the equations, let's just think for a moment about something that we take for granted. We are held to the ground just about right, not so strongly that we have to crawl to get to school (though an occasional exam may leave you crawling home) and not so lightly that we bump our heads on the ceiling when we take a step. It is also just about right so that we are held to the ground but not to each other (that would be awkward in any classroom) or to the objects around us (the phrase “catching a bus” would then take on a new meaning). The attraction obviously depends on how much “stuff” there is in ourselves and other objects: Earth has lots of “stuff” and produces a big attraction but another person has less “stuff” and produces a smaller (even negligible) attraction. Moreover, this “stuff” always attracts other “stuff,” never repelling it (or a hard sneeze could put us into orbit).

In the past people obviously knew that they were being pulled downward (especially if they tripped and fell over), but they figured that the downward force was unique to Earth and unrelated to the apparent movement of astronomical bodies across the sky. But in 1665, the 23-year-old Isaac Newton recognized that this force is responsible for holding the Moon in its orbit. Indeed he showed that every body in the universe attracts every other body. This tendency of bodies to move toward one another is called **gravitation**, and the “stuff” that is involved is the mass of each body. If the myth were true that a falling apple inspired Newton to his **law of gravitation**, then the attraction is between the mass of the apple and the mass of Earth. It is appreciable because the mass of Earth is so large, but even then it is only about 0.8 N. The attraction between two people standing near each other on a bus is (thankfully) much less (less than $1 \mu\text{N}$) and imperceptible.

The gravitational attraction between extended objects such as two people can be difficult to calculate. Here we shall focus on Newton's force law between two *particles* (which have no size). Let the masses be m_1 and m_2 and r be their separation. Then the magnitude of the gravitational force acting on each due to the presence of the other is given by

$$F = G \frac{m_1 m_2}{r^2} \quad (\text{Newton's law of gravitation}). \quad (13-1)$$

G is the **gravitational constant**:

$$\begin{aligned} G &= 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2 \\ &= 6.67 \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{s}^2. \end{aligned} \quad (13-2)$$



Courtesy NASA

Figure 13-1 The Andromeda Galaxy. Located 2.3×10^6 light-years from us, and faintly visible to the naked eye, it is very similar to our home galaxy, the Milky Way.

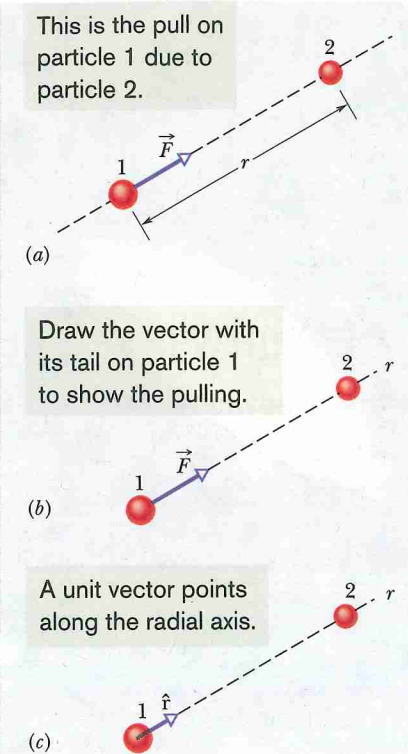


Figure 13-2 (a) The gravitational force \vec{F} on particle 1 due to particle 2 is an attractive force because particle 1 is attracted to particle 2. (b) Force \vec{F} is directed along a radial coordinate axis r extending from particle 1 through particle 2. (c) \vec{F} is in the direction of a unit vector \hat{r} along the r axis.

In Fig. 13-2a, \vec{F} is the gravitational force acting on particle 1 (mass m_1) due to particle 2 (mass m_2). The force is directed toward particle 2 and is said to be an *attractive force* because particle 1 is attracted toward particle 2. The magnitude of the force is given by Eq. 13-1. We can describe \vec{F} as being in the positive direction of an r axis extending radially from particle 1 through particle 2 (Fig. 13-2b). We can also describe \vec{F} by using a radial unit vector \hat{r} (a dimensionless vector of magnitude 1) that is directed away from particle 1 along the r axis (Fig. 13-2c). From Eq. 13-1, the force on particle 1 is then

$$\vec{F} = G \frac{m_1 m_2}{r^2} \hat{r}. \quad (13-3)$$

The gravitational force on particle 2 due to particle 1 has the same magnitude as the force on particle 1 but the opposite direction. These two forces form a third-law force pair, and we can speak of the gravitational force *between* the two particles as having a magnitude given by Eq. 13-1. This force between two particles is not altered by other objects, even if they are located between the particles. Put another way, no object can shield either particle from the gravitational force due to the other particle.

The strength of the gravitational force—that is, how strongly two particles with given masses at a given separation attract each other—depends on the value of the gravitational constant G . If G —by some miracle—were suddenly multiplied by a factor of 10, you would be crushed to the floor by Earth’s attraction. If G were divided by this factor, Earth’s attraction would be so weak that you could jump over a building.

Nonparticles. Although Newton’s law of gravitation applies strictly to particles, we can also apply it to real objects as long as the sizes of the objects are small relative to the distance between them. The Moon and Earth are far enough apart so that, to a good approximation, we can treat them both as particles—but what about an apple and Earth? From the point of view of the apple, the broad and level Earth, stretching out to the horizon beneath the apple, certainly does not look like a particle.

Newton solved the apple–Earth problem with the *shell theorem*:



A uniform spherical shell of matter attracts a particle that is outside the shell as if all the shell’s mass were concentrated at its center.

Earth can be thought of as a nest of such shells, one within another and each shell attracting a particle outside Earth’s surface as if the mass of that shell were at the center of the shell. Thus, from the apple’s point of view, Earth *does* behave like a particle, one that is located at the center of Earth and has a mass equal to that of Earth.

Third-Law Force Pair. Suppose that, as in Fig. 13-3, Earth pulls down on an apple with a force of magnitude 0.80 N. The apple must then pull up on Earth with a force of magnitude 0.80 N, which we take to act at the center of Earth. In the language of Chapter 5, these forces form a force pair in Newton’s third law. Although they are matched in magnitude, they produce different accelerations when the apple is released. The acceleration of the apple is about 9.8 m/s^2 , the familiar acceleration of a falling body near Earth’s surface. The acceleration of Earth, however, measured in a reference frame attached to the center of mass of the apple–Earth system, is only about $1 \times 10^{-25} \text{ m/s}^2$.



Checkpoint 1

A particle is to be placed, in turn, outside four objects, each of mass m : (1) a large uniform solid sphere, (2) a large uniform spherical shell, (3) a small uniform solid sphere, and (4) a small uniform shell. In each situation, the distance between the particle and the center of the object is d . Rank the objects according to the magnitude of the gravitational force they exert on the particle, greatest first.

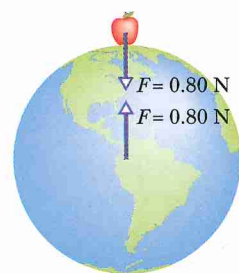


Figure 13-3 The apple pulls up on Earth just as hard as Earth pulls down on the apple.

13-2 GRAVITATION AND THE PRINCIPLE OF SUPERPOSITION

Learning Objectives

After reading this module, you should be able to . . .

13.04 If more than one gravitational force acts on a particle, draw a free-body diagram showing those forces, with the tails of the force vectors anchored on the particle.

13.05 If more than one gravitational force acts on a particle, find the net force by adding the individual forces as vectors.

Key Ideas

● Gravitational forces obey the principle of superposition; that is, if n particles interact, the net force $\vec{F}_{1,\text{net}}$ on a particle labeled particle 1 is the sum of the forces on it from all the other particles taken one at a time:

$$\vec{F}_{1,\text{net}} = \sum_{i=2}^n \vec{F}_{1i}$$

in which the sum is a vector sum of the forces \vec{F}_{1i} on particle 1 from particles 2, 3, . . . , n .

● The gravitational force \vec{F}_1 on a particle from an extended body is found by first dividing the body into units of differential mass dm , each of which produces a differential force $d\vec{F}$ on the particle, and then integrating over all those units to find the sum of those forces:

$$\vec{F}_1 = \int d\vec{F}.$$

Gravitation and the Principle of Superposition

Given a group of particles, we find the net (or resultant) gravitational force on any one of them from the others by using the **principle of superposition**. This is a general principle that says a net effect is the sum of the individual effects. Here, the principle means that we first compute the individual gravitational forces that act on our selected particle due to each of the other particles. We then find the net force by adding these forces vectorially, just as we have done when adding forces in earlier chapters.

Let’s look at two important points in that last (probably quickly read) sentence. (1) Forces are vectors and can be in different directions, and thus we must *add them as vectors*, taking into account their directions. (If two people pull on you in the opposite direction, their net force on you is clearly different than if they pull in the same direction.) (2) We *add* the individual forces. Think how impossible the world would be if the net force depended on some multiplying factor that varied from force to force depending on the situation, or if the presence of one force somehow amplified the magnitude of another force. No, thankfully, the world requires only simple vector addition of the forces.

For n interacting particles, we can write the principle of superposition for the gravitational forces on particle 1 as

$$\vec{F}_{1,\text{net}} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \vec{F}_{15} + \cdots + \vec{F}_{1n}. \quad (13-4)$$

Here $\vec{F}_{1,\text{net}}$ is the net force on particle 1 due to the other particles and, for example, \vec{F}_{13} is the force on particle 1 from particle 3. We can express this equation more compactly as a vector sum:

$$\vec{F}_{1,\text{net}} = \sum_{i=2}^n \vec{F}_{1i}. \quad (13-5)$$

Real Objects. What about the gravitational force on a particle from a real (extended) object? This force is found by dividing the object into parts small enough to treat as particles and then using Eq. 13-5 to find the vector sum of the forces on the particle from all the parts. In the limiting case, we can divide the extended object into differential parts each of mass dm and each producing a differential force $d\vec{F}$

Sample Problem 13.01 Net gravitational force, 2D, three particles

Figure 13-4a shows an arrangement of three particles, particle 1 of mass $m_1 = 6.0$ kg and particles 2 and 3 of mass $m_2 = m_3 = 4.0$ kg, and distance $a = 2.0$ cm. What is the net gravitational force $\vec{F}_{1,\text{net}}$ on particle 1 due to the other particles?

KEY IDEAS

(1) Because we have particles, the magnitude of the gravitational force on particle 1 due to either of the other particles is given by Eq. 13-1 ($F = Gm_1m_2/r^2$). (2) The direction of either gravitational force on particle 1 is toward the particle responsible for it. (3) Because the forces are not along a single axis, we cannot simply add or subtract their magnitudes or their components to get the net force. Instead, we must add them as vectors.

Calculations: From Eq. 13-1, the magnitude of the force \vec{F}_{12} on particle 1 from particle 2 is

$$F_{12} = \frac{Gm_1m_2}{a^2} = \frac{(6.67 \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{s}^2)(6.0 \text{ kg})(4.0 \text{ kg})}{(0.020 \text{ m})^2} = 4.00 \times 10^{-6} \text{ N}.$$

Similarly, the magnitude of force \vec{F}_{13} on particle 1 from particle 3 is

$$F_{13} = \frac{Gm_1m_3}{(2a)^2} = \frac{(6.67 \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{s}^2)(6.0 \text{ kg})(4.0 \text{ kg})}{(0.040 \text{ m})^2} = 1.00 \times 10^{-6} \text{ N}.$$

Force \vec{F}_{12} is directed in the positive direction of the y axis (Fig. 13-4b) and has only the y component F_{12} . Similarly, \vec{F}_{13} is directed in the negative direction of the x axis and has only the x component $-F_{13}$ (Fig. 13-4c). (Note something important: We draw the force diagrams with the tail of a force vector anchored on the particle experiencing the force. Drawing them in other ways invites errors, especially on exams.)

To find the net force $\vec{F}_{1,\text{net}}$ on particle 1, we must add the two forces as vectors (Figs. 13-4d and e). We can do so on a vector-capable calculator. However, here we note that $-F_{13}$ and F_{12} are actually the x and y components of $\vec{F}_{1,\text{net}}$. Therefore, we can use Eq. 3-6 to find first the magnitude and then the direction of $\vec{F}_{1,\text{net}}$. The magnitude is

$$F_{1,\text{net}} = \sqrt{(F_{12})^2 + (-F_{13})^2} = \sqrt{(4.00 \times 10^{-6} \text{ N})^2 + (-1.00 \times 10^{-6} \text{ N})^2} = 4.1 \times 10^{-6} \text{ N}. \quad \text{(Answer)}$$

Relative to the positive direction of the x axis, Eq. 3-6 gives the direction of $\vec{F}_{1,\text{net}}$ as

$$\theta = \tan^{-1} \frac{F_{12}}{-F_{13}} = \tan^{-1} \frac{4.00 \times 10^{-6} \text{ N}}{-1.00 \times 10^{-6} \text{ N}} = -76^\circ.$$

Is this a reasonable direction (Fig. 13-4f)? No, because the direction of $\vec{F}_{1,\text{net}}$ must be between the directions of \vec{F}_{12} and \vec{F}_{13} . Recall from Chapter 3 that a calculator displays only one of the two possible answers to a \tan^{-1} function. We find the other answer by adding 180° :

$$-76^\circ + 180^\circ = 104^\circ, \quad \text{(Answer)}$$

which is a reasonable direction for $\vec{F}_{1,\text{net}}$ (Fig. 13-4g).

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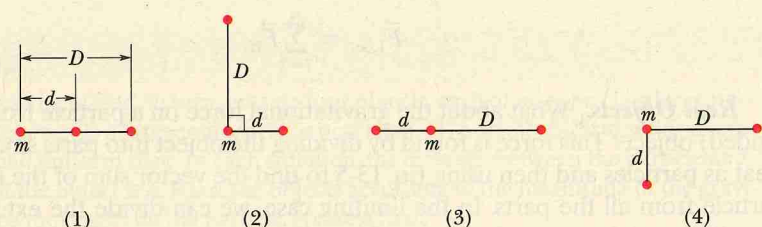
on the particle. In this limit, the sum of Eq. 13-5 becomes an integral and we have

$$\vec{F}_1 = \int d\vec{F}, \quad (13-6)$$

in which the integral is taken over the entire extended object and we drop the subscript “net.” If the extended object is a uniform sphere or a spherical shell, we can avoid the integration of Eq. 13-6 by assuming that the object’s mass is concentrated at the object’s center and using Eq. 13-1.

Checkpoint 2

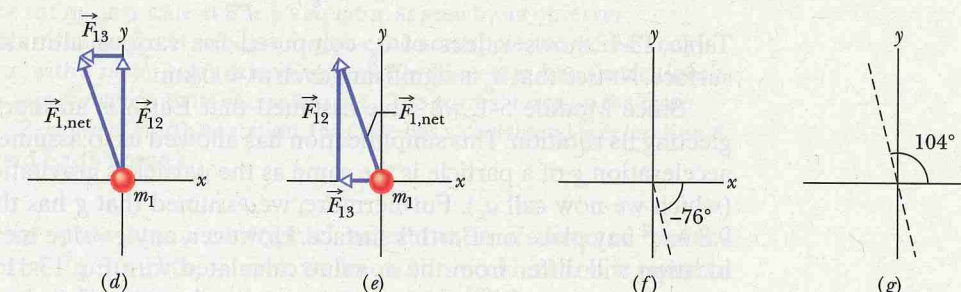
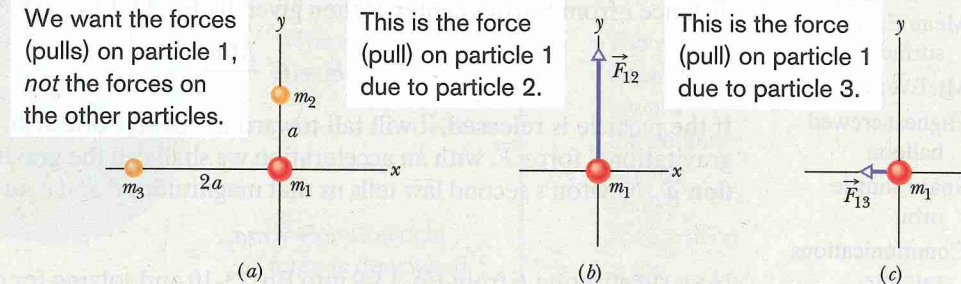
The figure shows four arrangements of three particles of equal masses. (a) Rank the arrangements according to the magnitude of the net gravitational force on the particle labeled m , greatest first. (b) In arrangement 2, is the direction of the net force closer to the line of length d or to the line of length D ?



We want the forces (pulls) on particle 1, not the forces on the other particles.

This is the force (pull) on particle 1 due to particle 2.

This is the force (pull) on particle 1 due to particle 3.



This is one way to show the net force on particle 1. Note the head-to-tail arrangement.

This is another way, also a head-to-tail arrangement.

A calculator’s inverse tangent can give this for the angle.

But this is the correct angle.

Figure 13-4 (a) An arrangement of three particles. The force on particle 1 due to (b) particle 2 and (c) particle 3. (d)–(g) Ways to combine the forces to get the net force magnitude and orientation. In WileyPLUS, this figure is available as an animation with voiceover.

13-3 GRAVITATION NEAR EARTH'S SURFACE

Learning Objectives

After reading this module, you should be able to . . .

13.06 Distinguish between the free-fall acceleration and the gravitational acceleration.

13.07 Calculate the gravitational acceleration near but outside a uniform, spherical astronomical body.

13.08 Distinguish between measured weight and the magnitude of the gravitational force.

Key Ideas

● The gravitational acceleration a_g of a particle (of mass m) is due solely to the gravitational force acting on it. When the particle is at distance r from the center of a uniform, spherical body of mass M , the magnitude F of the gravitational force on the particle is given by Eq. 13-1. Thus, by Newton’s second law,

$$F = ma_g,$$

which gives

$$a_g = \frac{GM}{r^2}.$$

● Because Earth’s mass is not distributed uniformly, because the planet is not perfectly spherical, and because it rotates, the actual free-fall acceleration \vec{g} of a particle near Earth differs slightly from the gravitational acceleration \vec{a}_g , and the particle’s weight (equal to mg) differs from the magnitude of the gravitational force on it.

Table 13-1 Variation of a_g with Altitude

Altitude (km)	a_g (m/s ²)	Altitude Example
0	9.83	Mean Earth surface
8.8	9.80	Mt. Everest
36.6	9.71	Highest crewed balloon
400	8.70	Space shuttle orbit
35 700	0.225	Communications satellite

Gravitation Near Earth's Surface

Let us assume that Earth is a uniform sphere of mass M . The magnitude of the gravitational force from Earth on a particle of mass m , located outside Earth a distance r from Earth's center, is then given by Eq. 13-1 as

$$F = G \frac{Mm}{r^2} \quad (13-9)$$

If the particle is released, it will fall toward the center of Earth, as a result of the gravitational force \vec{F} , with an acceleration we shall call the **gravitational acceleration** \vec{a}_g . Newton's second law tells us that magnitudes F and a_g are related by

$$F = ma_g \quad (13-10)$$

Now, substituting F from Eq. 13-9 into Eq. 13-10 and solving for a_g , we find

$$a_g = \frac{GM}{r^2} \quad (13-11)$$

Table 13-1 shows values of a_g computed for various altitudes above Earth's surface. Notice that a_g is significant even at 400 km.

Since Module 5-1, we have assumed that Earth is an inertial frame by neglecting its rotation. This simplification has allowed us to assume that the free-fall acceleration g of a particle is the same as the particle's gravitational acceleration (which we now call a_g). Furthermore, we assumed that g has the constant value 9.8 m/s² any place on Earth's surface. However, any g value measured at a given location will differ from the a_g value calculated with Eq. 13-11 for that location for three reasons: (1) Earth's mass is not distributed uniformly, (2) Earth is not a perfect sphere, and (3) Earth rotates. Moreover, because g differs from a_g , the same three reasons mean that the measured weight mg of a particle differs from the magnitude of the gravitational force on the particle as given by Eq. 13-9. Let us now examine those reasons.

- Earth's mass is not uniformly distributed.** The density (mass per unit volume) of Earth varies radially as shown in Fig. 13-5, and the density of the crust (outer section) varies from region to region over Earth's surface. Thus, g varies from region to region over the surface.
- Earth is not a sphere.** Earth is approximately an ellipsoid, flattened at the poles and bulging at the equator. Its equatorial radius (from its center point out to the equator) is greater than its polar radius (from its center point out to either north or south pole) by 21 km. Thus, a point at the poles is closer to the dense core of Earth than is a point on the equator. This is one reason the free-fall acceleration g increases if you were to measure it while moving at sea level from the equator toward the north or south pole. As you move, you are actually getting closer to the center of Earth and thus, by Newton's law of gravitation, g increases.
- Earth is rotating.** The rotation axis runs through the north and south poles of Earth. An object located on Earth's surface anywhere except at those poles must rotate in a circle about the rotation axis and thus must have a centripetal acceleration directed toward the center of the circle. This centripetal acceleration requires a centripetal net force that is also directed toward that center.

To see how Earth's rotation causes g to differ from a_g , let us analyze a simple situation in which a crate of mass m is on a scale at the equator. Figure 13-6a shows this situation as viewed from a point in space above the north pole.

Figure 13-6b, a free-body diagram for the crate, shows the two forces on the crate, both acting along a radial r axis that extends from Earth's center. The normal force \vec{F}_N on the crate from the scale is directed outward, in the positive direction of the r axis. The gravitational force, represented with its equivalent $m\vec{a}_g$, is directed inward. Because it travels in a circle about the center of Earth

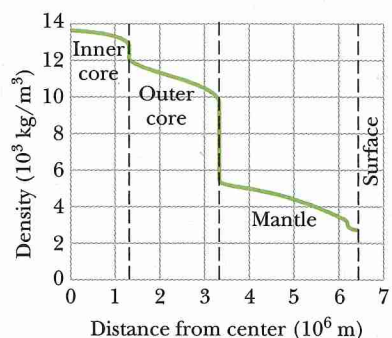


Figure 13-5 The density of Earth as a function of distance from the center. The limits of the solid inner core, the largely liquid outer core, and the solid mantle are shown, but the crust of Earth is too thin to show clearly on this plot.

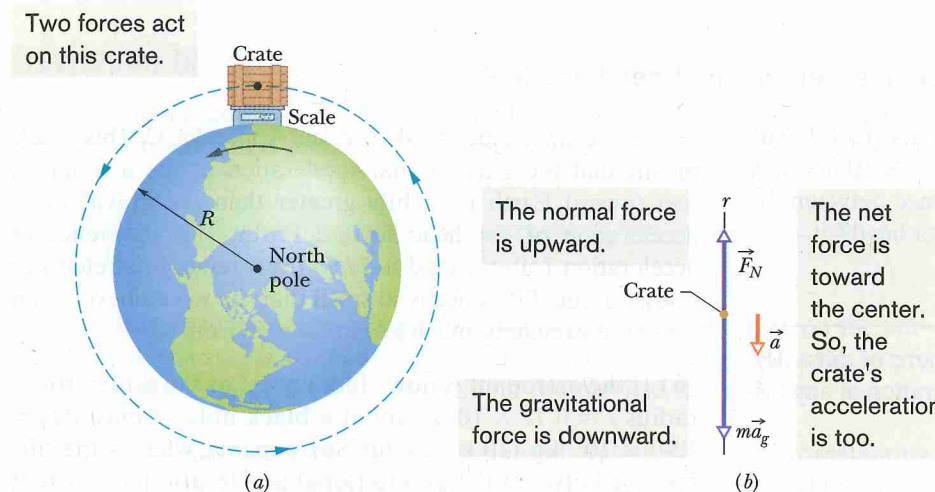


Figure 13-6 (a) A crate sitting on a scale at Earth's equator, as seen by an observer positioned on Earth's rotation axis at some point above the north pole. (b) A free-body diagram for the crate, with a radial r axis extending from Earth's center. The gravitational force on the crate is represented with its equivalent $m\vec{a}_g$. The normal force on the crate from the scale is \vec{F}_N . Because of Earth's rotation, the crate has a centripetal acceleration \vec{a} that is directed toward Earth's center.

as Earth turns, the crate has a centripetal acceleration \vec{a} directed toward Earth's center. From Eq. 10-23 ($a_r = \omega^2 r$), we know this acceleration is equal to $\omega^2 R$, where ω is Earth's angular speed and R is the circle's radius (approximately Earth's radius). Thus, we can write Newton's second law for forces along the r axis ($F_{\text{net},r} = ma_r$) as

$$F_N - ma_g = m(-\omega^2 R) \quad (13-12)$$

The magnitude F_N of the normal force is equal to the weight mg read on the scale. With mg substituted for F_N , Eq. 13-12 gives us

$$mg = ma_g - m(\omega^2 R), \quad (13-13)$$

which says

$$\left(\begin{array}{c} \text{measured} \\ \text{weight} \end{array} \right) = \left(\begin{array}{c} \text{magnitude of} \\ \text{gravitational force} \end{array} \right) - \left(\begin{array}{c} \text{mass times} \\ \text{centripetal acceleration} \end{array} \right).$$

Thus, the measured weight is less than the magnitude of the gravitational force on the crate, because of Earth's rotation.

Acceleration Difference. To find a corresponding expression for g and a_g , we cancel m from Eq. 13-13 to write

$$g = a_g - \omega^2 R, \quad (13-14)$$

which says

$$\left(\begin{array}{c} \text{free-fall} \\ \text{acceleration} \end{array} \right) = \left(\begin{array}{c} \text{gravitational} \\ \text{acceleration} \end{array} \right) - \left(\begin{array}{c} \text{centripetal} \\ \text{acceleration} \end{array} \right).$$

Thus, the measured free-fall acceleration is less than the gravitational acceleration because of Earth's rotation.

Equator. The difference between accelerations g and a_g is equal to $\omega^2 R$ and is greatest on the equator (for one reason, the radius of the circle traveled by the crate is greatest there). To find the difference, we can use Eq. 10-5 ($\omega = \Delta\theta/\Delta t$) and Earth's radius $R = 6.37 \times 10^6$ m. For one rotation of Earth, θ is 2π rad and the time period Δt is about 24 h. Using these values (and converting hours to seconds), we find that g is less than a_g by only about 0.034 m/s² (small compared to 9.8 m/s²). Therefore, neglecting the difference in accelerations g and a_g is often justified. Similarly, neglecting the difference between weight and the magnitude of the gravitational force is also often justified.

Sample Problem 13.02 Difference in acceleration at head and feet

(a) An astronaut whose height h is 1.70 m floats “feet down” in an orbiting space shuttle at distance $r = 6.77 \times 10^6$ m away from the center of Earth. What is the difference between the gravitational acceleration at her feet and at her head?

KEY IDEAS

We can approximate Earth as a uniform sphere of mass M_E . Then, from Eq. 13-11, the gravitational acceleration at any distance r from the center of Earth is

$$a_g = \frac{GM_E}{r^2}. \quad (13-15)$$

We might simply apply this equation twice, first with $r = 6.77 \times 10^6$ m for the location of the feet and then with $r = 6.77 \times 10^6$ m + 1.70 m for the location of the head. However, a calculator may give us the same value for a_g twice, and thus a difference of zero, because h is so much smaller than r . Here’s a more promising approach: Because we have a differential change dr in r between the astronaut’s feet and head, we should differentiate Eq. 13-15 with respect to r .

Calculations: The differentiation gives us

$$da_g = -2 \frac{GM_E}{r^3} dr, \quad (13-16)$$

where da_g is the differential change in the gravitational acceleration due to the differential change dr in r . For the astronaut, $dr = h$ and $r = 6.77 \times 10^6$ m. Substituting data into Eq. 13-16, we find

$$\begin{aligned} da_g &= -2 \frac{(6.67 \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{s}^2)(5.98 \times 10^{24} \text{ kg})}{(6.77 \times 10^6 \text{ m})^3} (1.70 \text{ m}) \\ &= -4.37 \times 10^{-6} \text{ m/s}^2, \quad (\text{Answer}) \end{aligned}$$

where the M_E value is taken from Appendix C. This result means that the gravitational acceleration of the astronaut’s feet toward Earth is slightly greater than the gravitational acceleration of her head toward Earth. This difference in acceleration (often called a *tidal effect*) tends to stretch her body, but the difference is so small that she would never even sense the stretching, much less suffer pain from it.

(b) If the astronaut is now “feet down” at the same orbital radius $r = 6.77 \times 10^6$ m about a black hole of mass $M_h = 1.99 \times 10^{31}$ kg (10 times our Sun’s mass), what is the difference between the gravitational acceleration at her feet and at her head? The black hole has a mathematical surface (*event horizon*) of radius $R_h = 2.95 \times 10^4$ m. Nothing, not even light, can escape from that surface or anywhere inside it. Note that the astronaut is well outside the surface (at $r = 229R_h$).

Calculations: We again have a differential change dr in r between the astronaut’s feet and head, so we can again use Eq. 13-16. However, now we substitute $M_h = 1.99 \times 10^{31}$ kg for M_E . We find

$$\begin{aligned} da_g &= -2 \frac{(6.67 \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{s}^2)(1.99 \times 10^{31} \text{ kg})}{(6.77 \times 10^6 \text{ m})^3} (1.70 \text{ m}) \\ &= -14.5 \text{ m/s}^2. \quad (\text{Answer}) \end{aligned}$$

This means that the gravitational acceleration of the astronaut’s feet toward the black hole is noticeably larger than that of her head. The resulting tendency to stretch her body would be bearable but quite painful. If she drifted closer to the black hole, the stretching tendency would increase drastically.

Gravitation Inside Earth

Newton’s shell theorem can also be applied to a situation in which a particle is located *inside* a uniform shell, to show the following:



A uniform shell of matter exerts no net gravitational force on a particle located inside it.

Caution: This statement does *not* mean that the gravitational forces on the particle from the various elements of the shell magically disappear. Rather, it means that the *sum* of the force vectors on the particle from all the elements is zero.

If Earth’s mass were uniformly distributed, the gravitational force acting on a particle would be a maximum at Earth’s surface and would decrease as the particle moved outward, away from the planet. If the particle were to move inward, perhaps down a deep mine shaft, the gravitational force would change for two reasons. (1) It would tend to increase because the particle would be moving closer to the center of Earth. (2) It would tend to decrease because the thickening shell of material lying outside the particle’s radial position would not exert any net force on the particle.

To find an expression for the gravitational force inside a uniform Earth, let’s use the plot in *Pole to Pole*, an early science fiction story by George Griffith. Three explorers attempt to travel by capsule through a naturally formed (and, of course, fictional) tunnel directly from the south pole to the north pole. Figure 13-7 shows the capsule (mass m) when it has fallen to a distance r from Earth’s center. At that moment, the *net* gravitational force on the capsule is due to the mass M_{ins} inside the sphere with radius r (the mass enclosed by the dashed outline), not the mass in the outer spherical shell (outside the dashed outline). Moreover, we can assume that the inside mass M_{ins} is concentrated as a particle at Earth’s center. Thus, we can write Eq. 13-1, for the magnitude of the gravitational force on the capsule, as

$$F = \frac{GmM_{\text{ins}}}{r^2}. \quad (13-17)$$

Because we assume a uniform density ρ , we can write this inside mass in terms of Earth’s total mass M and its radius R :

$$\begin{aligned} \text{density} &= \frac{\text{inside mass}}{\text{inside volume}} = \frac{\text{total mass}}{\text{total volume}}, \\ \rho &= \frac{M_{\text{ins}}}{\frac{4}{3}\pi r^3} = \frac{M}{\frac{4}{3}\pi R^3}. \end{aligned}$$

Solving for M_{ins} we find

$$M_{\text{ins}} = \frac{4}{3}\pi r^3 \rho = \frac{M}{R^3} r^3. \quad (13-18)$$

Substituting the second expression for M_{ins} into Eq. 13-17 gives us the magnitude of the gravitational force on the capsule as a function of the capsule’s distance r from Earth’s center:

$$F = \frac{GmM}{R^3} r. \quad (13-19)$$

According to Griffith’s story, as the capsule approaches Earth’s center, the gravitational force on the explorers becomes alarmingly large and, exactly at the center, it suddenly but only momentarily disappears. From Eq. 13-19 we see that, in fact, the force magnitude decreases linearly as the capsule approaches the center, until it is zero at the center. At least Griffith got the zero-at-the-center detail correct.

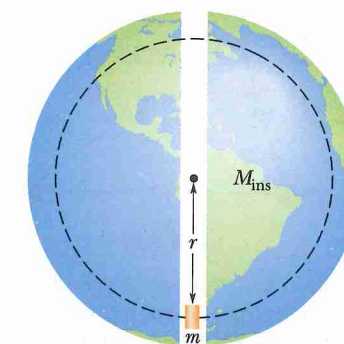


Figure 13-7 A capsule of mass m falls from rest through a tunnel that connects Earth’s south and north poles. When the capsule is at distance r from Earth’s center, the portion of Earth’s mass that is contained in a sphere of that radius is M_{ins} .

13-4 GRAVITATION INSIDE EARTH**Learning Objectives**

After reading this module, you should be able to . . .

13.09 Identify that a uniform shell of matter exerts no net gravitational force on a particle located inside it.

13.10 Calculate the gravitational force that is exerted on a particle at a given radius inside a nonrotating uniform sphere of matter.

Key Ideas

● A uniform shell of matter exerts no *net* gravitational force on a particle located inside it.

● The gravitational force \vec{F} on a particle inside a uniform solid sphere, at a distance r from the center, is due only to mass M_{ins} in an “inside sphere” with that radius r :

$$M_{\text{ins}} = \frac{4}{3}\pi r^3 \rho = \frac{M}{R^3} r^3,$$

where ρ is the solid sphere’s density, R is its radius, and M is its mass. We can assign this inside mass to be that of a particle at the center of the solid sphere and then apply Newton’s law of gravitation for particles. We find that the magnitude of the force acting on mass m is

$$F = \frac{GmM}{R^3} r.$$

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