$$y(x) = y_1(x) + y_2(x) + y_p(x).$$

(y_1 and y_2 are solutions of the homogeneous equation.)
Show that

$$y_p(x) = y_2(x) \int^x \frac{y_1(s)F(s)\,ds}{W\{y_1(s), y_2(s)\}} - y_1(x) \int^x \frac{y_2(s)F(s)\,ds}{W\{y_1(s), y_2(s)\}}$$

with $W\{y_1(s), y_2(s)\}$ the Wronskian of $y_1(s)$ and $y_2(s)$.
Hint: As in Ex. 8.5.18 let $y_p(x) = y_1(x)\,v(x)$ and develop a first order differential equation for $v'(x)$.

8.6 Nonhomogeneous Equation—Green's Function

The series substitution of Section 8.4 and the Wronskian double integral of Section 8.5 provide the most general solution of the *homogeneous*, linear, second-order differential equation. The specific solution, y_p, linearly dependent on the source term ($F(x)$ of Eq. 8.20*b*) may be cranked out by the variation of parameters method, Ex. 8.5.19. In this section, we turn to a different method of solution—Green's functions.

For a brief introduction to Green's function method, as applied to the solution of a nonhomogeneous partial differential equation, it is helpful to use the electrostatic analog. In the presence of charges the electrostatic potential ψ satisfies Poisson's nonhomogeneous equation (cf. Section 1.14)

$$\nabla^2\psi = -\frac{\rho}{\varepsilon_0}, \quad \text{(mks units)} \tag{8.76}$$

and Laplace's homogeneous equation,

$$\nabla^2\psi = 0, \tag{8.77}$$

in the absence of electric charge ($\rho = 0$). If the charges are point charges q_i, we know that the solution is

$$\psi = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i}, \tag{8.78}$$

a superposition of single-point charge solutions obtained from Coulomb's law for the force between two point charges q_1 and q_2,

$$\mathbf{F} = \frac{q_1 q_2 \mathbf{r}_0}{4\pi\varepsilon_0 r^2}. \tag{8.79}$$

By replacement of the discrete point charges with a smeared out distributed charge, charge density ρ, Eq. 8.78 becomes

$$\psi(r = 0) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r})}{r}\,d\tau \tag{8.80}$$

or, for the potential at $\mathbf{r} = \mathbf{r}_1$ away from the origin and the charge at $\mathbf{r} = \mathbf{r}_2$,

$$\psi(\mathbf{r}_1) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|}\,d\tau_2 \tag{8.81}$$

Dirac delta function. A formal derivation and generalization of this result is facilitated by using $\delta(x)$, the Dirac delta function, as in Section 1.15. For the one-dimensional case, the Dirac delta function is often defined by the following properties:

$$\delta(x) = 0, \qquad x \neq 0, \tag{8.82a}$$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1, \tag{8.82b}$$

and

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0). \tag{8.82c}$$

Here it is assumed that $f(x)$ is continuous at $x = 0$.

From these defining equations $\delta(x)$ must be an infinitely high, infinitely thin spike—as in the description of an impulsive force (Section 15.8) or charge density for a point charge.[1] The problem is that *no such function exists* in the usual sense of function. It is possible to approximate the delta function by a variety of functions, Eqs. 8.83*a*–8.83*d* and Figs. 8.3*a*–8.3*d*:

$$\delta_n(x) = \begin{cases} 0, & x < -\dfrac{1}{2n} \\ n, & -\dfrac{1}{2n} < x < \dfrac{1}{2n} \\ 0, & x > \dfrac{1}{2n} \end{cases} \tag{8.83a}$$

$$\delta_n(x) = \frac{n}{\sqrt{\pi}}\exp(-n^2x^2) \tag{8.83b}$$

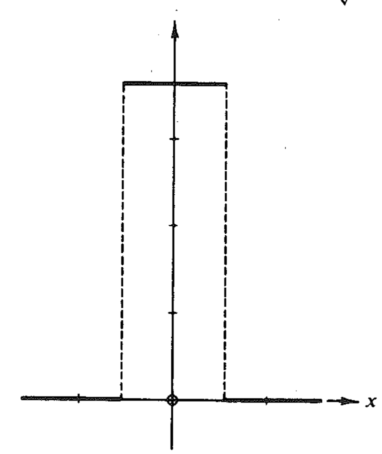

FIG. 8.3*a* δ-sequence function

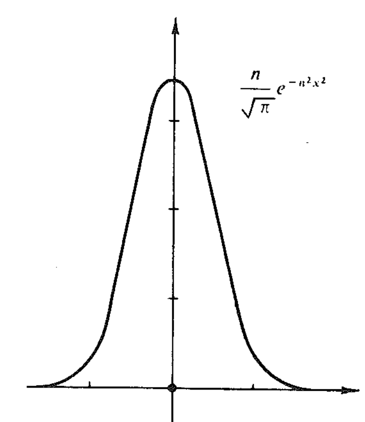

FIG. 8.3*b* δ-sequence function

[1] The delta function is frequently invoked to describe very short range forces such as nuclear forces.

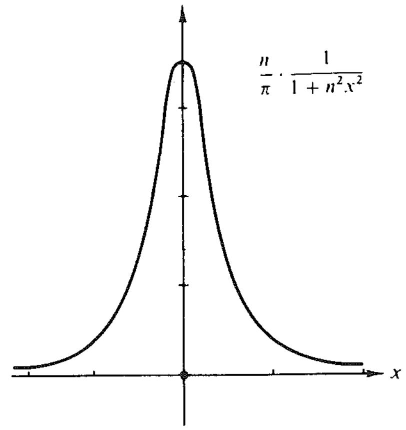

FIG. 8.3c δ-sequence function

$$\delta_n(x) = \frac{n}{\pi} \cdot \frac{1}{1+n^2x^2} \tag{8.83c}$$

$$\delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi}\int_{-n}^{n} e^{ixt}\,dt \tag{8.83d}$$

These approximations have varying degrees of usefulness. Equation 8.83*a* is useful in providing a simple derivation of the integral property, Eq. 8.82*c*. Equation 8.83*b* is convenient to differentiate. Its derivatives lead to the Hermite polynomials, Eq. 13.7. Equation 8.83*d* is particularly useful in Fourier analysis and in its applications to quantum mechanics.

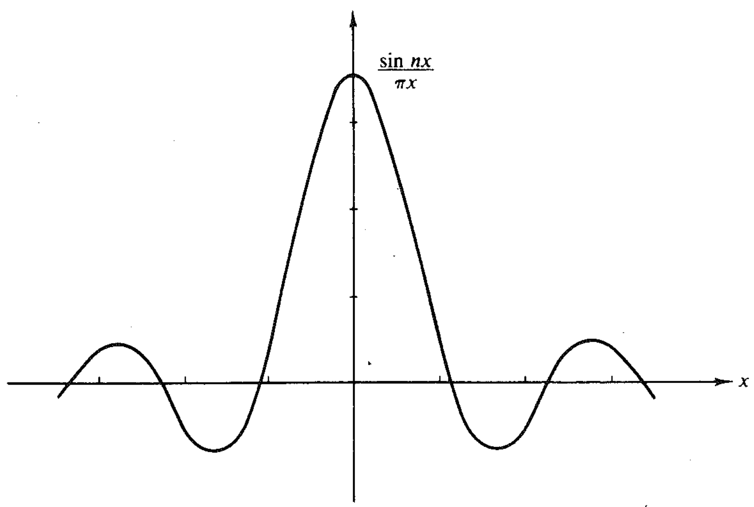

FIG. 8.3*d* δ-sequence function

For most physical purposes, such approximations are quite adequate. From a mathematical point of view, the situation is still unsatisfactory: the limits

$$\lim_{n\to\infty} \delta_n(x)$$

do not exist.

A way out of this difficulty is provided by the theory of distributions. Recognizing that Eq. 8.82*c* is the fundamental property, we focus our attention on it rather than on $\delta(x)$ itself. Equations 8.83*a*–8.83*d* with $n = 1, 2, 3, \ldots$ may be interpreted as *sequences* of normalized functions:

$$\int_{-\infty}^{\infty} \delta_n(x)\,dx = 1. \tag{8.84}$$

The sequence of integrals has the limit

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} \delta_n(x) f(x)\,dx = f(0). \tag{8.85a}$$

Note carefully that Eq. 8.85*a* is the limit of a sequence of integrals. Again, the limit of $\delta_n(x)$, $n \to \infty$, does not exist. (The limits for all four forms of $\delta_n(x)$ diverge at $x = 0$).

We may treat $\delta(x)$ consistently in the form

$$\int_{-\infty}^{\infty} \delta(x) f(x)\,dx = \lim_{n\to\infty}\int_{-\infty}^{\infty} \delta_n(x) f(x)\,dx. \tag{8.85b}$$

$\delta(x)$ is labeled a distribution (not a function) defined by the sequences $\delta_n(x)$ as indicated in Eq. 8.85*b*. We might emphasize that the integral on the left-hand side of Eq. 8.85*b* is not a Riemann integral.[1] It is a limit.

This distribution $\delta(x)$ is only one of an infinity of possible distributions, but it is the one we are interested in because of Eq. 8.82*c*.

We shall use $\delta(x)$ frequently and shall call it the Dirac delta function—for historical reasons. Remember that it is not really a function. It is essentially a shorthand notation, defined implicitly as the limit of integrals of a sequence, $\delta_n(x)$, according to Eq. 8.85*b*. It should be understood that our Dirac delta function has significance only as part of an integrand and never as an end result.

Shifting our singularity to the point $x = x'$, the Dirac delta function is written $\delta(x - x')$. Equation 8.82*c* becomes

$$\int_{-\infty}^{\infty} f(x)\,\delta(x - x')\,dx = f(x'), \tag{8.86}$$

As a description of a singularity at $x = x'$, the Dirac delta function may be written as $\delta(x - x')$ or as $\delta(x' - x)$. Going to three dimensions and using spherical polar coordinates, we obtain

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \delta(\mathbf{r}) r^2\,dr \sin\theta\,d\theta\,d\varphi = \iiint_{-\infty}^{\infty} \delta(x)\,\delta(y)\delta(z)\,dx\,dy\,dz = 1. \tag{8.87}$$

[1] It can be treated as a Stieltjes integral if desired. $\delta(x)\,dx$ is replaced by $dS(x)$, where $S(x)$ is the Heaviside step function (cf. Ex. 8.6.3).

This corresponds to a singularity (or source) at the origin. Again, if our source is at $\mathbf{r} = \mathbf{r}_1$, Eq. 8.87 becomes

$$\int\int\int \delta(\mathbf{r}_2 - \mathbf{r}_1) r_2^2 \, dr_2 \sin\theta_2 \, d\theta_2 \, d\varphi_2 = 1. \tag{8.88}$$

As already mentioned,

$$\delta(\mathbf{r}_2 - \mathbf{r}_1) = \delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{8.89}$$

Poisson's equation—Green's function solution. Returning to our electrostatic problem, we use ψ as the potential corresponding to the given distribution of charge and therefore satisfying Poisson's equation

$$\nabla^2\psi = -\frac{\rho}{\varepsilon_0}, \tag{8.90}$$

whereas a function φ, which we label a Green's function, is required to satisfy Poisson's equation with a point source at the point defined by $\mathbf{r}_2$:

$$\nabla^2\varphi = -\delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{8.91}$$

Physically, then, φ is the potential at $\mathbf{r}_1$ corresponding to a unit source (ε_0) at $\mathbf{r}_2$. By Green's theorem (Section 1.11)

$$\int(\psi \nabla^2\varphi - \varphi \nabla^2\psi)\, d\tau_2 = \int(\psi \nabla\varphi - \varphi \nabla\psi) \cdot d\boldsymbol{\sigma}. \tag{8.92}$$

Assuming that the integrand falls off faster than r^{-2}, we may simplify our problem by taking the volume so large that the surface integral vanishes, leaving

$$\int \psi \nabla^2\varphi \, d\tau_2 = \int \varphi \nabla^2\psi \, d\tau_2 \tag{8.93}$$

or by substituting in Eqs. 8.90 and 8.91,

$$-\int \psi(\mathbf{r}_2)\, \delta(\mathbf{r}_1 - \mathbf{r}_2)\, d\tau_2 = -\int \frac{\varphi(\mathbf{r}_1, \mathbf{r}_2)\, \rho(\mathbf{r}_2)}{\varepsilon_0} d\tau_2. \tag{8.94}$$

Integration by employing the defining property of the Dirac delta function (Eq. 8.82*c*), produces

$$\psi(\mathbf{r}_1) = \frac{1}{\varepsilon_0}\int \varphi(\mathbf{r}_1, \mathbf{r}_2)\, \rho(\mathbf{r}_2)\, d\tau_2. \tag{8.95}$$

Note that we have used Eq. 8.91 to eliminate $\nabla^2\varphi$ but that the function φ itself is still unknown. In Section 1.14, Gauss's law, we found that

$$\int \nabla^2\left(\frac{1}{r}\right) d\tau = \begin{cases} 0, \\ -4\pi, \end{cases} \tag{8.96}$$

0 if the volume did not include the origin and -4π if the origin were included. This result from Section 1.14 may be rewritten as

$$\nabla^2\left(\frac{1}{4\pi r}\right) = -\delta(\mathbf{r}), \quad \text{or} \quad \nabla^2\left(\frac{1}{4\pi r_{12}}\right) = -\delta(\mathbf{r}_1 - \mathbf{r}_2) \tag{8.97}$$

corresponding to a shift of the electrostatic charge from the origin to the position $\mathbf{r} = \mathbf{r}_2$. Here $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$, and the Dirac delta function $\delta(\mathbf{r}_1 - \mathbf{r}_2)$ vanishes unless $\mathbf{r}_1 = \mathbf{r}_2$. Therefore, in a comparison of Eqs. 8.91 and 8.97 the function φ (Green's function) is given by

$$\varphi(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|}. \tag{8.98}$$

The solution of our differential equation (Poisson's equation) is

$$\psi(\mathbf{r}_1) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} d\tau_2 \tag{8.99}$$

in complete agreement with Eq. 8.81.

In summary, Green's function, $\varphi(\mathbf{r}_1, \mathbf{r}_2)$, often written $G(\mathbf{r}_1, \mathbf{r}_2)$ as a reminder of the name, is a solution of Eq. 8.91. It enters in an integral solution of our differential equation, as in Eq. 8.81. For the simple, but important electrostatic case we obtain Green's function $G(\mathbf{r}_1, \mathbf{r}_2)$ by Gauss's law, comparing Eqs. 8.91 and 8.97. Finally, from the final solution (Eq. 8.99), it is possible to develop a physical interpretation of Green's function. It occurs as a weighting function or influence function which enhances or reduces the effect of the charge element $\rho(\mathbf{r}_2)\, d\tau_2$ according to its distance from the field point $\mathbf{r}_1$. Green's function, $G(\mathbf{r}_1, \mathbf{r}_2)$, gives the effect of a unit point source at $\mathbf{r}_2$ in producing a potential at $\mathbf{r}_1$. This is how it was introduced in Eq. 8.91; this is how it appears in Eq. 8.99.

An important property of Green's function is the symmetry of its two variables, that is,

$$G(\mathbf{r}_1, \mathbf{r}_2) = G(\mathbf{r}_2, \mathbf{r}_1). \tag{8.100}$$

Although this is obvious in the electrostatic case just considered, it can be proved under much more general conditions. In place of Eq. 8.91, let us require that $G(\mathbf{r}, \mathbf{r}_1)$ satisfy[1]

$$\nabla \cdot [p(\mathbf{r}) \nabla G(\mathbf{r}, \mathbf{r}_1)] + \lambda q(\mathbf{r})\, G(\mathbf{r}, \mathbf{r}_1) = -\delta(\mathbf{r} - \mathbf{r}_1), \tag{8.101}$$

corresponding to a mathematical point source at $\mathbf{r} = \mathbf{r}_1$. Here the functions $p(\mathbf{r})$ and $q(\mathbf{r})$ are well-behaved but otherwise arbitrary functions of $\mathbf{r}$. Green's function, $G(\mathbf{r}, \mathbf{r}_2)$, satisfies the same equation but the subscript 1 is replaced by subscript 2.

$$\nabla \cdot [p(\mathbf{r}) \nabla G(\mathbf{r}, \mathbf{r}_2)] + \lambda\, q(\mathbf{r})\, G(\mathbf{r}, \mathbf{r}_2) = -\delta(\mathbf{r} - \mathbf{r}_2). \tag{8.102}$$

Then $G(\mathbf{r}, \mathbf{r}_2)$ is a sort of potential at $\mathbf{r}$, created by a unit point source at $\mathbf{r}_2$. We multiply the equation for $G(\mathbf{r}, \mathbf{r}_1)$ by $G(\mathbf{r}, \mathbf{r}_2)$ and the equation for $G(\mathbf{r}, \mathbf{r}_2)$ by $G(\mathbf{r}, \mathbf{r}_1)$ and then subtract the two:

$$G(\mathbf{r}, \mathbf{r}_2)\nabla \cdot [p(\mathbf{r}) \nabla G(\mathbf{r}, \mathbf{r}_1)] - G(\mathbf{r}, \mathbf{r}_1) \nabla \cdot [p(\mathbf{r}) \nabla G(\mathbf{r}, \mathbf{r}_2)]$$
$$= -G(\mathbf{r}, \mathbf{r}_2)\, \delta(\mathbf{r} - \mathbf{r}_1) + G(\mathbf{r}, \mathbf{r}_1)\, \delta(\mathbf{r} - \mathbf{r}_2). \tag{8.103}$$

By integrating over whatever volume is involved, we obtain a surface integral by Green's theorem:

[1] Equation 8.101 is a three-dimensional version of the *self-adjoint* eigenvalue equation, Eq. 9.4.

$$\int_S [G(\mathbf{r}, \mathbf{r}_2)\, p(\mathbf{r})\, \nabla G(\mathbf{r}, \mathbf{r}_1) - G(\mathbf{r}, \mathbf{r}_1)\, p(\mathbf{r})\, \nabla G(\mathbf{r}, \mathbf{r}_2)] \cdot d\boldsymbol{\sigma} = -G(\mathbf{r}_1, \mathbf{r}_2) + G(\mathbf{r}_2, \mathbf{r}_1). \quad (8.104)$$

The terms on the right-hand side appear when we use the Dirac delta functions and carry out the volume integration. Under the requirement that Green's functions, $G(\mathbf{r}, \mathbf{r}_1)$ and $G(\mathbf{r}, \mathbf{r}_2)$, have the same values over the surface S and that their normal derivatives have the same values over the surfaces S, or that the Green's functions vanish (Dirichlet boundary conditions, Section 9.1)[1] over the surface S, the surface integral vanishes and

$$G(\mathbf{r}_1, \mathbf{r}_2) = G(\mathbf{r}_2, \mathbf{r}_1), \quad (8.105)$$

which shows that Green's function is symmetric. If the eigenfunctions are complex, boundary conditions corresponding to Eqs. 9.20–9.22 are appropriate. Equation 8.105 becomes

$$G(\mathbf{r}_1, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \quad (8.106)$$

Note that this symmetry property holds for Green's function in every equation in the form of Eq. 8.101. In Chapter 9 we shall call equations in this form self-adjoint. The symmetry is the basis of various reciprocity theorems; the effect of a charge at $\mathbf{r}_2$ on the potential at $\mathbf{r}_1$ is the same as the effect of a charge at $\mathbf{r}_1$ on the potential at $\mathbf{r}_2$.

This use of Green's functions is a powerful technique for solving many of the more difficult problems of mathematical physics. We shall return to it when we take up integral equations in Chapter 16.

EXERCISES

8.6.1 Let

$$\delta_n(x) = \begin{cases} 0, & x < -\dfrac{1}{2n}, \\ n, & -\dfrac{1}{2n} < x < \dfrac{1}{2n}, \\ 0, & \dfrac{1}{2n} < x. \end{cases}$$

Show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\, \delta_n(x)\, dx = f(0)$$

assuming that $f(x)$ is continuous at $x = 0$.

[1] Any attempt to demand that the normal derivatives vanish at the surface (Neumann's conditions, Section 9.1) leads to trouble with Gauss's Law. It is like demanding that $\int \mathbf{E} \cdot d\boldsymbol{\sigma} = 0$ when you know perfectly well that there is some electric charge inside the surface.

8.6.2 Verify that the sequence $\delta_n(x)$, based on the function

$$\delta_n = \begin{cases} 0, & x < 0 \\ ne^{-nx}, & x > 0, \end{cases}$$

is a delta sequence (satisfying Eq. 8.85*a*). Note that the singularity is at $+0$, the positive side of the origin.

8.6.3 (a) If we define a sequence $\delta_n(x) = n/(2\cosh^2 nx)$, show that

$$\int_{-\infty}^{\infty} \delta_n(x)\, dx = 1, \qquad \text{independent of } n.$$

(b) Continuing this analysis, show that

$$\int_{-\infty}^{x} \delta_n(x)\, dx = \frac{1}{2}[1 + \tanh nx] \equiv S_n(x)$$

and

$$\lim_{n\to\infty} S_n(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

This is the Heaviside unit step function.

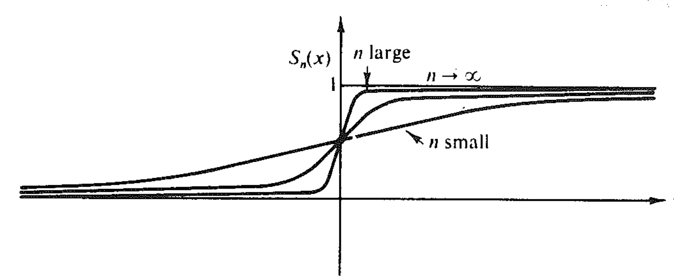

FIG. 8.4 $\frac{1}{2}[1 + \tanh nx]$ and the Heaviside unit step function

8.6.4 Using the Gauss error curve delta sequence (δ_n) show that

$$x \frac{d}{dx} \delta(x) = -\delta(x)$$

treating $\delta(x)$ and its derivative as in Eq. 8.85*b*.

8.6.5 Show that

$$\int_{-\infty}^{\infty} \delta'(x) f(x)\, dx = -f'(0).$$

Here we assume that $f'(x)$ is continuous at $x = 0$.

8.6.6 Prove that

$$\delta(f(x)) = \left| \frac{df(x)}{dx} \right|^{-1} \delta(x - x_0),$$

where x_0 is chosen so that $f(x_0) = 0$. *Hint*. Note that $\delta(f)\, df = \delta(x)\, dx$.