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Chapter 3: Index Notation

The rules of index notation:

(1) Any index may appear *once* or *twice* in any term in an equation

(2) A index that appears just once is called a *free index*.
The free indices must be the same on both sides of the equation.
Free indices take the values 1, 2 and 3

(3) A index that appears twice is called a *dummy index*.
Summation Convention: Dummy indices are summed over from 1 to 3
The name of a dummy index is not important.

$$\mathbf{a} \cdot \mathbf{b} = a_j b_j = a_l b_l = a_p b_p = a_1 b_1 + a_2 b_2 + a_3 b_3$$

(4) The Kronecker Delta:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

or

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Kronecker Delta is *symmetric* $\delta_{ij} = \delta_{ji}$. If one index on δ_{ij} is free and the other dummy then the action of δ_{ij} is to substitute the dummy index with the free index

$$\delta_{ij} a_j = a_i$$

If both indices are dummies then the δ_{ij} acts as scalar product.

(5) The Alternating Tensor:

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any of } i,\ j \text{ or } k \text{ are equal,} \\ 1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) = (1,3,2),\ (3,2,1) \text{ or } (2,1,3) \end{cases}$$

The Alternating Tensor is *antisymmetric*:

$$\epsilon_{ijk} = -\epsilon_{jik}$$

The Alternating Tensor is invariant under cyclic permutations of the indices:

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$$

(6) The vector product:

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk}\, a_j b_k$$

(7) The relation between δ_{ij} and ϵ_{ijk}:

$$\epsilon_{ijk}\, \epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Grad, Div and Curl and index notation

$$\text{grad} f = (\boldsymbol{\nabla} f)_i = \frac{\partial f}{\partial x_i}$$

$$(\boldsymbol{\nabla})_i = \frac{\partial}{\partial x_i}$$

$$\text{div}\, \boldsymbol{F} = \boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{\partial F_j}{\partial x_j}$$

$$(\text{curl}\, \boldsymbol{F})_i = (\boldsymbol{\nabla}\times\boldsymbol{F})_i = \epsilon_{ijk}\frac{\partial F_k}{\partial x_j}$$

$$(\boldsymbol{F}\cdot\boldsymbol{\nabla}) = F_j\frac{\partial}{\partial x_j}$$

Note: Here you cannot move the $\dfrac{\partial}{\partial x_j}$ around as it acts on everything that follows it.

Vector Differential Identities.
If $\boldsymbol{F}$ and $\boldsymbol{G}$ are vector fields and f and g are scalar fields then

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} f) = \nabla^2 f$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{F}) = 0$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla} f) = \boldsymbol{0}$$

$$\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f$$

$$\boldsymbol{\nabla}\cdot(f\boldsymbol{F}) = f\boldsymbol{\nabla}\cdot\boldsymbol{F} + \boldsymbol{F}\cdot\boldsymbol{\nabla} f$$

$$\boldsymbol{\nabla}\times(f\boldsymbol{F}) = f\boldsymbol{\nabla}\times\boldsymbol{F} + \boldsymbol{\nabla} f\times\boldsymbol{F}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) - \nabla^2\boldsymbol{F}$$

$$\boldsymbol{\nabla}(\boldsymbol{F}\cdot\boldsymbol{G}) = \boldsymbol{F}\times(\boldsymbol{\nabla}\times\boldsymbol{G}) + \boldsymbol{G}\times(\boldsymbol{\nabla}\times\boldsymbol{F}) + (\boldsymbol{F}\cdot\boldsymbol{\nabla})\boldsymbol{G} + (\boldsymbol{G}\cdot\boldsymbol{\nabla})\boldsymbol{F}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{F}\times\boldsymbol{G}) = \boldsymbol{G}\cdot(\boldsymbol{\nabla}\times\boldsymbol{F}) - \boldsymbol{F}\cdot(\boldsymbol{\nabla}\times\boldsymbol{G})$$

$$\boldsymbol{\nabla}\times(\boldsymbol{F}\times\boldsymbol{G}) = \boldsymbol{F}(\boldsymbol{\nabla}\cdot\boldsymbol{G}) - \boldsymbol{G}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) + (\boldsymbol{G}\cdot\boldsymbol{\nabla})\boldsymbol{F} - (\boldsymbol{F}\cdot\boldsymbol{\nabla})\boldsymbol{G}$$