Fluids: A brief introduction

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1 The convective derivative

Following a fluid element as it moves, the total time derivative of a property of that element, say A, is called the CONVECTIVE DERIVATIVE (also known as the *material derivative* or the *substantial derivative*), and is written

$$\frac{DA}{Dt} \equiv \frac{\partial A}{\partial t} + (\vec{v} \cdot \nabla)A, \tag{1}$$

where $\vec{v}$ is the velocity of the fluid element. See Fig. 1. In a given flow field, however, it is customary to adopt the Eulerian specification where quantities such as A are described as a field, i.e., a function of space and time, $A(\vec{r}, t)$. This means that any property of a fluid element changes in time for two reasons: first because the field A is an explicit function of time, but second because the element is moving in space.

The definition of the convective derivative holds for vector fields as well, and one of the most important vector fields is the velocity itself, $\vec{v}(\vec{r}, t)$. It is important because $D\vec{v}/Dt$ is the acceleration of a fluid element and is the quantity needed in expressing Newton's second law for the fluid element. (This is the Lagrangian specification, where each fluid element is followed, and the convective derivative is needed to convert between the two specifications.) In this case, we have

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}. \tag{2}$$

This relation holds for any field, vector or scalar.

Example

Consider a point source of fluid with a constant rate of mass entry, $Q = dm/dt =$ constant, a simple model for the solar wind. A distance r from the point source, conservation of mass requires

$$Q = (4\pi r^2)\rho v_r, \tag{3}$$

where the quantity ρv_r is the "mass flux," i.e., mass per-unit-area per-unit-time, and $\vec{v} = v_r \hat{r}$ is the time-independent VELOCITY FIELD

$$v_r(r) = \frac{Q}{4\pi \rho r^2} \tag{4}$$

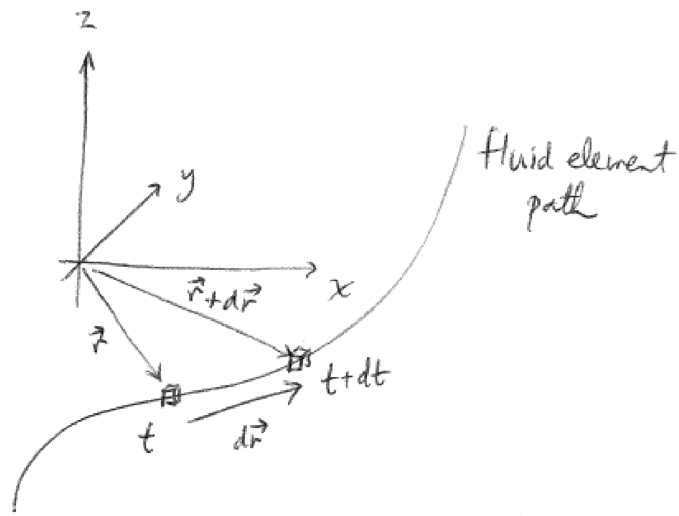

Figure 1: Motion of a fluid element.

which in this example is spherically symmetric and is only a function of r. With this spherical symmetry, the convective derivative of $\vec{v}$ is

$$\frac{D\vec{v}}{Dt} = v_r \left(\frac{\partial v_r}{\partial r} \right) \hat{r} = - \left(\frac{Q}{4\pi\rho} \right)^2 \frac{2}{r^5} \hat{r} \tag{5}$$

where I have assumed that the density ρ is constant. This is the acceleration of a fluid element as a function of position, but we can obtain the acceleration as a function of time in the following manner. Since v_r is only a function of r, we can use separation of variables to solve the differential equation $v_r = dr/dt$

$$\int_{t_0}^{t} dt' = \int_{r_0}^{r} \frac{4\pi\rho}{Q} r'^2 dr' \tag{6}$$

and obtain

$$r(t) = \left[r_0^3 + \frac{3Q}{4\pi\rho} (t - t_0) \right]^{1/3} . \tag{7}$$

Taking a time derivative dr/dt gives the radial velocity as a function of time for a fluid element, and a second time derivative gives the acceleration

$$a_r(t) = \frac{d^2 r}{dt^2} = -\frac{2}{9} \left(\frac{3Q}{4\pi\rho} \right)^2 \left[r_0^3 + \frac{3Q}{4\pi\rho} (t - t_0) \right]^{-5/3} , \tag{8}$$

where the density ρ is still constant. You can obtain the same answer by inserting the solution for r sinto the convective derivative of $\vec{v}$ (see Problem 1).

Note that if $r_0 = 0$, i.e., the element starts at the origin, then as $t \to t_0$ the acceleration becomes singular. This, of course, is a limitation of our simplified model, and the fact that $r_0 \to 0$ is not realistic.

Problems

1. For a point source of mass with constant Q and a radially symmetric constant ρ, we obtained
$$\frac{D\vec{v}}{Dt} = a_r\hat{r} = -\left(\frac{Q}{4\pi\rho}\right)^2 \frac{2}{r^5}\,\hat{r}. \tag{9}$$
Obtain an expression for $a_r(t)$ by inserting $r(t)$ into the convective derivative and compare it to the result obtained by direct differentiation, d^2r/dt^2.

2. Perform the same analysis as in the Example and in Problem 1, but for the case where the density depends on the radial position as a power law, i.e., $\rho(r) = A/r^m$, where A is a constant (Q is still constant). This is more realistic as it allows for the density to decrease as the fluid element moves away from the source. That is, (a) solve for $v_r(r)$ and then $r(t)$ by integrating $dt = dr/v_r$, and (b) obtain an expression for a_r in two different ways, (1) by differentiating twice *and* (2) by using the convective derivative. Compare your two answers.

2 The Euler equation

If we apply Newton's second law $\vec{F} = d\vec{p}/dt$ to a fluid element of volume V_0, we obtain

$$-(\nabla P)V_0 = \frac{D}{Dt}\left(\rho V_0 \vec{v}\right), \tag{10}$$

where $P(\vec{r}, t)$ is the pressure field. That the left-hand-side is the force can be seen from a one-dimensional treatment: If the fluid element is a small cube of extent dx with a cross-sectional area A, then the net force in the $\hat{x}$ direction due to the pressure difference is $[-(P + dP) + P]A = -(dP/dx)V_0$, where $V_0 = Adx$. Similar treatment of the other dimensions results in the gradient operator. Since the fluid element has constant mass, the quantity ρV_0 is constant, and we are left with the Euler equation

$$\rho\left(\frac{\partial}{\partial t}\vec{v} + \vec{v}\cdot\nabla\vec{v}\right) = -\nabla P + \vec{F}_{ext} \tag{11}$$

where $\vec{F}_{ext}$ are "external" forces (per-unit-volume), which includes not only gravity and fictitious forces like the Coriolis force, but also "internal" forces like viscosity and surface tension.

Example

Again consider our point source Q with constant density. We can calculate the pressure P as a function of radial position r from Eq. (11), whose radial component in spherical coordinates becomes

$$-\frac{dP}{dr} = \rho\frac{D\vec{v}}{Dt} = -\left(\frac{Q}{4\pi}\right)^2 \frac{2}{\rho r^5}. \tag{12}$$

Integrating from ∞ to r gives

$$P(r) = P_\infty - \left(\frac{Q}{4\pi}\right)^2 \frac{1}{2\rho r^4}, \tag{13}$$

where $P_\infty = P(r \to \infty)$. Notice that P_∞ must be large so that P remains finite near the origin. This pressure must be high in order to slow down the fluid as it convects outward. "Note that seemingly innocuous assumptions (constant density and constant mass rate) have generated a completely unphysical situation...This is how many problems in fluid dynamics challenge the intuition; the physical implications of many assumptions are often not clear until an actual calculation is done."[1]

If the density varies inversely with r^2 (see Problem 3), then you can show that both the pressure and velocity are independent of r! This still is not a good model for the solar wind, but at least the pressure is not increasing with r. However, if the fluid is a perfect gas, the temperature must be increasing with r, and we would be left with including a heating mechanism in our model.

Problems

3. Solve for the pressure as a function of radius if the density is a power law, as in Problem 2.

4. Show that the Euler equation (with no external forces) can be written in flux conservative form

$$\frac{\partial}{\partial t}(\rho\vec{v}) + \nabla \cdot (\rho\vec{v}\vec{v} + P\mathsf{I}) = 0, \tag{14}$$

where I is the unit tensor and

$$P\mathsf{I} = \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix} \tag{15}$$

is the pressure tensor for a symmetric, isotropic medium.

5. Find an expression for the pressure as a function of position in a fluid of density ρ at rest in a constant gravitational field. That is, let $\vec{F}_{ext} = \rho\vec{g} = -\rho g\hat{z}$, which is the force (per-unit-volume) exerted on a fluid element.

[1] Stanley M. Flatté, *Fluid Dynamics for Natural Scientists*, 1987.

3 The continuity equation

The conservation of mass is expressed via the CONTINUITY EQUATION

$$\frac{\partial}{\partial t}(\rho) + \nabla \cdot (\rho \vec{v}) = 0. \tag{16}$$

Integrating over a volume V_0 and invoking the divergence theorem gives

$$\frac{dm}{dt} = \frac{d}{dt}\int_{V_0} \rho \, dV = -\int_{A_0} (\rho \vec{v}) \cdot d\hat{A}, \tag{17}$$

where A_0 is the (closed) area that encloses the volume V_0, and m is the mass of the fluid in that volume. Equation (17) expresses the fact that any change in mass within V_0 must be compensated for by a flux of mass out of V_0 through the surface area. Therefore, this continuity equation expresses the principle of the conservation of mass.

There are other, useful, ways to write the continuity equation, Eq. (16). Expanding the second term using the product rule,[2]

$$\nabla \cdot (\rho \vec{v}) = \rho \nabla \cdot \vec{v} + \vec{v} \cdot \nabla \rho, \tag{18}$$

and grouping two terms to form the convective derivative, results in

$$\frac{D\rho}{Dt} + (\nabla \cdot \vec{v})\,\rho = 0. \tag{19}$$

If the flow in incompressible, that is, the density of each fluid element remains constant as it flows, then $D\rho/Dt = 0$. Under these conditions, Eq. (19) says that $\nabla \cdot \vec{v} = 0$. Therefore, a velocity field that has zero divergence means that the flow is incompressible.

Example

We have already used the concept of mass conservation in Eq. (3). When the flow field is time stationary and spherically symmetric, i.e., where there is no dependence on t, θ or ϕ, and where the velocity has only a radial component, $\vec{v} = v_r \hat{r}$, Eq. (16) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho v_r\right) = 0, \tag{20}$$

or

$$r^2 \rho v_r = C, \tag{21}$$

where C is a constant (i.e., not a function of r). A comparison with Eq. (3) shows that $C = Q/4\pi$.

[2] In index notation, this reads

$$\frac{\partial}{\partial x_i}(\rho v_i) = \rho \frac{\partial v_i}{\partial x_i} + v_i \frac{\partial \rho}{\partial x_i}.$$

4 Stellar models

We now apply these fluid equations to stars and obtain a simple model of a self-gravitating, spherically symmetric, hot gas. At this point, we won't explicitly include the energy source (fusion) that counterbalances gravity and keeps the star from collapsing, but we will include it indirectly by assuming a temperature profile $T(r)$ that, if maintained, will allow the star to be in static equilibrium.

Static equilibrium

If we assume that our star is spherically symmetric and in static equilibrium, i.e., there is no angular variation $\partial/\partial\phi = \partial/\partial\theta = 0$ nor time variation $\partial/\partial t = 0$, and nothing is moving $\vec{v} = 0$, then the Euler equation is — in spherical coordinates —

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}, \tag{22}$$

where the right-hand-side is the gravitational force per-unit-volume exerted on a fluid element a distance r from the center. According to Newton's shell theorems, M_r is the mass interior to the radius r,

$$M_r \equiv \int_0^r 4\pi r'^2 \rho(r')\, dr' \tag{23}$$

which can be written in differential form

$$\frac{dM_r}{dr} = 4\pi r^2 \rho(r). \tag{24}$$

We now have two (differential) equations for three functions of position, ρ, P, and M_r, so we need a third, constitutive relationship, and the simplest is the so-called POLYTROPIC RELATION

$$P = K\rho^\alpha, \tag{25}$$

where α is called the polytropic exponent. The polytropic exponent can be considered a generalization of the adiabatic exponent.[3] For historical reasons, the exponent α is sometimes

[3]Recall that for an adiabatic process, a fluid system follows a thermodynamic path in the P-V plane that satisfies

$$PV^\gamma = \text{constant},$$

where $\gamma = C_P/C_V$ is the ratio of heat capacities. The adiabatic condition can be expressed in terms of the density ρ in the following way

$$P \propto \frac{1}{V^\gamma} \propto \rho^\gamma,$$

which has the same form as Eq. (25). A polytropic process (or a polytropic change of state) is one in which the heat capacity C remains constant. It does not have to be at constant pressure nor constant volume. You can show (see Problem 8) that the polytropic index can be expressed as

$$\alpha = \frac{C_P - C}{C_V - C},$$

so that $\alpha = \gamma$ if the heat capacity is zero, which is exactly the case for an adiabatic process.

expressed as $\alpha = 1 + \frac{1}{n}$, and n is called the polytropic index. The reason is that for ideal gases the polytropic relation is equivalent to $\rho \propto T^n$, and the ρ-T plane played a major role in the early development of thermodynamics.[4] The justification for assuming the polytropic relation is elegantly stated by Arthur Stanley Eddington, England's premier astrophysicist in the early 20th century, who stated

> In general, whether the gas is perfect or imperfect, any value of the pressure can be made to correspond to a given density by assigning an appropriate temperature; our procedure thus amounts to imposing a particular temperature distribution on the star. This will only correspond to possible actual conditions if the temperature distribution is such that it can maintain itself automatically.[5]

Our three equations that we must solve for the three variables are therefore Eqs. (22), (24), and (25). These can be expressed more elegantly by making them dimensionless. That is, making the following change of variables

$$M_r = qM \qquad r = xR \qquad \rho = \rho_0 z^n \tag{26}$$

where $M = \int_0^R \rho dV$ is the total mass of the star, R is the radius of the star, z is a dimensionless temperature (see Footnote 4), and ρ_0 is some density scale, as yet undetermined. The pressure can be expressed as

$$P = K\rho^\alpha = K\rho^{1+\frac{1}{n}} = K\left(\rho_0 z^n\right)^{1+\frac{1}{n}} = \left(K\rho_0^{1+\frac{1}{n}}\right) z^{n+1}, \tag{27}$$

and we can define $P_0 \equiv K\rho_0^{1+\frac{1}{n}}$ as the central pressure. Using these new variable definitions, our two differential equations become

$$\frac{dz}{dx} = -A\frac{q}{x^2} \qquad \frac{dq}{dx} = Bx^2 z^n, \tag{28}$$

[4] The ideal gas law, $PV = nRT$ can be written

$$P = \left(\frac{R}{M}\right)\rho T,$$

where M is the molar mass. Combining this with Eq. (25) gives $\rho \propto T^n$.

[5] A. S. Eddington, *The Internal Constitution of the Stars*, Dover 1959; originally published 1926, Cambridge University Press. In practice, the temperature distribution is determined by two more differential equations. First, the "luminosity" that flows through a sphere of radius r, L_r

$$\frac{dL_r}{dr} = \varepsilon 4\pi r^2 \rho,$$

where ε is the energy generated per-unit-mass per-unit-time, or the power generated per-unit-mass at radius r. The physical mechanism of energy generation must be determined, for example nuclear fusion, before $\varepsilon(r)$ can be chosen. The second equation must determine the temperature gradient, or must express the fact that the luminosity — because it is energy flow — must be consistent with the temperature gradient. For the case where radiation (photons) carries the energy, this becomes

$$L_r = -4\pi r^2 \frac{4acT^3}{3\kappa\rho}\frac{dT}{dr},$$

where a is the Stefan-Boltzmann constant and κ is the absorption coefficient. Note that energy flows in a direction *opposite* to the temperature gradient.

where

$$A = \frac{GM}{KR\rho_0^{1/n}(n+1)} \qquad\qquad B = \frac{4\pi\rho_0 R^3}{M}. \tag{29}$$

We are free to choose the two unknowns K and ρ_0, and a logical choice is to let both $A = 1$ and $B = 1$. We can now eliminate q from these two first-order ODEs which results in one second-order ODE, known as the Lane-Emden equation for stellar equilibrium

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dz}{dx}\right) = -z^n. \tag{30}$$

Lane-Emden functions

In general, the solutions to the Lane-Emden equation of index n are denoted z_n, and are known as the "Lane-Emden functions of index n." There are three values of n ($n = 0, 1, 5$) for which solutions are known in closed form. They are

$$z_0 = 1 - \frac{x^2}{6} \qquad\qquad z_1 = \frac{\sin x}{x} \qquad\qquad z_5 = \left(1 + \frac{x^2}{3}\right)^{-1/2}$$

Normally, the solution to a second-order ordinary differential equation depends on two arbitrary constants of integration. However, in this case, one of the constants must be set to zero in order to obtain physically meaningful solutions (i.e., z must be finite at the origin), and the other constant is determined by setting the boundary condition $z = 1$ at $x = 0$. This is equivalent to defining $\rho_0 \equiv \rho(r = 0)$ as the central density.

Case 1: $n = 0$

For this solution, the (scaled) temperature is $z = 1 - x^2/6$, and decreases from $z = 1$ at the star's center to $z = 5/6$ at the star's radius $x = 1$. The density turns out to be constant

$$\rho = \rho_0 z^0 = \rho_0, \tag{31}$$

and the (scaled) mass function q can be found from Eq. (28)

$$\frac{dq}{dx} = x^2 z^0 = x^2, \tag{32}$$

and integration gives $q = x^3/3$, which is zero at the center — as it must be — and you can check that after integration this gives the correct value for the total mass of the star.

Problems

6. Starting with the equations for hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho\frac{GM_r}{r^2} \qquad\qquad \frac{dM_r}{dr} = 4\pi r^2\rho$$

and the polytropic relation $P = K\rho^{1+\frac{1}{n}}$, Derive the Lane-Emden equation

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dz}{dx}\right) = -z^n, \tag{33}$$

where $M_r = qM$, $r = xR$, and $\rho = \rho_0 z^n$. NOTE: You will need to set the following sets of constants to unity

$$\frac{4\pi\rho_0 R^3}{M} = 1 \qquad\qquad \frac{GM}{RK\rho_0^{1/n}(n+1)} = 1.$$

These two equations simply determine the values of the two arbitrary constants ρ_0 and K.

7. Prove, by direct substitution, that the three Lane-Emden functions given above (z_0, z_1, and z_5) solve the Lane-Emden equation with the appropriate polytropic index n.

8. Show that the polytropic exponent can be written

$$\alpha = \frac{C_P - C}{C_V - C}.$$

Use the fact that the first law of thermodynamics, $dU = đQ - PdV$, can be expressed as $C_V dT = CdT - PdV$ because the change in internal energy is always $dU = C_V dT$ and the heat energy absorbed is, for a general process, $đQ = CdT$.