Chapter 4

One Fluid MHD

A. Heuristic Considerations

In the preceding two chapters we have derived the Vlasov equation as the lowest (first order) approximation to the microscopic kinetic equation and then obtained the two-fluid equations by taking velocity moments of the Vlasov equation and invoking the Maxwellian closeure approximation. We also showed how the Vlasov and fluid equations can be heuristcially "derived" directly from conservation of particles in the six dimensional phase space and from mass, momentum, and energy conservation, respectively. The equations of magnetohydrodynamics (MHD) can also be obtained either deductively, from first principles, or heuristically. In this chapter we shall first use an heuristic approach to anticipate the form to be expected for one fluid MHS and then show that equations of this character can, indeed, be derived from the two fluid MHD formulation of Chapter 3 with suitable approximations.

For a neutral fluid, with mass density ρ_m, velocity $\mathbf{u}$ and scalar pressure p, conservation of mass, momentum and energy give equations similar to those obtained from a single species in Chapter 3, i.e., (3.5), (3.14), and (3.21) with one difference: for a neutral fluid there is, of course, no force corresponding to the $nq\mathbf{E}$ term of (3.14). However, if the fluid is electrically conducting, then its motion may give rise to an electrical current density, $\mathbf{j}$, and a consequent force density $\mathbf{j} \times \mathbf{B}/c$ in the presence of a magnetic field. Thus, we expect equations of the form

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u}) = 0 \tag{4.1}$$

$$\rho_m \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \nabla p = \frac{1}{c} \mathbf{j} \times \mathbf{B} \tag{4.2}$$

$$p \rho_m^{-\gamma} = \text{constant}. \tag{4.3}$$

As usual, we adjoin to these dynamic equations the Maxwell equations (for a neutral fluid)

$$\nabla \cdot \mathbf{E} = 0, \tag{4.4a}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4.4b}$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{4.4c}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}. \tag{4.4d}$$

Since MHD is valid for low frequencies, the displacement current term is often a small correction and many treatments of MHD omit it altogether. We shall retain it in general, but drop it when appropriate.

Considering Eqs. (4.1) through (4.4) as a description of the temporal evolution of the system, we see that they are incomplete. The evolution of ρ_m (in terms of ρ_m and $\mathbf{u}$) is determined by (4.1) and that of $\mathbf{u}$ (in terms of ρ_m, $\mathbf{u}$, p, $\mathbf{j}$, and $\mathbf{B}$) is given by (4.2). From (4.3) we can solve for p in terms of ρ_m, and Ampere's Law (omitting the displacement current) gives $\mathbf{j}$ in terms of $\mathbf{B}$, but the evolution of $\mathbf{B}$ according to Faraday's Law depends on $\mathbf{E}$, so some further relation involving $\mathbf{E}$ is required to close the system of equations. (Alternatively, if we retain the displacement current term, Ampere's Law serves to determine the evolution of $\mathbf{E}$ in terms of $\mathbf{j}$ and $\mathbf{B}$, but we then lack an equation to determine $\mathbf{j}$.) The missing link is supplied by a "constitutive" equation expressing $\mathbf{j}$ in terms of $\mathbf{E}$ and $\mathbf{B}$, introduced in an *ad hoc* or phenomenological basis similar to that used for the dielectric constant ε in the electrodynamics of uniform continuous media. The simplest assumption is that of scalar conductivity: we assume that in the local rest frame of the fluid the current density $\tilde{\mathbf{j}}$ and the field $\tilde{\mathbf{E}}$ are proportional, $\tilde{\mathbf{j}} = \sigma\tilde{\mathbf{E}}$. The corresponding quantities in the lab frame then satisfy

$$\mathbf{j} = \tilde{\mathbf{j}} = \sigma\tilde{\mathbf{E}} = \sigma\left(\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{B}\right). \tag{4.5}$$

Equations (4.1) through (4.5) constitute the simplest form of conventional, one-fluid MHD. In the next section, we show how to derive, from two fluid MHD, a set of equations similar to (4.1) through (4.5) which have been obtained here by macroscopic and phenomenological considerations.

B. Derivation from Two Fluid MHD

We start from the equations n the from [3.14] through [3.16] and define ρ_m and $\mathbf{u}$

$$\rho_m \equiv n_e m + n_i M, \tag{4.6}$$
$$\rho_m\mathbf{u} \equiv n_e m\mathbf{u}_e + n_i M\mathbf{u}_i. \tag{4.7}$$

Our first simplifying approximation is to assume charge neutrality

$$n_e = n_i = n \tag{4.8}$$

so that

$$\rho_m \approx nM \tag{4.9}$$

and

$$\mathbf{u} = \mathbf{u}_i + \frac{m}{M}\mathbf{u}_e. \tag{4.10}$$

As usual, we neglect corrections of order m/M. In the absence of external sources, we have

$$\mathbf{j} = en(\mathbf{u}_i - \mathbf{u}_e). \tag{4.11}$$

Then (4.10) and (4.11) can be solved for

$$\mathbf{u}_i = \mathbf{u} + \delta\mathbf{w} \tag{4.12}$$

and

$$\mathbf{u}_e = \mathbf{u} - \mathbf{w} \tag{4.13}$$

where $\mathbf{w} \equiv \mathbf{j}/ne$ and $\delta \equiv m/M$.[1]

We now take linear combinations of [3.14] through [3.16] for the two species. We shall retain the collisional momentum transfer terms, $\mathbf{P}$, even though we have not derived them in a rigorous way, since only then do we obtain the conventional form of one-fluid MHD. We shall assume that only electron-ion collisions are involved and these give rise to a simple "frictional" drag force,

$$\mathbf{P}_e = -\mathbf{P}_i = nm_e\nu(\mathbf{u}_i - \mathbf{u}_e) \tag{4.14}$$

where ν is an electron-ion collision frequency. (In a subsequent chapter [which one?] we shall derive the collisional terms in a rigorous way, discuss the circumstances under which this simple form is valid, and show how ν can be computed.) Adding the continuity equations, (3.23), for each species, weighted by the respective masses, gives the expected continuity equation (4.1), or equivalently,

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0. \tag{4.15}$$

Similarly, it follows from the energy equations (3.25) for each species that

$$pn^{-\gamma} = \text{constant}, \tag{4.16}$$

where $p = p_e + p_i$ and we have assumed $\gamma_e = \gamma_i = \gamma$. (In general, γ has the value appropriate to a three-dimensional, monatomic gas, $\gamma = 5/3$.) Note that (4.15) and (4.16) are formally the same as the continuity and energy equations (3.5) and (3.22). However, in the latter case n, $\mathbf{u}$ and p refer to a particular species (the species label having been suppressed for notational convenience), whereas here p denotes total pressure; $\mathbf{u}$ is defined by (4.7); and $n = n_e = n_i$.

There remain only the momentum equations. Taking their sum and using (4.12) and (4.13) to simplify the convective term yields

$$nM\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p - \frac{1}{c}\mathbf{j}\times\mathbf{B} = \left(\frac{m}{e^2}\right)\mathbf{j}\cdot\nabla\left(\frac{\mathbf{j}}{n}\right). \tag{4.17}$$

The right side of (4.17) can be neglected, and we recover the conventional form (4.2) if we make the very reasonable assumption that $Mu^2 \gg m(j/ne)^2$, i.e., that the kinetic energy of fluid flow is much greater than that associated with current flow. (We also assume that the scale lengths associated with the spatial variations of these two quantities are not so disparate as to reverse the sense of this inequality.)

Finally, we take the difference of the momentum equations for each species, multiplied by e and divided by the respective masses. Some terms cancel while others are of order m/M and can be dropped. A term proportional to $\mathbf{j}\times\mathbf{B}$ appears and it can be eliminated using (4.2). The result is

$$\left(\frac{m}{ne^2}\right)\frac{\partial \mathbf{j}}{\partial t} + \frac{\nabla p_i}{ne} + \left(\frac{M}{e}\right)\frac{d\mathbf{u}}{dt} + \left(\frac{m\nu}{ne^2}\right)\mathbf{j} \quad - \quad \left(\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{B}\right) =$$

[1]Of course, strictly speaking (4.12) and (4.13) should be $(1+\delta)\mathbf{u}_i = \mathbf{u} + \delta\mathbf{w}$ and $(1+\delta)\mathbf{u}_e = \mathbf{u} - \mathbf{w}$, but we neglect δ compared with unity.

$$\left(\frac{m}{e}\right)\left[\mathbf{w}\cdot\nabla\mathbf{w} \quad - \quad \mathbf{w}\cdot\nabla\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{w}-\left(\frac{\mathbf{w}}{n}\right)\nabla\cdot(n\mathbf{u})\right] \tag{4.18}$$

The approximations conventionally made in deriving the MHD equations are that

$$\frac{w\tau}{L} \ll 1 \text{ and } \frac{u\tau}{L} \ll 1 \tag{4.19}$$

where L is a typical scale length for n, $\mathbf{u}$, and $\mathbf{j}$, $L \sim u/|\nabla u| \sim n/|\nabla n| \sim j/|\nabla j|$, and τ is the time scale on which $\mathbf{j}$ varies, $\tau \sim j/|\partial \mathbf{j}/\partial t|$. Then the right hand side of (4.18) can be neglected compared to $\partial \mathbf{j}/\partial t$. [If $\nu\tau \gg 1$, we justify the neglect of these terms by comparing them to $\nu\mathbf{j}$, i.e., by replacing τ by ν^{-1} in (4.19).] This gives the "Generalized Ohm's Law,"

$$\left(\mathbf{E}+\frac{1}{c}\mathbf{u}\times\mathbf{B}\right)=\eta\mathbf{j}+\left(\frac{m}{ne^2}\right)\frac{\partial\mathbf{j}}{\partial t}+\left(\frac{M}{e}\right)\frac{d\mathbf{u}}{dt}+\frac{\nabla p_i}{ne}, \tag{4.20}$$

where we have defined the resistivity η and its reciprocal, the electrical conductivity σ, as

$$\eta=\frac{1}{\sigma}=\frac{m\nu}{ne^2}=\frac{4\pi\nu}{\omega_p^2}. \tag{4.21}$$

In many cases of physical interest, we can neglect some of the terms on the right hand side of (4.20). Keeping only the first term gives the Simple Ohm's Law (4.5) with electrical conductivity, σ, given by (4.21). Thus the first term on the right side of (4.20) is associated with ohmic resistivity. The second arises from electron inertia; the third, from ion inertia; and the last from non-zero ion pressure. Neglecting all terms on the right gives what is called the "infinite conductivity" approximation

$$\left(\mathbf{E}+\frac{1}{c}\mathbf{u}\times\mathbf{B}\right)=0. \tag{4.22}$$

In this particularly simple limit, the one-fluid MHD equations become reasonably tractable and their consequences have been explored in considerable detail.

To summarize this section, the simplest form of MHD, sometimes termed "ideal MHD," is described by the equations

$$\frac{d\rho_m}{dt}+\rho_m\nabla\cdot\mathbf{u}=\frac{\partial\rho_m}{\partial t}+\nabla\cdot(\rho_m\mathbf{u})=0 \tag{4.23a}$$

$$\rho_m\left(\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)+\nabla p=\frac{1}{c}\mathbf{j}\times\mathbf{B} \tag{4.23b}$$

$$\frac{d}{dt}\left(p\rho_m^{-\gamma}\right)=0 \tag{4.23c}$$

$$\nabla\times\mathbf{B}=\frac{4\pi}{c}\mathbf{j}+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \tag{4.23d}$$

$$\frac{\partial\mathbf{B}}{\partial t}=-c\nabla\times\mathbf{E} \tag{4.23e}$$

$$\mathbf{E}+\frac{1}{c}\mathbf{u}\times\mathbf{B}=0 \tag{4.23f}$$

(In most applications, the displacement current can be dropped, implying closed current circuits $\nabla \cdot \mathbf{j} = 0$.) Of course, the assumption of infinite conductivity inevitably excludes important physical phenomena, so it is often necessary to consider "resistive MHD," meaning the equations (4.23) with the last equation replaced by

$$\mathbf{E} + \frac{1}{c}\mathbf{u} \times \mathbf{B} = \eta \mathbf{j}. \tag{4.24}$$

Still more accurate versions of MHD have also been studied, e.g., those obtained by using some of the additional terms in the generalized Ohm's law (4.20) or by allowing a pressure tensor p rather than the scalar pressure resulting from Maxwellian closure. While these more complicated versions of MHD can be useful, it is often easier to deal with the physics omitted from the ideal or resistive MHD by going back to the two fluid equations or to the Vlasov equation, at least for the ions, rather than patching up the MHD equations. In any case, it is useful to express the results in terms of connections to the simpler ideal or resistive MHD formulations.

At present, a large body of MHD literature, including many developments of considerable mathematical elegance, exists. As can be seen from the block diagram in Fig. 2.1, one-fluid MHD is quite far "down the line" of approximations, but it forms an indispensable guide to the physics of complex magnetic geometries, like those found in controlled fusion experiments (e.g., tokamaks, mirrors and pinches) or in astrophysical and geophysical problems (e.g., the solar wind, planetary magnetospheres and pulsars). In Section C we discuss a few simple consequences of the MHD equations and in Section D we explore some of the physics contained in these equations using the same device as in the previous chapter, i.e., the linearization about simple equilibrium solutions.

C. Elementary General Properties of the MHD Equations

For resistive MHD (i.e., using 4.24) we have from Faraday's Law (4.4c) and Ampere's Law (4.4d)

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \left(\frac{\mathbf{j}}{\sigma} - \frac{1}{c}\mathbf{u} \times \mathbf{B}\right) = \left(\frac{c^2}{4\pi\sigma}\right)\nabla^2\mathbf{B} + \nabla \times (\mathbf{u} \times \mathbf{B}). \tag{4.25}$$

In the absence of flow, this reduces to a classical diffusion equation with a diffusion coefficient

$$D = \frac{c^2}{4\pi\sigma}. \tag{4.26}$$

It follows that an initial magnetic field, and the currents which support it, will decay due to Ohmic dissipation on a time scale $\tau = L^2/D$, where L is a typical scale length.

For $\mathbf{u} \neq 0$, the relative magnitude of the two terms on the right side of (4.25) will be of order

$$R_m = \frac{4\pi\sigma u L}{c^2}, \tag{4.27}$$

which, by analogy with the Reynolds number encountered in viscous flow of a neutral gas, is called the magnetic Reynolds number. When $R_m \ll 1$, diffusion dominates. When $R_m \gg 1$, then (4.25) can be approximated by

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}), \tag{4.28}$$

the equation of "frozen-in magnetic flux." The reason for this appellation becomes clear when we consider an arbitrary closed curve, C, moving with the fluid and the rate of change of magnetic flux, ϕ, through any surface, S, bounded by C

$$\frac{d\phi}{dt} = \frac{d}{dt}\int_S \mathbf{B} \cdot d\vec{\sigma} = \int_S d\vec{\sigma} \cdot \left(\frac{\partial \mathbf{B}}{\partial t}\right) + \int_C \mathbf{B} \cdot (\mathbf{u} \times d\mathbf{s}), \tag{4.29}$$

where the last term takes into account the motion of C.[2] Stoke's law gives

$$\frac{d\phi}{dt} = \int_S d\vec{\sigma} \cdot \left[\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{u})\right], \tag{4.30}$$

which, according to Eq. (4.28), vanishes. The physical reason is clear: motion of the fluid induces electric fields, and the resultant currents, in the infinite conductivity limit, generate a magnetic field just sufficient to keep ϕ constant as C moves. If we represent the magnetic field by lines of magnetic flux, then the picture of magnetic field lines moving with ("frozen into") the fluid is certainly consistent with $d\phi/dt = 0$, and can be useful in providing a physical understanding of complex MHD phenomena.

As a simple example of the use of the frozen-in concept we give a heuristic discussion of Alfven waves in a magnetized plasma, a phenomenon we will examine more formally, and in more detail, in Section D. The Lorentz force term, $\mathbf{j} \times \mathbf{B}/c$, can, with neglect of the displacement current, be written as

$$\frac{1}{c}\mathbf{j} \times \mathbf{B} = \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} = \nabla \cdot \left(\frac{\mathbf{BB}}{4\pi} - \frac{B^2}{8\pi}\,\mathsf{I}\right), \tag{4.31}$$

i.e., as the divergence of the magnetic portion of the usual Maxwell stress tensor $\mathsf{S} = \mathbf{BB}/4\pi - (B^2/8\pi)\mathsf{I}$, corresponding to an isotropic magnetic pressure, $B^2/8\pi$, and a tension along the lines, of magnitude $B^2/4\pi$ per unit area. If the field lines are frozen to the fluid, and vice versa, then a flux tube of area A will experience a tension force $T = AB^2/4\pi$ and the plasma "frozen" to it will endow it with a mass per-unit-length $\rho_m A$. Thus, we might expect the field lines to behave like a stretched string or wire, in that a perturbation transverse to $\mathbf{B}_0$ would propagate along the line with phase velociy

$$c_A = \sqrt{\frac{T}{\rho_m A}} = \sqrt{\frac{B^2}{4\pi\rho_m}}. \tag{4.32}$$

This expectation is confirmed, by both experiments and by a systematic theoretical analysis (given in Section D.). The velocity c_A is the Alfven velocity and the waves are known as Alfven waves, after Hannes Alfven who first predicted them.

[2] This is known as the Liebniz integral rule. —ed.

Another general an important property of the ideal MHD equations (4.23) is the energy conservation law. To derive it, we take the dot product of $\mathbf{u}$ with the momentum equation and show that each term can be expressed as the sum of a time derivative and a divergence. For the first term we have

$$\mathbf{u} \cdot \rho_m \frac{d\mathbf{u}}{dt} = \frac{\rho_m}{2} \frac{\partial u^2}{\partial t} + \rho_m \mathbf{u} \cdot \left(\frac{\nabla u^2}{2} \right) = \frac{\partial}{\partial t} \left(\frac{\rho_m u^2}{2} \right) + \nabla \cdot \left(\frac{\rho_m \mathbf{u} u^2}{2} \right), \tag{4.33}$$

where we have made use of the continuity equation; the second term gives

$$\begin{aligned} \mathbf{u} \cdot \nabla p = \frac{dp}{dt} - \frac{\partial p}{\partial t} &= \left(\frac{\gamma p}{\rho_m} \right) \frac{d\rho_m}{dt} - \frac{\partial p}{\partial t} \\ &= -\gamma p \nabla \cdot \mathbf{u} - \frac{\partial p}{\partial t} \\ &= -\gamma \nabla \cdot (p\mathbf{u}) + \gamma \mathbf{u} \cdot \nabla p - \frac{\partial p}{\partial t} \\ &= \frac{1}{\gamma - 1} \left(\frac{\partial p}{\partial t} + \gamma \nabla \cdot (p\mathbf{u}) \right), \end{aligned} \tag{4.34}$$

where we have again used the energy equation and the continuity equation (and, in the last step, collected the $\mathbf{u} \cdot \nabla p$ terms on one side of the equation and solved for $\mathbf{u} \cdot \nabla p$); and from the third term we obtain

$$\mathbf{u} \cdot \frac{1}{c} \mathbf{j} \times \mathbf{B} = -\frac{1}{c} \mathbf{j} \cdot \mathbf{u} \times \mathbf{B} \tag{4.35}$$

$$= \left(\frac{c}{4\pi} \right) \nabla \times \mathbf{B} \cdot \mathbf{E} \tag{4.36}$$

It follows immediately that

$$\mathbf{u} \cdot \left(\rho_m \frac{d\mathbf{u}}{dt} + \nabla p - \frac{1}{c} \mathbf{j} \times \mathbf{B} \right) = \frac{\partial U}{\partial t} + \nabla \cdot \mathsf{S} = 0, \tag{4.37}$$

where

$$U = \frac{\rho_m u^2}{2} + \frac{p}{\gamma - 1} + \frac{B^2}{8\pi} \tag{4.38}$$

is the energy density (kinetic plus compressional plus magnetic) and

$$\mathsf{S} = \frac{\rho_m u^2}{2} \mathbf{u} + \frac{p \gamma \mathbf{u}}{\gamma - 1} + \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \tag{4.39}$$

is the energy flux, the last term being just the Poynting flux. For a volume V enclosed by a surface on which S vanishes (e.g., because the surface recedes to infinity), we have

$$\frac{d}{dt} \int_V U \, d\tau = 0. \tag{4.40}$$

D. Linearized Ideal MHD

The simplest equilibrium (i.e., time-independent) solution of the MHD equations (4.23) is that corresponding to a uniform stationary plasma ($n = n_0 =$ constant, $p = p_0 =$ constant, $\mathbf{u} = 0$) in a uniform magnetic field ($\mathbf{B} = \mathbf{B}_0 =$ constant). We shall use the technique of simple-minded plane wave substitution (cf. Chapter 3, Section D) to find the dispersion equation for the resulting system of equations; from our discussion in Chapter 3, it is clear the results may be extended to more physically posed problems with external sources or specified initial values. Thus, we set

$$n = n_0 + n_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{4.41a}$$

$$\mathbf{u} = \mathbf{u}_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{4.41b}$$

Suppressing the subscript 1 where it is not needed, i.e., on $\mathbf{u}_1$ and $\mathbf{E}_1$ (since $\mathbf{u}_0 = \mathbf{E}_0 = 0$) we obtain from (4.23)

$$-\omega n_1 + n_0 \mathbf{k}\cdot\mathbf{u} = 0 \tag{4.42a}$$

$$-n_0 M\omega\mathbf{u} + Mc_s^2 n_1 \mathbf{k} = \frac{-i\mathbf{j}\times\mathbf{B}_0}{c} \tag{4.42b}$$

$$c_s^2 = \frac{\gamma p_0}{n_0 M} \tag{4.42c}$$

$$\mathbf{k}\times\mathbf{B}_1 = \frac{-4\pi i}{c}\mathbf{j} - \frac{\omega\mathbf{E}}{c} \tag{4.42d}$$

$$\mathbf{k}\times\mathbf{E} = \frac{\omega\mathbf{B}_1}{c} \tag{4.42e}$$

$$\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{B}_0 = 0 \tag{4.42f}$$

Before examining the general solutions of this set of equations, we consider the simplest case: waves propagating along the magnetic field $\mathbf{B}_0$ in the ideal MHD limit and with the displacement current neglected. For the transverse component of $\mathbf{u}$, i.e., the one perpendicular to $\mathbf{k}$ and $\mathbf{B}_0$, we have from the momentum equation (4.42b)

$$\mathbf{u}_t = i\frac{\mathbf{j}\times\mathbf{B}_0}{\rho_0\omega c} \tag{4.43}$$

where $\rho_0 = n_0 M$. Using Ampere's law (4.42d) and Faraday's law (4.42e) we have

$$\mathbf{u}_t = -\frac{kB_0 c}{4\pi\rho_0\omega^2}\,\mathbf{k}\times\mathbf{E}, \tag{4.44}$$

and substituting for $\mathbf{E}$ from (4.42f) gives

$$\mathbf{u}_t\left[1 - \left(\frac{kc_A}{\omega}\right)^2\right] = 0 \tag{4.45}$$

where

$$c_A = \sqrt{\frac{B_0^2}{4\pi\rho_0}} \tag{4.46}$$

is the Alfven speed defined earlier. Thus, waves with $\mathbf{u}_t \neq 0$ are possible only if $(\omega/k)^2 = c_A^2$ and they are transverse ($\mathbf{E}$, $\mathbf{B}$, and $\mathbf{u}$ are all normal to $\mathbf{B}_0$). A physical picture of these Alfven waves in terms of stretched field lines with plasma "frozen" to them was given in Section C..

We now return to the full set of linearized MHD equations (4.42). It is easy to see that when $\omega/kc \ll 1$ (phase velocity small compared to c) we can neglect the displacement current in (4.42d):

$$-4\pi i\mathbf{j} = \frac{\mathbf{k}\times(\mathbf{k}\times\mathbf{E})c^2}{\omega} + \omega\mathbf{E} = \frac{(\omega^2 - k^2c^2)\mathbf{E}}{\omega} \approx \frac{-k^2c^2\mathbf{E}}{\omega}. \tag{4.47}$$

We can then substitute (4.42a) and (4.42d) into (4.42b), obtaining an equation involving $\mathbf{u}$ and $\mathbf{B}_1$

$$\omega^2\mathbf{u} - c_s^2(\mathbf{k}\cdot\mathbf{u})\,\mathbf{k} = \frac{i\omega\mathbf{j}\times\mathbf{B}_0}{\rho_0 c} = \frac{-\omega(\mathbf{k}\times\mathbf{B}_1)\times\mathbf{B}_0}{4\pi\rho_0}. \tag{4.48}$$

Taking the scalar product of this with $\mathbf{k}$ gives

$$\mathbf{k}\cdot\mathbf{u} = \left[\frac{k^2\omega(\omega^2 - k^2c^2)}{4\pi\rho_0}\right]\ \mathbf{B}_1\cdot\mathbf{B}_0 \tag{4.49}$$

and hence an explicit expression for $\mathbf{u}$ in terms of $\mathbf{B}_1$

$$\mathbf{u} = \frac{\omega^2(\mathbf{B}_0\cdot\mathbf{B}_1)\,\mathbf{k} - (\omega^2 - k^2c_s^2)(\mathbf{k}\cdot\mathbf{B}_0)\,\mathbf{B}_1}{4\pi\rho_0\omega(\omega^2 - k^2c_s^2)}. \tag{4.50}$$

An expression for $\mathbf{B}_1$ in terms of $\mathbf{u}$ follows immediately from (4.42e) and (4.42f)

$$\mathbf{B}_1 = \frac{-\mathbf{k}\times(\mathbf{u}\times\mathbf{B}_0)}{\omega}. \tag{4.51}$$

Finally, substituting (4.50) into (4.51) gives

$$\mathbf{B}_1\left[1 - \left(\frac{kc_A\cos\theta}{\omega}\right)^2\right] = \hat{\mathbf{b}}\cdot\mathbf{B}_1\frac{(\hat{\mathbf{b}} - \hat{\mathbf{k}}\cos\theta)k^2c_A^2}{(\omega^2 - k^2c_s^2)} \tag{4.52}$$

where we have set

$$\hat{\mathbf{k}} = \frac{\mathbf{k}}{k} \qquad \hat{\mathbf{b}} = \frac{\mathbf{B}_0}{B_0} \qquad \cos\theta = \hat{\mathbf{k}}\cdot\hat{\mathbf{b}} \tag{4.53}$$

Introducing the phase velocity of the waves, $V = \omega/k$, we have

$$\mathbf{B}_1\left[1 - \left(\frac{c_A\cos\theta}{V}\right)^2\right] = (\hat{\mathbf{b}}\cdot\mathbf{B}_1)c_A^2\frac{(\hat{\mathbf{b}} - \hat{\mathbf{k}}\cos\theta)}{(V^2 - c_s^2)}. \tag{4.54}$$

1. Shear Alfven Wave

One solution of (4.54) is obtained when $\mathbf{B}_1 \cdot \hat{\mathbf{b}} = 0$ (i.e., the perturbed magnetic field $\mathbf{B}_1$ is perpendicular to $\mathbf{B}_0$) and

$$V^2 = c_A^2 \cos^2\theta. \tag{4.55}$$

It follows from (4.50) and (4.51) that $\hat{\mathbf{b}} \cdot \mathbf{B}_1 = 0$ implies $\mathbf{k} \cdot \mathbf{u} = 0$ and hence a fluid motion that is incompressible ($n_1 = 0$) but has finite shear [e.g., $u_y(x)$ or $u_z(x)$ if $\mathbf{k} = k\hat{\mathbf{x}}$]. This solution is therefore called the **shear Alfven wave**. The dispersion relation (4.55) can be written

$$\omega^2 = k_\parallel^2 c_A^2 \tag{4.56}$$

showing that the frequency is independent of

$$\mathbf{k}_\perp = \hat{\mathbf{b}} \times (\mathbf{k} \times \hat{\mathbf{b}}). \tag{4.57}$$

This means that any superposition of $\mathbf{k}_\perp$ with given k_z will have the same frequency. In particular, the wave can be localized to a single field line or flux tube, which can be thought of as oscillating independently of any others. We also see that the group velocity is always equal to c_A and directed along $\mathbf{B}_0$

$$\mathbf{v}_g = \frac{\partial \omega}{\partial \mathbf{k}} = \frac{\partial(\mathbf{k} \cdot \hat{\mathbf{b}}\, c_A)}{\partial \mathbf{k}} = c_A \hat{\mathbf{b}}. \tag{4.58}$$

2. Fast and Slow Alfven Waves

If $\mathbf{B}_1 \cdot \hat{\mathbf{b}} \neq 0$, then taking the dot product of (4.54) with $V^2\hat{\mathbf{b}}$ gives

$$(\mathbf{B}_1 \cdot \hat{\mathbf{b}})F(V) \equiv \mathbf{B}_1 \cdot \hat{\mathbf{b}} \left[(V^2 - c_A^2\cos^2\theta)(V^2 - c_s^2) - V^2 c_A^2 \sin^2\theta\right] = 0 \tag{4.59}$$

and hence the dispersion relation $F(V) = 0$ whose solutions are

$$V^2 = \frac{1}{2}\left\{c_s^2 + c_A^2 \pm \sqrt{(c_s^2 + c_A^2)^2 - 4c_s^2 c_A^2 \cos^2\theta}\right\}. \tag{4.60}$$

These solutions are called the **fast wave** (+ sign) and the **slow wave** (− sign) and a discussion of their properties is conveniently divided into two cases:

a) low β; $c_s \leq c_A$

We note that

$$\frac{c_s^2}{c_A^2} = \frac{4\pi n\gamma(T_e + T_i)}{B^2} = \frac{\gamma}{2}(\beta_e + \beta_i), \tag{4.61}$$

where the quantity

$$\beta = \frac{nT}{B^2/8\pi}, \tag{4.62}$$

the ratio of kinetic to magnetic pressure for each species, is one of the basic MHD parameters. Thus, this case is referred to as "low β." For $\theta = 0$, (4.60) gives

$$V^2 = \frac{1}{2}\left\{c_s^2 + c_A^2 \pm (c_s^2 - c_A^2)\right\} \tag{4.63}$$

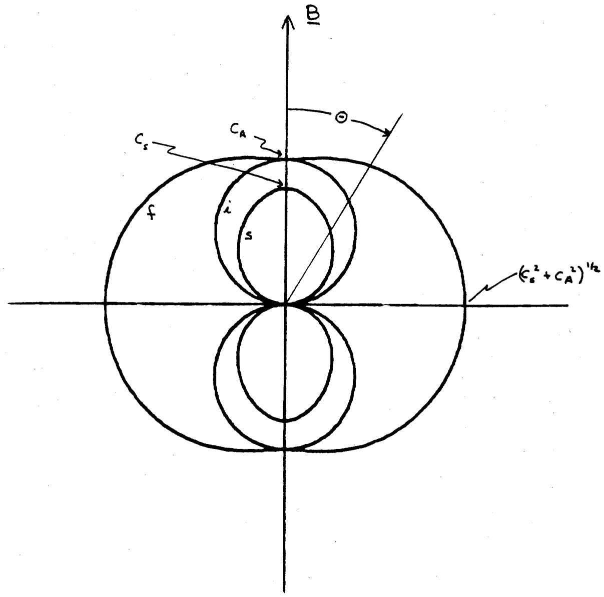

Figure 4.1: Polar plot of V versus θ for slow, intermediate, and fast MHD waves at low β. Here, $c_s/c_A = 0.8$.

so the fast wave has velocity c_A, and the slow wave has velocity c_s. For $\theta = 90°$, the fast wave has velocity

$$V^2 = c_s^2 + c_A^2. \tag{4.64}$$

This speed is called the magnetosonic speed, since the restoring forces are partially magnetic pressure, $B^2/8\pi$, and partially acoustic pressure (nT). The slow wave has $V = 0$. For other values of θ, the solutions (4.60) are best represented graphically by a polar plot of V versus θ as shown in Fig. 4.1. Since

$$\left(c_s^2 + c_A^2\right)^2 - 4c_s^2 c_A^2 \cos^2\theta \geq \left(c_s^2 - c_A^2\right)^2 \tag{4.65}$$

it follows from (4.60) that, for $c_s^2 < c_A^2$,

$$V_s \leq c_s < c_A \leq V_f \tag{4.66}$$

a property obvious from Fig. 4.1. We have also shown in Fig. 4.1 the shear wave solution (4.55); it is sometimes called the **intermediate wave** because, denoting the speeds of the three solutions by V_s, V_i, and V_f, we have the relation

$$V_s < V_i < V_f, \tag{4.67}$$

a relation valid for all θ. [To prove this, we substitute $V = V_i = c_A \cos\theta$ into the function $F(V)$ defined in (4.59)

$$F(V_i) = V_i^4 - (c_s^2 + c_A^2)V_i^2 + c_s^2 c_A^2 \cos^2\theta = (V_i^2 - V_s^2)(V_i^2 - V_f^2). \tag{4.68}$$

Since $F(V_i) = -c_a^4 \cos^2\theta \sin^2\theta \leq 0$, it follows that $(V_i - V_s)(V_i - V_f) \leq 0$, which proves (4.67). (The other possibility, $V_f < V_i < V_s$, is ruled out since (4.66) requires $V_s \leq V_f$.)].

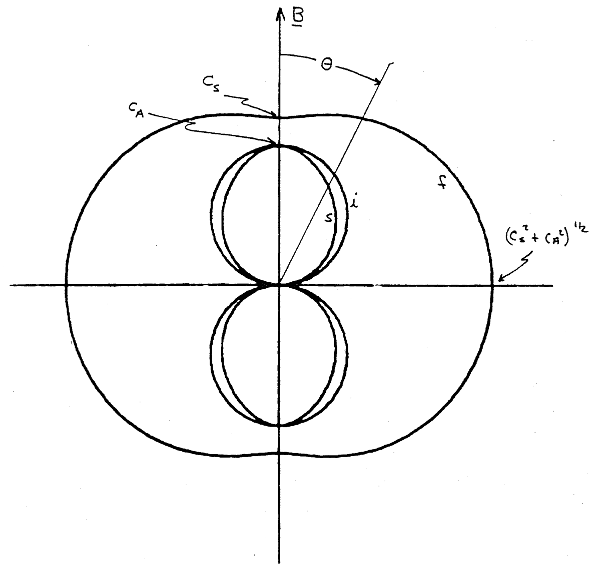

Figure 4.2: Polar plot of V versus θ for slow, intermediate, and fast MHD waves at high β. Here, $c_s/c_A = 1.2$.

b) high β; $c_A \leq c_s$

The polar plot for this case is shown in Fig. 4.2. Since (4.60) is symmetric under the interchange $c_s \leftrightarrow c_A$, arguments like those of the preceding paragraph show that in this case, $V_s \leq c_A < c_s \leq V_f$, and that (4.67) is still valid.

For the intermediate or shear wave, it follows from (4.50) that $\mathbf{u}$ is parallel to $\mathbf{B}_1$ and both are perpendicular to the plane spanned by $\mathbf{k}$ and $\hat{\mathbf{b}}$. If $\mathbf{k}$ is not parallel to $\hat{\mathbf{b}}$, then $\mathbf{E} = -(\mathbf{u} \times \mathbf{B}_0)/c$ will have components both parallel and perpendicular to $\mathbf{k}$, resulting in **mixed** polarization, neither electrostatic ($\mathbf{k} \times \mathbf{E} = 0$) nor purely electromagnetic ($\mathbf{k} \cdot \mathbf{E} = 0$). For the fast and slow waves, it follows from (4.50) and (4.51) that $\mathbf{u}$ lies in the plane of $\mathbf{k}$ and $\hat{\mathbf{b}}$. Then $\mathbf{E}$ is parallel to $\mathbf{k} \times \hat{\mathbf{b}}$ so $\mathbf{k} \cdot \mathbf{E} = 0$ and the waves are purely electromagnetic. In contrast to the shear wave they are compressional, $n_1 \neq 0$, since $\mathbf{k} \cdot \mathbf{u} \neq 0$. In the cold plasma limit, $c_s \to 0$, there are only two waves, the fast (compressible) wave and the intermediate (shear) wave. In this situation, the latter is very often referred to as the "slow" wave but it would be preferable to use the terms intermediate or shear wave to avoid confusion with the slow wave which is present when $c_s \neq 0$.

It is to be emphasized that ideal MHD gives correctly only the very low frequency, long wavelength behavior of the plasma. Many important wave phenomena are totally absent from this approximation and it is only when we consider the two-fluid description of waves in a magnetized plasma (Chapter 8) that we get a satisfactory perspective.