Dimensional Analysis

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"Your units are wrong!" cried the teacher.
"Your church weighs six joules — what a feature!
And the people inside
Are four hours wide,
And eight gauss away from the preacher."

— David Morin

1 Introduction

We can obtain approximations to the solutions of many differential equations (without solving them) by using a technique called *dimensional analysis*.[1] There are primarily two methods that can be used, a "guess" as to which physical variables are important, and a scaling of the governing differential equation. The first method will be shown below, and applied to Planck units in Section 2. The second method will be used in Section 3 to obtain the Reynolds number. Finally, several examples will be given in Section 4.

To indicate how the first method works, let's try to determine the period of a simple pendulum *without* actually solving the differential equation that results from Newton's second law. The first step is to ask yourself "What quantities can the period possibly depend on?" In this case, I would include the properties of the pendulum, such as the string length ℓ, the mass of the bob m, and the maximum amplitude θ_0. The period also might depend on the strength of the restoring force, characterized by the acceleration due to gravity g. The second step is to write down the period as a product of these quantities raised to arbitrary powers

$$T = \ell^a \, m^b \, g^c \, \theta_0^d. \tag{1}$$

Now, at the very least the dimensions *must* match on both sides of any valid equation, so I can express the dimensions of (1) as

$$[T]^1 = [L]^a \, [M]^b \, \left(\frac{[L]}{[T]^2}\right)^c, \tag{2}$$

where I've used $[M]$, $[L]$, and $[T]$, for the dimensions of mass, length, and time, respectively. Notice that the parameter d doesn't appear because θ_0 is a dimensionless quantity, and will

[1] An excellent treatise on this technique is P. W. Bridgman, *Dimensional Analysis*, Yale University Press, 1922. It is also called "similitude."

give us no dimensional information. Equating powers of each dimension results in three equations for the powers of mass, length, and time, respectively

$$\begin{aligned} b &= 0 \\ a + c &= 0 \\ -2c &= 1. \end{aligned} \tag{3}$$

Solving these equations results in $a = \frac{1}{2}$, $b = 0$, and $c = -\frac{1}{2}$, and, therefore, the formula for the period must be of the form

$$T = f(\theta_0)\sqrt{\frac{\ell}{g}}. \tag{4}$$

Again, since the parameter θ_0 is dimensionless, the functional form of the dependence of T on θ_0 is arbitrary. A study of a simple pendulum shows that if θ_0 is small, $f(\theta_0) \approx 2\pi$, and that f is an increasing function of θ_0.

2 Planck units and the Buckingham π theorem

In 1899, Max Planck devised a complete system of so-called "natural" units that was based on the three fundamental physical quantities $\hbar$, G and c. He realized[2] that if we rescaled our usual SI units of meter, kilogram, and second, these quantities would no longer appear in physical equations — they would effectively be rescaled to unity. Noting that the dimensions of these physical constants are $[c] = \frac{[L]}{[T]}$, $[G] = \frac{[L]^3}{[M][T]^2}$, and $[\hbar] = \frac{[M][L]^2}{[T]}$, Planck determined length, mass, and time scales, ℓ_p, m_p, and t_p, which are called the Planck length, Planck mass, and Planck time, respectively

$$\begin{aligned} \ell_p &= \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-35} \text{ m} \\ t_p &= \sqrt{\frac{\hbar G}{c^5}} \approx 5.4 \times 10^{-44} \text{ s} \\ m_p &= \sqrt{\frac{\hbar c}{G}} \approx 2.2 \times 10^{-8} \text{ kg} \end{aligned} \tag{5}$$

These can be obtained using the 'product of powers' method above.

The usefulness of Planck units arises when physicists investigate quantum gravity. Our best theory of gravity is General Relativity, which includes both G (because this sets the scale for the gravitational force) and c (because it is a relativistic theory). In addition, all of our quantum theories involve $\hbar$, Planck's constant. Therefore, if we wish to construct a quantum theory of gravity, then we should expect that all three constants would be involved, and Planck units would set the scales at which quantum gravitational effects might start to be important. For instance, ℓ_p says that we need to look at subatomic phenomenon with a

[2] Planck wasn't the first to do this. In 1881, George Johnstone Stoney used c, G, e (or $e^2/4\pi\epsilon_0$ depending on your unit system) to determine length, mass, and time scales.

'microscopy' capable of 'seeing' down to 10^{-35} m in order to see such effects. Unfortunately, we are a long way from this experimental precision.

A general method to determine how many scales can be obtained in any given situation is that of Buckingham's π theorem.[3] It states that

> Any dimensionally homogeneous equation connecting N physical values, the dimensions of which are expressed by n fundamental units, can be reduced to a function relationship between π dimensionless numbers, where $\pi = N - n$.

For the case of Planck units, $N = 3$ (c, G, $\hbar$), $n = 3$ ($[M]$, $[L]$, $[T]$), and $N - n = 0$. We have no dimensionless quantities! But what if we include another dimension*full* quantity, such as ℓ_p, a length? Then we have $N = 4$ physical quantities (c, G, $\hbar$, ℓ_p) that are still expressible in $n = 3$ independent units, and so there is $N - n = 1$ dimensionless quantity, and that quantity is

$$\frac{\ell_p}{\sqrt{\frac{\hbar G}{c^3}}} = k. \tag{6}$$

This dimensionless quantity is equal to some (unknown) pure number k, which is usually set to unity.

3 Reynolds number

The second method that can be used to obtain dimensionless quantities, and hence gain insight into a system using dimensional analysis, is to force the governing differential equation to be dimensionless. Then, if there are any free parameters, solutions to the equation with the same value of those parameters will be identical.

This idea can be illustrated simply by looking at the equation of motion for a particle of mass m orbiting a fixed mass M located at the origin

$$m\frac{d^2\vec{r}}{dt^2} = -\frac{GMm}{r^2}\hat{r}. \tag{7}$$

In principle, we would need to solve this equation for each value of M and initial position of m (of course, m itself divides out) to solve for the motion of all the planets, say. However, if we scale all lengths to L and all times to T

$$\bar{r} = \frac{r}{L} \qquad\qquad \bar{t} = \frac{t}{T} \tag{8}$$

with L and T yet to be determined (they will depend on the system that we are interested in modeling), we can obtain a dimensionless equation that needs to be solved only once. Velocities, of course, will be scaled by L/T. Now if we express (7) in terms of the dimensionless variables $\bar{r}$ and $\bar{t}$ we have

$$\frac{d^2\vec{\bar{r}}}{d\bar{t}^2} = -\left(\frac{T^2}{L^3}GM\right)\frac{1}{\bar{r}^2}\hat{r}. \tag{9}$$

[3]See, e.g., N. P. Cheremisinoff, *Practical fluid mechanics for engineers and scientists*, Technomic Publishing Company, 1990. ERAU library: `TA 357 .C44 1990`

The equation is now dimensionless, and it can be made even simpler if we set the coefficient in the parentheses to unity

$$\left(\frac{T^2}{L^3}GM\right) = 1. \tag{10}$$

We are free to do this, because we have the freedom to choose both L and T. This then gives

$$\frac{d^2\vec{\bar{r}}}{d\bar{t}^2} = -\frac{1}{\bar{r}^2}\hat{r}. \tag{11}$$

Now, solving (11) solves all problems at once, for any central mass M and for any size orbit. In fact, (10) is nothing but Kepler's Third Law (to within a constant), which states that the orbital period squared is proportional to the semi major axis cubed.

3.1 Navier-Stokes

If we apply this method to the incompressible Navier-Stokes equation, the Reynolds number appears. Given

$$\rho\frac{D\vec{v}}{Dt} = -\nabla p + \mu\nabla^2\vec{v}, \tag{12}$$

where ρ is the density and μ is the dynamic viscosity of the fluid, we can scale all the variables as in (8). However, in addition we'll need

$$\bar{v} = \frac{v}{L/T} \qquad \bar{\nabla} = L\nabla \qquad \bar{p} = \frac{p}{M/LT^2} \tag{13}$$

where an additional mass scaling, M, is needed for the pressure. It appears that there are two 'parameters' in (12), ρ and μ, and we would have to solve (12) for each system of different ρ, μ as well as size and speed of the flow. However, after scaling the Navier-Stokes equation, we are free make choices about the scaling factors and reduce the parameters to one. The scaled equation becomes

$$\left(\frac{L^3}{M}\rho\right)\frac{D\vec{\bar{v}}}{D\bar{t}} = -\bar{\nabla}\bar{p} + \left(\frac{LT}{M}\mu\right)\bar{\nabla}^2\vec{\bar{v}}. \tag{14}$$

We can set the first coefficient in parentheses to unity by choosing the scaled for mass, $M = L^3\rho$. The second coefficient becomes $T\mu/L^2\rho$, and our equation is

$$\frac{D\vec{\bar{v}}}{D\bar{t}} = -\bar{\nabla}\bar{p} + \left(\frac{T\mu}{L^2\rho}\right)\bar{\nabla}^2\vec{\bar{v}}. \tag{15}$$

We now have a choice. It would be nice to set $T\mu/L^2\rho = 1$ to obtain one equation without any parameters, just like (11), which means that we have set the relation between the length and time scales, L and T. However, if we want to be able to model systems with different sizes and flow speeds, then we can replace that last coefficient with a dimensionless number

$$\frac{T\mu}{L^2\rho} = \frac{1}{\mathrm{Re}} \tag{16}$$

where Re is the "Reynolds number." It is usually written as

$$\mathrm{Re} = \frac{\rho L V}{\mu}, \tag{17}$$

where V is a typical velocity scale, and L, of course, is a typical length scale of the system. As can be seen from the final version of the Navier-Stokes equation

$$\frac{D\bar{\vec{v}}}{D\bar{t}} = -\bar{\nabla}\bar{p} + \left(\frac{1}{\mathrm{Re}}\right)\bar{\nabla}^2\bar{\vec{v}}, \tag{18}$$

the value of the Reynolds number for a specific system characterizes the effect of the inertial forces relative to viscous forces. That is, when Re is large, viscous forces play no role, and the flow is typically turbulent. However, for small Re, viscosity is important, and the flow is typically laminar.

3.2 The magnetic Reynolds number

In the approximation to the Maxwell-fluid equations known as resistive magnetohydrodynamics, we have from Faraday's Law, Ampere's Law, and Ohm's Law,[4]

$$\frac{\partial \vec{B}}{\partial t} = \left(\frac{c^2}{4\pi\sigma}\right)\nabla^2\vec{B} + \nabla\times\left(\vec{u}\times\vec{B}\right), \tag{19}$$

where σ is the conductivity of the medium. In the absence of flow (i.e., $\vec{u} = 0$) this reduces to a classical diffusion equation with a diffusion coefficient

$$D = \frac{c^2}{4\pi\sigma}. \tag{20}$$

It follows that an initial magnetic field, and the currents which support it, will decay due to Ohmic dissipation on a time scale $\tau = L^2/D$, where L is a typical scale length.

For $\vec{u} \neq 0$, the relative magnitude of the two terms on the right side of (19) will be of order (exercise: show this)

$$R_m = \frac{4\pi\sigma u L}{c^2}, \tag{21}$$

[4]Maxwell's equations consist of Gauss's Laws for both the electric and magnetic fields

$$\nabla\cdot\vec{E} = 4\pi\rho, \qquad \nabla\cdot\vec{B} = 0,$$

Faraday's Law and Ampere's Law,

$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \qquad \nabla\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}.$$

Finally, we need Ohm's Law

$$\vec{E} + \frac{1}{c}\vec{u}\times\vec{B} = \vec{j}/\sigma$$

to complete the set.

which, by analogy with the Reynolds number encountered in viscous flow of a neutral gas, is called the "magnetic Reynolds number." When $R_m \ll 1$, diffusion dominates. When $R_m \gg 1$, then (19) can be approximated by

$$\frac{\partial \vec{B}}{\partial t} \approx \nabla \times \left(\vec{u} \times \vec{B}\right), \tag{22}$$

the equation of "frozen-in magnetic flux." In this case, the magnetic field lines move with the plasma.

4 Examples

There are a large number of examples that illustrate these principles. Here are just a few.

4.1 Are sound waves isothermal or adiabatic?

The pressure and density variations in sound waves imply that there will also be temperature variations. However, when adjacent regions have different temperatures, as in alternating phases of a wave (the peaks and troughs will have slightly different temperatures), those regions would like to equilibrate and have the same temperature. However, do the sound wave oscillations occur slowly enough for the heat energy to diffuse from a region of higher T to that of lower T and make the waves essentially isothermal? Or do they occur too quickly for the heat energy to flow, and therefore the waves are adiabatic?

One might think that high frequency sound waves oscillate so quickly that the heat energy doesn't have time to flow the half wavelength distance from a temperature maximum to a temperature minimum, making them adiabatic. And in conjunction, low frequency sound waves oscillate so slowly that there's plenty of time for temperature equilibration, making them isothermal. However, this turns out to be exactly opposite of the truth. High frequency waves have short wavelengths, which means that the heat energy doesn't have far to diffuse, and they are isothermal, and low frequency waves have very long wavelengths making it difficult for the heat energy to diffuse the long distance quickly enough, so they are adiabatic. These counter-intuitive results can easily be shown with our tool of dimensional analysis.

The diffusion of heat is, of course, governed by the heat equation

$$\frac{\partial T}{\partial t} = D\nabla^2 T, \tag{23}$$

where the diffusion coefficient has dimensions $[D] = [L]^2/[T]$. Unlike convection, which travels at a constant speed, and therefore it takes twice as long to move twice the distance, in diffusion it takes four times as long to move twice the distance. This is the reason why the long wavelength wins out at low frequencies, implying adiabaticity.

In the spirit of dimensional analysis, the only length scale in the problem is the wavelength, λ, and the only time scale is the frequency, f. This means that our dimensionless quantity must be

$$\frac{D}{\lambda^2 f}. \tag{24}$$

If this quantity is large, then the heat energy diffuses fast enough to make the waves isothermal. Otherwise, they are adiabatic. So our condition for isothermal waves is

$$\frac{D}{\lambda^2 f} > 1, \tag{25}$$

or, since the phase velocity of the wave is $v_{ph} = \lambda f$,

$$f > \frac{v_{ph}^2}{D} \equiv f_c, \tag{26}$$

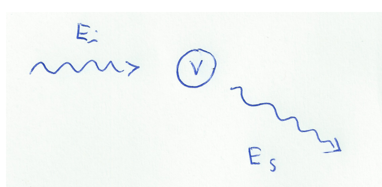
where f_c is a critical frequency above which the sound waves are isothermal. For sound waves in air, the phase velocity is approximately constant ($\sim$ 340 m/s). There is a small amount of dispersion (i.e., when the phase velocity depends on the frequency), but it is a small effect. For the heat equation[5]

$$D = \frac{k}{\rho C_V}, \tag{27}$$

where k is the thermal diffusivity, ρ is the density, and C_V is the specific heat at constant volume. Inserting values for air, I get that the critical frequency, f_c, is on the order of 1000 Hz.

4.2 Rayleigh scattering

It is well known that the sky is blue because the sunlight impinging on the atmosphere undergoes Rayleigh scattering. Since the scattered intensity is proportional to f^4, where f is the frequency of the light, blue light is scattered more strongly than red (it has a shorter wavelength, and hence a higher frequency, by about a factor of 2, hence the scattering is stronger by about a factor of 16). This is also the reason why sunsets and sunrises are red. This f^4 behavior can be understood using dimensional analysis.



Consider an incident electromagnetic (EM) waves with uniform amplitude E_i impinging on a particle of volume $V \sim L^3$. If the wavelength of the light is large, $\lambda \gg L$, then the entire

[5]This arises from the continuity equation for energy

$$\rho C_V \frac{\partial T}{\partial t} + \nabla \cdot \vec{q} = 0,$$

where $\vec{q}$ is the flux of heat energy, along with Fourier's Law for heat conduction

$$\vec{q} = -k\nabla T.$$

volume V will oscillate in phase and hence the scattered amplitude, E_s, will be proportional to V

$$E_s \sim L^3. \tag{28}$$

Also, in contrast to the incident plane wave, the scattered wave is emitted in all directions, and hence the scattered intensity, I_s (which is proportional to the square of the amplitude), must fall off as the inverse square of the distance r from the scatterer

$$I_s \sim E_s^2 \sim \frac{1}{r^2}. \tag{29}$$

Finally, the scattered amplitude must also be proportional to the incident amplitude, $E_s \sim E_i$. Putting this all together leaves us with the wrong dimensions, and the only parameter with dimensions of length is the wavelength, so the final version must be

$$I_s \sim E_s^2 \sim E_i^2 \frac{L^6}{r^2 \lambda^4} \sim f^4, \tag{30}$$

where the last proportionality was obtained because $c = f\lambda$. Hence we obtain the prominent feature of Rayleigh scattering.

4.3 Atomic Radius

Applying dimensional analysis to the hydrogen atom, an electron "orbiting" a proton, I hypothesize that the possible quantities that determine its size are its properties, such as e, a measure of the strength of the electric force between the proton and electron; m_e, a measure of the electron's inertia;[6] and ϵ_0, also a measure of the strength of the attractive electric force. (It could be true that the gravitational force plays a role—if so, then we'd need to include G. But the gravitational force is very weak compared with the electric force, so we have good reason to ignore it.) If you include only these three parameters, you'll find that the resulting system of equations for the values of the exponents is over-determined, i.e., there are no solutions (see Problem 1). However, if you include a fourth quantity, h, Planck's constant, then you'll find that the problem is solvable. The fact that a solution requires the inclusion of h means that the size of the atom is quantum mechanical in nature, and that without quantum mechanics, atoms would not have a well-defined size. In other words, it implies that quantum physics *must* play an important role in atomic structure. Our second step is now clear: the atomic radius r can be expressed as a product

$$r = e^a\, m_e^b\, \epsilon_0^c\, h^d. \tag{31}$$

To express this dimensionally, we need to introduce another dimension, charge, which I'll denote by $[Q]$. Since, in the SI system, ϵ_0 has the units $\mathrm{C^2/Nm^2}$, and h has the units J s, I can express the dimensions of (31) as

$$[L]^1 = [Q]^a\, [M]^b\, \left(\frac{[Q]^2[T]^2}{[M][L]^3}\right)^c \left(\frac{[M][L]^2}{[T]}\right)^d. \tag{32}$$

[6] I'll make the approximation that the proton is very massive and therefore stationary.

Again, equating powers of each dimension results in the set of equations for mass, length, time, and charge, respectively

$$\begin{aligned} b - c + d &= 0 \\ 2d - 3c &= 1 \\ 2c - d &= 0 \\ a + 2c &= 0, \end{aligned} \tag{33}$$

allows us to determine that $a = -2$, $b = -1$, $c = 1$, and $d = 2$. Therefore the formula for the size r of the atom is

$$r = \mathcal{C}\,\frac{\epsilon_0 h^2}{m_e e^2}. \tag{34}$$

As is the case with all dimensional analysis, there is no way to determine the dimensionless numerical factor $\mathcal{C}$, but we'll see that if $\mathcal{C} = 1/\pi$, then r equals the Bohr radius. Interestingly, the fourth equation in (33) requires $a = -2c$, which means that e and ϵ_0 always occur as e^2/ϵ_0. This makes sense because they both determine the strength of the electric force between two particles of charge e.

Have all possible dependencies been considered? For example, are relativistic effects important? Should we include the speed of light c in our analysis? Problem 2 investigates this possibility.

4.4 Electromagnetic mass.

In the early 20th century, a quantity called the "classical electron radius" was determined. It is

$$r_e = \frac{e^2}{4\pi\epsilon_0 m_e c^2} \approx 2.8 \times 10^{-15} \text{ m}. \tag{35}$$

This was obtained by setting the energy necessary to assemble the electron's charge e equal to $m_e c^2$, the idea being that the energy of assembly went toward the rest energy of the electron. However, experimentally, the "size" of the electron is less than 10^{-22} m! In any case, this classical radius (classical here meaning 'not quantum') is a useful quantity that describes some of the electron's interactions. The technique of dimensional analysis can be used to determine this radius. Consider a sphere of electric charge q and radius r. What quantities might its mass depend on? Since the work necessary to assemble the charge depends on the electric force, the mass should depend on q, r, and also ϵ_0 (which determines the strength of the electric force). If we set the electron mass equal to the product

$$m_e = q^a\, r^b\, \epsilon_0^f, \tag{36}$$

then the set of equations for the powers are four equations for only three unknowns, a, b, and c

$$\begin{aligned} -f &= 1 \\ b - 3f &= 0 \\ 2f &= 0 \\ a + 2f &= 0. \end{aligned} \tag{37}$$

The system is obviously overdetermined (e.g., the first and third equations are inconsistent). What does this mean physically? We originally guessed that since the sphere of charge was in empty space, the only possibilities were those that dealt with the electric force. But empty space *does* have properties, in the sense that the electric force is carried by photons, which travel at the speed of light. Therefore, we should include c in our list of possible dependencies, and use the formula

$$m_e = q^a\, r^b\, \epsilon_0^f\, c^d. \tag{38}$$

In Problem 3 you can show that the electromagnetic mass must take the form

$$m_e = \mathcal{D}\frac{q^2}{\epsilon_0 r c^2}, \tag{39}$$

where $\mathcal{D}$ is an unknown dimensionless constant. This is identical to the result in (35) above for the classical electron radius.

Problem 1. Attempt to obtain the size of the hydrogen atom using dimensional analysis assuming that r depends only on e, m_e, and ϵ_0. What mathematical problems do you encounter?

Problem 2. Attempt to obtain the size of the hydrogen atom using dimensional analysis assuming that r depends on e, m_e, ϵ_0, as well as h and c. Do you encounter any mathematical problems in this case?

Problem 3. Use dimensional analysis to show that the mass of a sphere of charge in empty space takes the form of (39) if you assume dependences of the form of (38).

Solution 1. Our guess for the form of r is

$$r = e^a \, m_e^b \, \epsilon_0^c.$$

As in the text, equating powers of dimensions on each side of the equation gives a set similar to (33), except that all the values of d are zero

$$\begin{aligned} b - c &= 0 \\ -3c &= 1 \\ 2c &= 0 \\ a + 2c &= 0, \end{aligned}$$

Immediately we see that the second and third equation are not consistent. In fact, the entire system is overdetermined, as there are too many equations for the number of free variables. The set of equations is not linearly independent.

Solution 2. Now we have to add a term to our guess for r

$$r = e^a \, m_e^b \, \epsilon_0^c \, h^d \, c^e.$$

This means that equating dimensions, as in (32), results in

$$[L]^1 = [Q]^a \, [M]^b \left(\frac{[Q]^2 [T]^2}{[M][L]^3} \right)^c \left(\frac{[M][L]^2}{[T]} \right)^d \left(\frac{[L]}{[T]} \right)^e.$$

Now, however, we have an underdetermined system, which means that there's not enough information to solve for all 5 exponents.

$$\begin{aligned} b - c + d &= 0 \\ 2d - 3c + e &= 1 \\ 2c - d - e &= 0 \\ a + 2c &= 0, \end{aligned}$$

The last equation again tells us that e and ϵ_0 must be grouped as e^2/ϵ_0. The first equation minus the sum of the second and third results in $b = -1$, as before. But I then obtain only two equations for the final three exponents, $c = d - 1$ and $c - e = 1$. This system is underdetermined. A more likely solution to the problem of introducing relativistic effects is to assume that the radius is

$$r = e^a \, m_e^b \, \epsilon_0^c \, h^d \, f\left(\frac{v}{c}\right),$$

where f is a dimensionless function of $\beta = v/c$. This will result in (34) multiplied by f, and other considerations will be needed to determine the functional form of f.

Solution 3. With the addition of the factor of c^d, Eqs. (37) become

$$\begin{aligned} -f &= 1 \\ b - 3f + d &= 0 \\ 2f - d &= 0 \\ a + 2f &= 0 \end{aligned}$$

These four equations with four unknowns can be easily solved: $a = 2$, $b = -1$, $f = -1$, and $d = -2$, which results in (39).