Charged particle dynamics

M. A. Reynolds

October 27, 2016

1 Introduction

This second project will allow you to become further acquainted with the Matlab programming environment, learn about solving coupled differential equations, and modify previously written code. You will again write a report, fixing any mistakes that you made in your previous report.

2 Project Requirements

Step 1 — cyclotron motion Download the file ParticlePush.m from the course web page and run it. This main program solves Newton's second law for a charged particle in a uniform magnetic field and plots the results. You will compare the numerical solution for pure cyclotron motion with the solution derived in class.

(a) The analytical solution for velocity derived in class for pure cyclotron motion was:

$$\vec{v}(t) = v_\perp \sin(\omega_c t)\hat{x} + v_\perp \cos(\omega_c t)\hat{y} + v_\parallel \hat{z}.$$

Integrate this solution to produce an expression for the position vector $\vec{r}$ as a function of time.

(b) Alter ParticlePush.m to plot the analytical solution alongside the one obtained using the Runge-Kutta integration technique. Make sure to use the same magnetic field value as in the program and include a plot legend with distinguishable line types for the numerical and analytical solutions. Verify that both solutions are in reasonable agreement. For this exercise they should be almost identical.

(c) Alter the charge and mass variables in ParticlePush.m so that you can simulate an electron for the same initial conditions. Run the program, and plot the results alongside the analytical expression, using values for electrons this time. Demonstrate again that the numerical and analytical results are consistent.

NOTE: Your results should be summarized in two plots (one for the proton and one for electrons and the overplotted analytical solutions). These plots must be included in your report (i.e., integrated — with captions — into the pdf file). Your derivation for the position as a function of time should also be included in your report.

Step 2 — ∇B motion Here, you will simulate gradient-B drift motion for a proton.

(a) Modify the magnetic field subroutine in ParticlePush.m to include a magnetic field with a gradient.[1] The simplest way to do this is to keep the magnetic field purely in the z-direction (as in the previous section) but to include a linear variation (e.g., with respect to x) so that a constant gradient exists (in the x-direction in this case).

$$\vec{B}(x,y,z) = B_0 \left(1 + \frac{x}{L}\right) \hat{z},$$

where $L = 2$ m. As in Step 1, use a reference magnetic field of $B_0 = 45000$ nT. Modify the initial conditions so that there is no parallel-to-B component of velocity (to simplify the visualization of results) and assume that the particle starts at the origin with velocity $v_y = 4000$ m/s. Plot the results and verify that the particle drifts in the correct direction while also undergoing cyclotron motion. Note that since the particle remains in the $z = 0$ plane, a 2D plot of y vs. x is sufficient.

(b) Construct an approximate analytical solution for the particle position vector vs. time by adding together the cyclotron motion solution with an average grad-B drift. Use the magnetic field above to compute the gradient. For computing other quantities (e.g., magnetic fields and moments) in the gradient-B drift formula, use average values of the magnetic field. These can be calculated by using the average position. Plot your analytical solutions alongside the numerical solution that you have calculated in part (a) and verify that the two solutions are approximately equal.

(c) Run the program for two more different initial velocities — half and double the values used before. Make plots of the particle paths from these two simulations. Discuss the dependence of your results on initial velocity and interpret the results in terms of Larmor radii and gradient-B drift speeds. Discuss the consistency of your simulation with the basic gradient-B drift formula.

NOTE: Your results should be summarized in two plots (one comparing numerical and analytical results and another showing the results of halving and doubling the energy). Your derivation for the position as a function of time for a proton undergoing grad-B drift should also be included in your report.

Step 3 — magnetic mirror motion (a) Modify the magnetic field subroutine in ParticlePush.m to describe a magnetic field with a "bottle" configuration, i.e., a region of weak field surrounded by a strong-field region. One simple way to do this is to specify the z component of the magnetic field as a quadratic function of z

$$B_z = B_0 \left(1 + \frac{z^2}{a^2}\right),$$

where $a = 300$ m and $B_0 = 10\ \mu$T.

[1] There are two places in the RK2 routine where the magnetic field structure must be inserted. First, at the very beginning of the loop, and here B depends on the previous values of r and v, which are called "rprev" and "vprev". Second, after new values of the position and velocity are computed, the electric and magnetic field values must be evaluated there, as functions of 'r' and 'v'.

(b) Use the divergence-free constraint of Gauss's law for magnetism ($\nabla \cdot \vec{B} = 0$) to determine the radial component of the magnetic field B_ρ, as we did in class. You can assume that the magnetic field has azimuthal symmetry. You will need to convert the cylindrical components of the magnetic field vector into Cartesian components $B_x(x, y, z)$ and $B_y(x, y, z)$, as needed by the program.

(c) Choose the initial position of your proton to be the origin, as before, but have the initial velocity components be $v_{y0} = v_{z0} = 10,000$ m/s. You will need to modify the total runtime (variable 'lt') to be more than 10 cyclotron periods (50-80 should be sufficient). Plot the particle path and verify that it mirrors correctly. Estimate the period (in seconds) of the mirror motion by producing an additional plot of the particle velocity vs. time along the z-direction (the axis of the bottle) and reading the period of the oscillation from the graph. You may need to adjust the runtime so that the particle mirrors a few times so you can easily estimate frequency and period. Take care to avoid division by zero when computing the x and y components of the magnetic field.

(d) The parallel equation of motion can actually be solved analytically for the special form of the magnetic field we have chosen. Set up an equation of motion for the z-component of the velocity by including the mirror force

$$F_z = -\mu \frac{\partial B_z}{\partial z}$$

in Newton's second law. Solve your differential equation (remember that the magnetic moment is constant of motion!) and use the results to calculate the what the frequency and period of motion along the axis of the bottle should be. Compare this prediction of the oscillation period with that obtained from the simulation results.

NOTE: Your results should be summarized in two plots (one showing the simulated mirror motion and another showing the z-component of velocity vs. time). Your derivation for the radial and Cartesian components of the magnetic field should be included along with your analysis of the parallel equation of motion.

Step 4 — motion in a dipole magnetic field (a) Modify the magnetic field subroutine in PARTICLEPUSH.M to describe a dipole magnetic field. Convert the form for dipole magnetic field, as discussed in class, into Cartesian coordinates. i.e., obtain the following functions, $B_x(x, y, z)$, $B_y(x, y, z)$, and $B_z(x, y, z)$. Perhaps the best way to do this is to use the Cartesian form of the unit vectors $\hat{r}$ and $\hat{\lambda}$ plug in values for r, $\sin\lambda$, and $\cos\lambda$, in terms of x, y, and z. Be sure to include this derivation in your report.

(b) Modify the magnetic field subroutine in PARTICLEPUSH.M so that it calculates the dipole magnetic field. The magnetic moment of the Earth[2] is $M = 7.94 \times 10^{22}$ A m^2. Set the initial position of the particle to be in the equatorial plane ($z = 0$) at a distance of $r = 2R_E$ from the center of the earth (which is at the origin). Adjust the initial velocity so that the proton has equal velocity components $v_{x0} = v_{y0} = v_{z0}$ and a total

[2]Or, if you prefer, you can set the field strength at the Earth's surface at the equator to be $B_0 = 3 \times 10^5$ T.

kinetic energy of 400 keV. Note the special initial conditions you use in your report. Alter the runtime of the program so that the particle can be seen to mirror and drift azimuthally (due to grad-B and curvature drift). I have found that 1250 cyclotron periods is sufficient. Plot the results for the particle path. Estimate the mirror bounce period from the plot (be sure to explain how you obtained your estimate in the report).

(c) Adjust the runtime of the program so that you can see the particle execute one full orbit around the Earth. Plot the path and estimate the orbit time. Due to the large number of outputs for the path, it may be faster to plot only one out of every 1000 points along the path to see the drift motion. This is controlled by the variable 'itout.' I have found that about 250,000 cyclotron periods are required to get a full orbit and that the program takes several minutes to run under these circumstances. Estimate the time the particle takes to orbit the Earth.

(d) Compute analytically the gradient-curvature drift velocity for a particle with the parameters used in your simulation. Remember that the radius of curvature of a field line that crosses the equatorial plane at $r = r_0$ is $r_0/3$. Assume that the particle describes a circular path around the Earth and compute the time it takes to complete one orbit. Compare this estimate against the simulation results. This calculation involves several significant approximations, but you should still be able to get within a factor of 2-10 of the true result given by the simulation.

NOTE: Your results should be summarized in two plots (one showing the particle mirroring in the dipole field and another showing drift motion around the earth. Your derivation for the Cartesian components of the dipole sfield and your calculations involving gradient-curvature drift should also be included.

3 References

You don't need to use the "bibliography" environment for your references, but you can type them out by hand.