10. Bessel Functions of Fractional Order

Mathematical Properties

10.1. Spherical Bessel Functions

Definitions

Differential Equation

10.1.1

$$z^2w''+2zw'+[z^2-n(n+1)]w=0 \qquad (n=0, \pm 1, \pm 2, \ldots)$$

Particular solutions are the *Spherical Bessel functions of the first kind*

$$j_n(z)=\sqrt{\tfrac{1}{2}\pi/z}\,J_{n+\frac{1}{2}}(z),$$

the *Spherical Bessel functions of the second kind*

$$y_n(z)=\sqrt{\tfrac{1}{2}\pi/z}\,Y_{n+\frac{1}{2}}(z),$$

and the *Spherical Bessel functions of the third kind*

$$h_n^{(1)}(z)=j_n(z)+iy_n(z)=\sqrt{\tfrac{1}{2}\pi/z}\,H_{n+\frac{1}{2}}^{(1)}(z),$$

$$h_n^{(2)}(z)=j_n(z)-iy_n(z)=\sqrt{\tfrac{1}{2}\pi/z}\,H_{n+\frac{1}{2}}^{(2)}(z).$$

The pairs $j_n(z)$, $y_n(z)$ and $h_n^{(1)}(z)$, $h_n^{(2)}(z)$ are linearly independent solutions for every n. For general properties see the remarks after **9.1.1.**

Ascending Series (See 9.1.2, 9.1.10)

10.1.2

$$j_n(z)=\frac{z^n}{1\cdot 3\cdot 5\ldots(2n+1)}\left\{1-\frac{\frac{1}{2}z^2}{1!(2n+3)}+\frac{(\frac{1}{2}z^2)^2}{2!(2n+3)(2n+5)}-\ldots\right\}$$

10.1.3

$$y_n(z)=-\frac{1\cdot 3\cdot 5\ldots(2n-1)}{z^{n+1}}\left\{1-\frac{\frac{1}{2}z^2}{1!(1-2n)}+\frac{(\frac{1}{2}z^2)^2}{2!(1-2n)(3-2n)}-\ldots\right\}$$

$$(n=0, 1, 2, \ldots)$$

Limiting Values as $z\to 0$

10.1.4 $$z^{-n}j_n(z)\to\frac{1}{1\cdot 3\cdot 5\ldots(2n+1)}$$

10.1.5

$$z^{n+1}y_n(z)\to -1\cdot 3\cdot 5\ldots(2n-1) \qquad (n=0, 1, 2, \ldots)$$

Wronskians

10.1.6 $$W\{j_n(z), y_n(z)\}=z^{-2}$$

10.1.7

$$W\{h_n^{(1)}(z), h_n^{(2)}(z)\}=-2iz^{-2} \qquad (n=0, 1, 2, \ldots)$$

Representations by Elementary Functions

10.1.8

$$j_n(z)=z^{-1}[P(n+\tfrac{1}{2}, z)\sin(z-\tfrac{1}{2}n\pi)+Q(n+\tfrac{1}{2}, z)\cos(z-\tfrac{1}{2}n\pi)]$$

10.1.9

$$y_n(z)=(-1)^{n+1}z^{-1}[P(n+\tfrac{1}{2}, z)\cos(z+\tfrac{1}{2}n\pi)-Q(n+\tfrac{1}{2}, z)\sin(z+\tfrac{1}{2}n\pi)]$$

$$P(n+\tfrac{1}{2}, z)=1-\frac{(n+2)!}{2!\Gamma(n-1)}(2z)^{-2}+\frac{(n+4)!}{4!\Gamma(n-3)}(2z)^{-4}-\ldots$$

$$=\sum_0^{[\frac{1}{2}n]}(-1)^k(n+\tfrac{1}{2}, 2k)(2z)^{-2k}$$

$$Q(n+\tfrac{1}{2}, z)=\frac{(n+1)!}{1!\Gamma(n)}(2z)^{-1}-\frac{(n+3)!}{3!\Gamma(n-2)}(2z)^{-3}+\frac{(n+5)!}{5!\Gamma(n-4)}(2z)^{-5}-\ldots$$

$$=\sum_0^{[\frac{1}{2}(n-1)]}(-1)^k(n+\tfrac{1}{2}, 2k+1)(2z)^{-2k-1}$$

$$(n=0, 1, 2, \ldots)$$

$$(n+\tfrac{1}{2}, k)=\frac{(n+k)!}{k!\Gamma(n-k+1)}$$

$n \backslash k$	1	2	3	4	5
1	2				
2	6	12			
3	12	60	120		
4	20	180	840	1680	
5	30	420	3360	15120	30240

10.1.10

$$j_n(z)=f_n(z)\sin z+(-1)^{n+1}f_{-n-1}(z)\cos z$$

$$f_0(z)=z^{-1},\qquad f_1(z)=z^{-2}$$

$$f_{n-1}(z)+f_{n+1}(z)=(2n+1)z^{-1}f_n(z)$$

$$(n=0,\ \pm 1,\ \pm 2,\ \ldots)$$

The Functions $j_n(z)$, $y_n(z)$ for $n=0,\ 1,\ 2$

10.1.11 $$j_0(z)=\frac{\sin z}{z}$$

$$j_1(z)=\frac{\sin z}{z^2}-\frac{\cos z}{z}$$

$$j_2(z)=\left(\frac{3}{z^3}-\frac{1}{z}\right)\sin z-\frac{3}{z^2}\cos z$$ *

10.1.12

$$y_0(z)=-j_{-1}(z)=-\frac{\cos z}{z}$$

$$y_1(z)=j_{-2}(z)=-\frac{\cos z}{z^2}-\frac{\sin z}{z}$$

$$y_2(z)=-j_{-3}(z)=\left(-\frac{3}{z^3}+\frac{1}{z}\right)\cos z-\frac{3}{z^2}\sin z$$ *

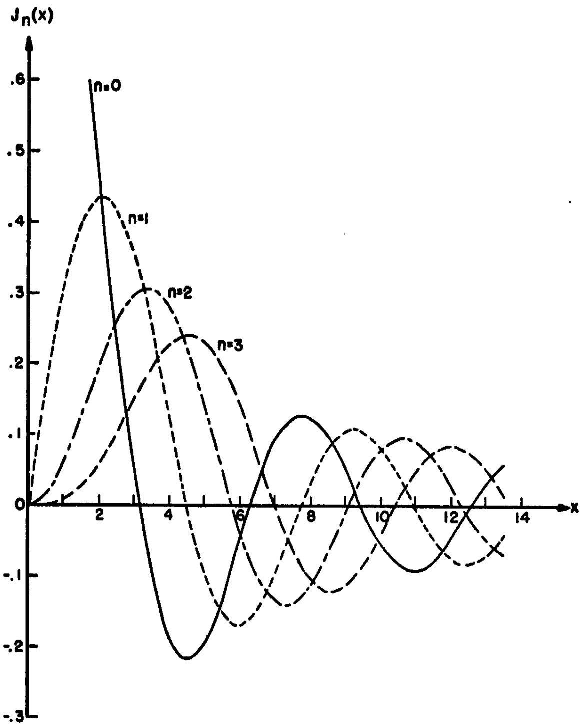

FIGURE 10.1. $j_n(x)$. $n=0(1)3$.

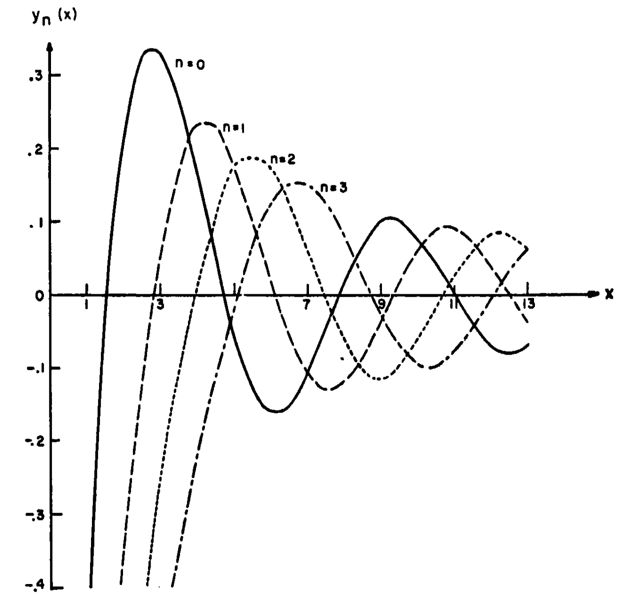

FIGURE 10.2. $y_n(x)$. $n=0(1)3$.

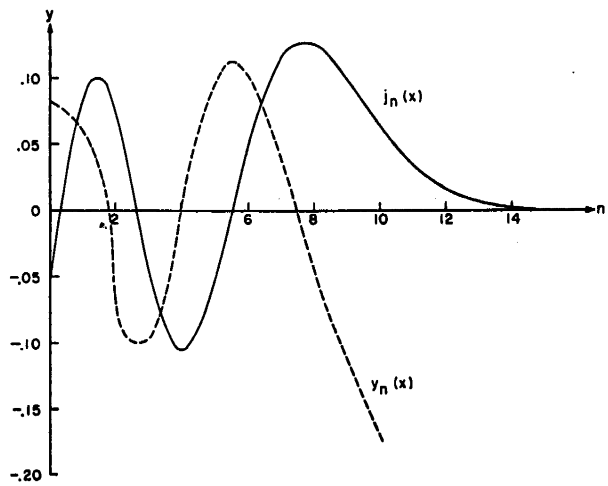

FIGURE 10.3. $j_n(x)$, $y_n(x)$. $x=10$.

Poisson's Integral and Gegenbauer's Generalization

10.1.13 $$j_n(z)=\frac{z^n}{2^{n+1}n!}\int_0^{\pi}\cos(z\cos\theta)\sin^{2n+1}\theta\,d\theta$$

(See **9.1.20.**)

10.1.14

$$=\frac{1}{2}(-i)^n\int_0^{\pi}e^{iz\cos\theta}P_n(\cos\theta)\sin\theta\,d\theta$$

$$(n=0,\ 1,\ 2,\ \ldots)$$

*See page II.

Spherical Bessel Functions of the Second and Third Kind

10.1.15

$$y_n(z)=(-1)^{n+1}j_{-n-1}(z) \qquad (n=0, \pm 1, \pm 2, \ldots)$$

10.1.16

$$h_n^{(1)}(z)=i^{-n-1}z^{-1}e^{iz}\sum_0^n (n+\tfrac{1}{2}, k)(-2iz)^{-k}$$

10.1.17

$$h_n^{(2)}(z)=i^{n+1}z^{-1}e^{-iz}\sum_0^n (n+\tfrac{1}{2}, k)\,(2iz)^{-k}$$ *

10.1.18

$$h_{-n-1}^{(1)}(z)=i(-1)^n h_n^{(1)}(z)$$

$$h_{-n-1}^{(2)}(z)=-i(-1)^n h_n^{(2)}(z) \qquad (n=0, 1, 2, \ldots)$$

Elementary Properties

Recurrence Relations

$f_n(z): j_n(z), y_n(z), h_n^{(1)}(z), h_n^{(2)}(z)$

$(n=0, \pm 1, \pm 2, \ldots)$

10.1.19 $$f_{n-1}(z)+f_{n+1}(z)=(2n+1)z^{-1}f_n(z)$$

10.1.20 $$nf_{n-1}(z)-(n+1)f_{n+1}(z)=(2n+1)\frac{d}{dz}f_n(z)$$

10.1.21 $$\frac{n+1}{z}f_n(z)+\frac{d}{dz}f_n(z)=f_{n-1}(z)$$

(See **10.1.23.**)

10.1.22 $$\frac{n}{z}f_n(z)-\frac{d}{dz}f_n(z)=f_{n+1}(z)$$

(See **10.1.24.**)

Differentiation Formulas

$f_n(z): j_n(z), y_n(z), h_n^{(1)}(z), h_n^{(2)}(z)$

$(n=0, \pm 1, \pm 2, \ldots)$

10.1.23 $$\left(\frac{1}{z}\frac{d}{dz}\right)^m [z^{n+1}f_n(z)]=z^{n-m+1}f_{n-m}(z)$$

10.1.24 $$\left(\frac{1}{z}\frac{d}{dz}\right)^m [z^{-n}f_n(z)]=(-1)^m z^{-n-m}f_{n+m}(z)$$

$(m=1, 2, 3, \ldots)$

Rayleigh's Formulas

10.1.25

$$j_n(z)=z^n\left(-\frac{1}{z}\frac{d}{dz}\right)^n \frac{\sin z}{z}$$

10.1.26

$$y_n(z)=-z^n\left(-\frac{1}{z}\frac{d}{dz}\right)^n \frac{\cos z}{z} \qquad (n=0, 1, 2, \ldots)$$

*See page II.

Modulus and Phase

$$j_n(z)=\sqrt{\tfrac{1}{2}\pi/z}\,M_{n+\frac{1}{2}}(z)\cos\theta_{n+\frac{1}{2}}(z),$$

$$y_n(z)=\sqrt{\tfrac{1}{2}\pi/z}\,M_{n+\frac{1}{2}}(z)\sin\theta_{n+\frac{1}{2}}(z)$$

(See **9.2.17.**)

10.1.27

$$(\tfrac{1}{2}\pi/z)\,M^2_{n+\frac{1}{2}}(z)=\frac{1}{z^2}\sum_0^n \frac{(2n-k)!(2n-2k)!}{k![(n-k)!]^2}(2z)^{2k-2n}$$

(See **9.2.28.**)

10.1.28 $$(\tfrac{1}{2}\pi/z)M^2_{1/2}(z)=j_0^2(z)+y_0^2(z)=z^{-2}$$

10.1.29

$$(\tfrac{1}{2}\pi/z)M^2_{3/2}(z)=j_1^2(z)+y_1^2(z)=z^{-2}+z^{-4}$$

10.1.30

$$(\tfrac{1}{2}\pi/z)M^2_{5/2}(z)=j_2^2(z)+y_2^2(z)=z^{-2}+3z^{-4}+9z^{-6}$$

Cross Products

10.1.31 $$j_n(z)y_{n-1}(z)-j_{n-1}(z)y_n(z)=z^{-2}$$

10.1.32

$$j_{n+1}(z)y_{n-1}(z)-j_{n-1}(z)y_{n+1}(z)=(2n+1)z^{-3}$$

10.1.33

$$j_0(z)j_n(z)+y_0(z)y_n(z)$$

$$=z^{-2}\sum_0^{[\frac{1}{2}n]}(-1)^k 2^{n-2k}\left(k+\tfrac{1}{2}\right)_{n-2k}\binom{n-k}{k}z^{2k-n}$$

$(n=0, 1, 2, \ldots)$

Analytic Continuation

10.1.34 $$j_n(ze^{m\pi i})=e^{mn\pi i}j_n(z)$$

10.1.35 $$y_n(ze^{m\pi i})=(-1)^m e^{mn\pi i}y_n(z)$$

10.1.36 $$h_n^{(1)}(ze^{(2m+1)\pi i})=(-1)^n h_n^{(2)}(z)$$

10.1.37 $$h_n^{(2)}(ze^{(2m+1)\pi i})=(-1)^n h_n^{(1)}(z)$$

10.1.38 $$h_n^{(l)}(ze^{2m\pi i})=h_n^{(l)}(z)$$

$(l=1, 2;\ m, n=0, 1, 2, \ldots)$

Generating Functions

10.1.39

$$\frac{1}{z}\sin\sqrt{z^2+2zt}=\sum_0^\infty \frac{(-t)^n}{n!}y_{n-1}(z) \qquad (2|t|<|z|)$$

10.1.40 $$\frac{1}{z}\cos\sqrt{z^2-2zt}=\sum_0^\infty \frac{t^n}{n!}j_{n-1}(z)$$

Derivatives With Respect to Order

10.1.41

$$\left[\frac{\partial}{\partial\nu}j_\nu(x)\right]_{\nu=0}=(\tfrac{1}{2}\pi/x)\{\mathrm{Ci}(2x)\sin x-\mathrm{Si}(2x)\cos x\}$$

10.1.42

$$\left[\frac{\partial}{\partial\nu}j_\nu(x)\right]_{\nu=-1}=(\tfrac{1}{2}\pi/x)\{\mathrm{Ci}(2x)\cos x+\mathrm{Si}(2x)\sin x\}$$

10.1.43

$$\left[\frac{\partial}{\partial\nu}y_\nu(x)\right]_{\nu=0}=(\tfrac{1}{2}\pi/x)\{\mathrm{Ci}(2x)\cos x+[\mathrm{Si}(2x)-\pi]\sin x\}$$

10.1.44

$$\left[\frac{\partial}{\partial\nu}y_\nu(x)\right]_{\nu=-1}=(\tfrac{1}{2}\pi/x)\{\mathrm{Ci}(2x)\sin x-[\mathrm{Si}(2x)-\pi]\cos x\}$$

Addition Theorems and Degenerate Forms

r, ρ, θ, λ arbitrary complex; $R=\sqrt{(r^2+\rho^2-2r\rho\cos\theta)}$

10.1.45 $$\frac{\sin\lambda R}{\lambda R}=\sum_0^\infty(2n+1)j_n(\lambda r)j_n(\lambda\rho)P_n(\cos\theta)$$

***10.1.46** $$-\frac{\cos\lambda R}{\lambda R}=\sum_0^\infty(2n+1)j_n(\lambda r)y_n(\lambda\rho)P_n(\cos\theta)\qquad |re^{\pm i\theta}|<|\rho|$$

10.1.47 $$e^{iz\cos\theta}=\sum_0^\infty(2n+1)e^{\frac{1}{2}n\pi i}j_n(z)P_n(\cos\theta)$$

10.1.48

$$J_0(z\sin\theta)=\sum_0^\infty(4n+1)\frac{(2n)!}{2^{2n}(n!)^2}j_{2n}(z)P_{2n}(\cos\theta)$$

Duplication Formula

10.1.49

$$j_n(2z)=$$

* $$-n!z^{n+1}\sum_0^n\frac{2n-2k+1}{k!(2n-k+1)!}j_{n-k}(z)y_{n-k}(z)$$

Some Infinite Series Involving $j_n^2(z)$

10.1.50 $$\sum_0^\infty(2n+1)j_n^2(z)=1$$

10.1.51 $$\sum_0^\infty(-1)^n(2n+1)j_n^2(z)=\frac{\sin 2z}{2z}$$

10.1.52 $$\sum_0^\infty j_n^2(z)=\frac{\mathrm{Si}(2z)}{2z}$$

*See page II.

Fresnel Integrals

10.1.53

$$C(\sqrt{2x/\pi})=\frac{1}{2}\int_0^x J_{-\frac{1}{2}}(t)dt$$

$$=\sqrt{2}[\cos\tfrac{1}{2}x\sum_0^\infty(-1)^nJ_{2n+\frac{1}{2}}(\tfrac{1}{2}x)$$

$$+\sin\tfrac{1}{2}x\sum_0^\infty(-1)^nJ_{2n+3/2}(\tfrac{1}{2}x)]$$

10.1.54

$$S(\sqrt{2x/\pi})=\frac{1}{2}\int_0^x J_{\frac{1}{2}}(t)dt$$

$$=\sqrt{2}[\sin\tfrac{1}{2}x\sum_0^\infty(-1)^nJ_{2n+\frac{1}{2}}(\tfrac{1}{2}x)$$

$$-\cos\tfrac{1}{2}x\sum_0^\infty(-1)^nJ_{2n+3/2}(\tfrac{1}{2}x)].$$

(See also **11.1.1, 11.1.2.**)

Zeros and Their Asymptotic Expansions

The zeros of $j_n(x)$ and $y_n(x)$ are the same as the zeros of $J_{n+\frac{1}{2}}(x)$ and $Y_{n+\frac{1}{2}}(x)$ and the formulas for $j_{\nu,s}$ and $y_{\nu,s}$ given in **9.5** are applicable with $\nu=n+\frac{1}{2}$. There are, however, no simple relations connecting the zeros of the derivatives. Accordingly, we now give formulas for $a'_{n,s}$, $b'_{n,s}$, the s-th positive zero of $j'_n(z)$, $y'_n(z)$, respectively; $z=0$ is counted as the first zero of $j'_0(z)$.

(Tables of $a'_{n,s}$, $b'_{n,s}$, $j_n(a'_{n,s})$, $y_n(b'_{n,s})$ are given in [10.31].)

Elementary Relations

$$f_n(z)=j_n(z)\cos\pi t+y_n(z)\sin\pi t$$

(t a real parameter, $0\leq t\leq 1$)

If τ_n is a zero of $f'_n(z)$ then

10.1.55 $$f_n(\tau_n)=[\tau_n/(n+1)]f_{n-1}(\tau_n)$$

(See **10.1.21.**)

10.1.56 $$=(\tau_n/n)f_{n+1}(\tau_n)$$

(See **10.1.22.**)

10.1.57 $$=\left\{\frac{1}{\pi}[\tau_n^2-n(n+1)]\frac{d\tau_n}{d\tau}\right\}^{-\frac{1}{2}}$$