MECHANICS

THIRD EDITION

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What part of the field comes from the immediately adjacent matter and what part from more distant matter?

20. Show that the gravitational field equations (6.21), (6.31), and (6.33) are satisfied by the field intensity and potential which you calculated in Problem 5.

***21.** (a) Show that $\delta\mathcal{G}$ found in Problem 17(b) satisfies Laplace's equation (6.35). This, together with the fact that $\delta\mathcal{G}$ has the same angular dependence as the mass density which produces it, suggests that the formula given for $\delta\mathcal{G}$ may actually be valid everywhere outside the earth.

b) To show this, consider Poisson's equation (6.33) with $\rho = f(r)\,(1-3\cos^2\theta)$. Show that a solution $\mathcal{G} = h(r)\,(1-3\cos^2\theta)$ will satisfy Eq. (6.33) with this form of ρ, provided

$$\frac{d^2h}{dr^2}+\frac{2}{r}\frac{dh}{dr}-\frac{6h}{r^2} = -4\pi Gf.$$

c) Show that $h = r^{-3}$ satisfies this equation in the region where $f = 0$. Can you complete the proof that the formula for $\delta\mathcal{G}$ found in Problem 17(b) is in fact valid everywhere outside the earth?

CHAPTER 7

MOVING COORDINATE SYSTEMS

7.1 MOVING ORIGIN OF COORDINATES

Let a point in space be located by vectors $\boldsymbol{r}$, $\boldsymbol{r}^*$ with respect to two origins of coordinates O, O^*, and let O^* be located by a vector $\boldsymbol{h}$ with respect to O (Fig. 7.1). Then the relation between the coordinates $\boldsymbol{r}$ and $\boldsymbol{r}^*$ is given by

$$\boldsymbol{r} = \boldsymbol{r}^*+\boldsymbol{h}, \tag{7.1}$$

$$\boldsymbol{r}^* = \boldsymbol{r}-\boldsymbol{h}. \tag{7.2}$$

In terms of rectangular coordinates, with axes x^*, y^*, z^* parallel to axes x, y, z, respectively, these equations can be written:

$$x = x^*+h_x, \qquad y = y^*+h_y, \qquad z = z^*+h_z; \tag{7.3}$$

$$x^* = x-h_x, \qquad y^* = y-h_y, \qquad z^* = z-h_z. \tag{7.4}$$

Now if the origin O^* is moving with respect to the origin O, which we regard as fixed, the relation between the velocities relative to the two systems is obtained by differentiating Eq. (7.1):

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}^*}{dt}+\frac{d\boldsymbol{h}}{dt}$$

$$= \boldsymbol{v}^*+\boldsymbol{v}_h, \tag{7.5}$$

where $\boldsymbol{v}$ and $\boldsymbol{v}^*$ are the velocities of the moving point relative to O and O^*, and $\boldsymbol{v}_h$ is the velocity of O^* relative to O. We are supposing that the axes x^*, y^*, z^* remain parallel to x, y, z. This is called a *translation* of the starred coordinate system with respect to the unstarred system. Written out in cartesian components, Eq. (7.5) becomes the time derivative of Eq. (7.3). The relation between relative accelerations is

$$\boldsymbol{a} = \frac{d^2\boldsymbol{r}}{dt^2} = \frac{d^2\boldsymbol{r}^*}{dt^2}+\frac{d^2\boldsymbol{h}}{dt^2}$$

$$= \boldsymbol{a}^*+\boldsymbol{a}_h. \tag{7.6}$$

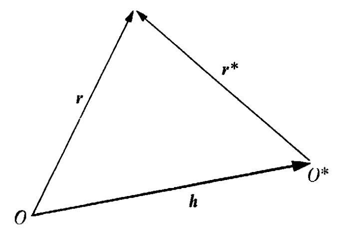

Fig. 7.1 Change of origin of coordinates.

Again these equations can easily be written out in terms of their rectangular components.

Newton's equations of motion hold in the fixed coordinate system, so that we have, for a particle of mass m subject to a force $\boldsymbol{F}$:

$$m\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{F}. \tag{7.7}$$

Using Eq. (7.6), we can write this equation in the starred coordinate system:

$$m\frac{d^2\boldsymbol{r}^*}{dt^2} + m\boldsymbol{a}_h = \boldsymbol{F}. \tag{7.8}$$

If O^* is moving at constant velocity relative to O, then $\boldsymbol{a}_h = \boldsymbol{0}$, and we have

$$m\frac{d^2\boldsymbol{r}^*}{dt^2} = \boldsymbol{F}. \tag{7.9}$$

Thus Newton's equations of motion, if they hold in any coordinate system, hold also in any other coordinate system moving with uniform velocity relative to the first. This is the Newtonian principle of relativity. It implies that, so far as mechanics is concerned, we cannot specify any unique fixed coordinate system or *frame of reference* to which Newton's laws are supposed to refer; if we specify one such system, any other system moving with constant velocity relative to it will do as well. This property of Eq. (7.7) is sometimes expressed by saying that Newton's equations of motion remain *invariant* in form, or that they are *covariant*, with respect to uniform translations of the coordinates. The concept of frame of reference is not quite the same as that of a coordinate system, in that if we make a change of coordinates that does not involve the time, we do not regard this as a change of frame of reference. A frame of reference includes all coordinate systems at rest with respect to any particular one. The principle of (special) relativity proposed by Einstein asserts that this relativity principle is not restricted to mechanics, but holds for all physical phenomena. The special theory of relativity is the result of the application of this principle to all types of phenomena, particularly electromagnetic phenomena. It turns out that this can only be done by modifying Newton's equations of motion slightly and, in fact, even Eqs. (7.5) and (7.6) require modification as we shall see in Chapter 13.

For any motion of O^*, we can write Eq. (7.8) in the form

$$m\frac{d^2\boldsymbol{r}^*}{dt^2} = \boldsymbol{F} - m\boldsymbol{a}_h. \tag{7.10}$$

This equation has the same form as the equation of motion (7.7) in a fixed coordinate system, except that in place of the force $\boldsymbol{F}$, we have $\boldsymbol{F} - m\boldsymbol{a}_h$. The term $-m\boldsymbol{a}_h$ we may call a fictitious force. We can treat the motion of a mass m relative to a moving coordinate system using Newton's equations of motion if we add this fictitious force to the actual force which acts. From the point of view of classical mechanics, it is not a force at all, but part of the mass times acceleration transposed to the other side of the equation. The essential distinction is that the real forces $\boldsymbol{F}$ acting on m depend on the positions and motions of other bodies, whereas the fictitious force depends on the acceleration of the starred coordinate system with respect to the fixed coordinate system. In the general theory of relativity, terms like $-m\boldsymbol{a}_h$ are regarded as legitimate forces in the starred coordinate system, on the same footing with the force $\boldsymbol{F}$, so that in all coordinate systems the same law of motion holds. This, of course, can only be done if it can be shown how to deduce the force $-m\boldsymbol{a}_h$ from the positions and motions of other bodies. The program is not so simple as it may seem from this brief outline, and modifications in the laws of motion are required to carry it through.†

7.2 ROTATING COORDINATE SYSTEMS

We now consider coordinate systems x, y, z and x^*, y^*, z^* whose axes are rotated relative to one another as in Fig. 7.2, where, for the present, the origins of the two sets of axes coincide. Introducing unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ associated with axes x, y, z, and unit vectors $\hat{\boldsymbol{x}}^*$, $\hat{\boldsymbol{y}}^*$, $\hat{\boldsymbol{z}}^*$ associated with axes x^*, y^*, z^*, we can express the position vector $\boldsymbol{r}$ in terms of its components along either set of axes:

$$\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}, \tag{7.11}$$

$$\boldsymbol{r} = x^*\hat{\boldsymbol{x}}^* + y^*\hat{\boldsymbol{y}}^* + z^*\hat{\boldsymbol{z}}^*. \tag{7.12}$$

Note that since the origins now coincide, a point is represented by the same vector $\boldsymbol{r}$ in both systems; only the components of $\boldsymbol{r}$ are different along the different axes. The relations between the coordinate systems can be obtained by taking the dot product of either the starred or the unstarred unit vectors with Eqs. (7.11)

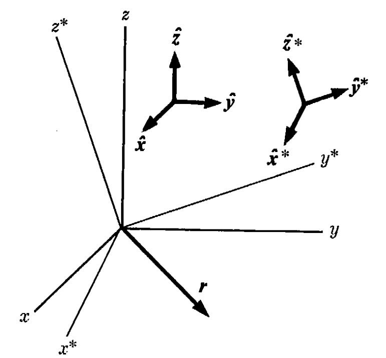

Fig. 7.2 Rotation of coordinate axes.

†P. G. Bergmann, *Introduction to the Theory of Relativity*. New York: Prentice-Hall, 1946. (Part 2.)

and (7.12). For example, if we compute $\hat{x}\cdot\boldsymbol{r}$, $\hat{y}\cdot\boldsymbol{r}$, $\hat{z}\cdot\boldsymbol{r}$, from Eqs. (7.11) and (7.12) and equate the results, we obtain

$$\begin{aligned} x &= x^*(\hat{x}^*\cdot\hat{x})+y^*(\hat{y}^*\cdot\hat{x})+z^*(\hat{z}^*\cdot\hat{x}),\\ y &= x^*(\hat{x}^*\cdot\hat{y})+y^*(\hat{y}^*\cdot\hat{y})+z^*(\hat{z}^*\cdot\hat{y}),\\ z &= x^*(\hat{x}^*\cdot\hat{z})+y^*(\hat{y}^*\cdot\hat{z})+z^*(\hat{z}^*\cdot\hat{z}). \end{aligned} \tag{7.13}$$

The dot products $(\hat{x}^*\cdot\hat{x})$, etc., are the cosines of the angles between the corresponding axes. Similar formulas for x^*, y^*, z^* in terms of x, y, z can easily be obtained by the same process. Equations (7.11), (7.12), and (7.13) do not depend on the fact that the vector $\boldsymbol{r}$ is drawn from the origin. Analogous formulas apply in terms of the components of any vector $\boldsymbol{A}$ along the two sets of axes. If the starred axes are rotating, the cosines of the angles between starred and unstarred axes are functions of time.

The time derivative of any vector $\boldsymbol{A}$ was defined by Eq. (3.52): p 85

$$\frac{d\boldsymbol{A}}{dt} = \lim_{\Delta t\to 0}\frac{\boldsymbol{A}(t+\Delta t)-\boldsymbol{A}(t)}{\Delta t}. \tag{7.14}$$

In attempting to apply this definition in the present case, we encounter a difficulty if the coordinate systems are rotating with respect to each other. A vector which is constant in one coordinate system is not constant in the other, but rotates. The definition requires us to subtract $\boldsymbol{A}(t)$ from $\boldsymbol{A}(t+\Delta t)$. During the time Δt, coordinate system x^*, y^*, z^* has rotated relative to x, y, z, so that at time $t+\Delta t$, the two systems will not agree as to which vector is (or was) $\boldsymbol{A}(t)$, i.e., which vector is in the same position that $\boldsymbol{A}$ was in at time t. The result is that the time derivative of a given vector will be different in the two coordinate systems. Let us use d/dt to denote the time derivative with respect to the unstarred coordinate system, which we regard as fixed, and d^*/dt to denote the time derivative with respect to the rotating starred coordinate system. We make this distinction with regard to vectors only; there is no ambiguity with regard to numerical quantities, and we denote their time derivatives by d/dt, or by a dot, which will have the same meaning in all coordinate systems. Let the vector $\boldsymbol{A}$ be given by

$$\boldsymbol{A} = A_x\hat{x}+A_y\hat{y}+A_z\hat{z}, \tag{7.15}$$

$$\boldsymbol{A} = A_x^*\hat{x}^*+A_y^*\hat{y}^*+A_z^*\hat{z}^*. \tag{7.16}$$

The unstarred time derivative of $\boldsymbol{A}$ may be obtained by differentiating Eq. (7.15), regarding $\hat{x}$, $\hat{y}$, $\hat{z}$ as constant vectors in the fixed system:

$$\frac{d\boldsymbol{A}}{dt} = \dot{A}_x\hat{x}+\dot{A}_y\hat{y}+\dot{A}_z\hat{z}. \tag{7.17}$$

Similarly, the starred derivative of $\boldsymbol{A}$ is given in terms of its starred components by

$$\frac{d^*\boldsymbol{A}}{dt} = \dot{A}_x^*\hat{x}^*+\dot{A}_y^*\hat{y}^*+\dot{A}_z^*\hat{z}^*. \tag{7.18}$$

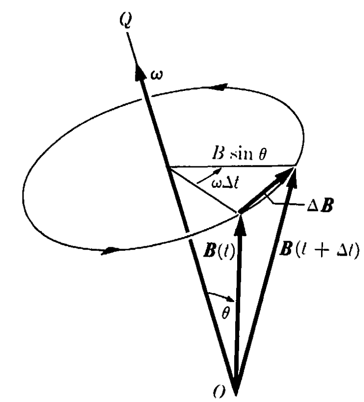

Fig. 7.3 Time derivative of a rotating vector.

We may regard Eqs. (7.17) and (7.18) as the definitions of unstarred and starred time derivatives of a vector. We can also obtain a formula for d/dt in starred components by taking the unstarred derivative of Eq. (7.16), remembering that the unit vectors $\hat{x}^*$, $\hat{y}^*$, $\hat{z}^*$ are moving relative to the unstarred system, and have time derivatives:

$$\frac{d\boldsymbol{A}}{dt} = \dot{A}_x^*\hat{x}^*+\dot{A}_y^*\hat{y}^*+\dot{A}_z^*\hat{z}^*+A_x^*\frac{d\hat{x}^*}{dt}+A_y^*\frac{d\hat{y}^*}{dt}+A_z^*\frac{d\hat{z}^*}{dt}. \tag{7.19}$$

A similar formula could be obtained for $d^*\boldsymbol{A}/dt$ in terms of its unstarred components. FIRST A SPECIAL CASE.

Let us now suppose that the starred coordinate system is rotating about some axis OQ through the origin, with an angular velocity ω (Fig. 7.3). We define the *vector angular velocity* $\boldsymbol{\omega}$ as a vector of magnitude ω directed along the axis OQ in the direction of advance of a right-hand screw rotating with the starred system. Consider a vector $\boldsymbol{B}$ at rest in the starred system. Its starred derivative is zero, and we now show that its unstarred derivative is

$$\frac{d\boldsymbol{B}}{dt} = \boldsymbol{\omega}\times\boldsymbol{B}. \tag{7.20}$$

In order to subtract $\boldsymbol{B}(t)$ from $\boldsymbol{B}(t+\Delta t)$, we draw these vectors with their tails together, and it will be convenient to place them with their tails on the axis of rotation. (The time derivative depends only on the components of $\boldsymbol{B}$ along the axes, and not on the position of $\boldsymbol{B}$ in space.) We first verify from Fig. 7.3 that the direction of $d\boldsymbol{B}/dt$ is given correctly by Eq. (7.20), recalling the definition [Eq. (3.24) and Fig. 3.11] of the cross product. The magnitude of $d\boldsymbol{B}/dt$ as given by Eq. (7.20) is

$$\left|\frac{d\boldsymbol{B}}{dt}\right| = |\boldsymbol{\omega}\times\boldsymbol{B}| = \omega B\sin\theta. \tag{7.21}$$

This is the correct formula, since it can be seen from Fig. 7.3 that, when Δt is small,

$$|\Delta\boldsymbol{B}| = (B\sin\theta)(\omega\,\Delta t).$$

NOW, GENERAL CASE

When Eq. (7.20) is applied to the unit vectors $\hat{x}^*$, $\hat{y}^*$, $\hat{z}^*$, Eq. (7.19) becomes, if we make use of Eqs. (7.18) and (7.16):

$$\frac{dA}{dt} = \frac{d^*A}{dt} + A_x^*(\omega \times \hat{x}^*) + A_y^*(\omega \times \hat{y}^*) + A_z^*(\omega \times \hat{z}^*)$$
$$= \frac{d^*A}{dt} + \omega \times A. \tag{7.22}$$

This is the fundamental relationship between time derivatives for rotating coordinate systems. It may be remembered by noting that the time derivative of any vector in the unstarred coordinate system is its derivative in the starred system plus the unstarred derivative it would have if it were at rest in the starred system. Equation (7.22) applies even when the angular velocity vector ω is changing in magnitude and direction with time. Taking the derivative of right and left sides of Eq. (7.22), and applying Eq. (7.22) again to A and d^*A/dt, we have for the second time derivative of any vector A:

$$\frac{d^2A}{dt^2} = \frac{d}{dt}\left(\frac{d^*A}{dt}\right) + \left[\omega \times \frac{dA}{dt}\right] + \frac{d\omega}{dt} \times A$$
$$= \frac{d^{*2}A}{dt^2} + \omega \times \frac{d^*A}{dt} + \left[\omega \times \left(\frac{d^*A}{dt} + \omega \times A\right)\right] + \frac{d\omega}{dt} \times A$$
$$= \frac{d^{*2}A}{dt^2} + 2\omega \times \frac{d^*A}{dt} + \omega \times (\omega \times A) + \frac{d\omega}{dt} \times A. \tag{7.23}$$

Since $\omega \times A = 0$ if ω is parallel to A, [Eq. (3.29)], the starred and unstarred derivatives of any vector parallel to the axis of rotation are the same, according to Eq. (7.22). In particular,

$$\frac{d\omega}{dt} = \frac{d^*\omega}{dt}.$$

It is to be noted that the vector ω on both sides of this equation is the angular velocity of the starred system relative to the unstarred system, although its time derivative is calculated with respect to the unstarred system on the left side, and with respect to the starred system on the right. The angular velocity of the unstarred system relative to the starred system will be $-\omega$.

We now show that the relations derived above for a rotating coordinate system are perfectly general, in that they apply to any motion of the starred axes relative to the unstarred axes. Let the unstarred rates of change of the starred unit vectors be given in terms of components along the starred axes by

$$\frac{d\hat{x}^*}{dt} = a_{11}\hat{x}^* + a_{12}\hat{y}^* + a_{13}\hat{z}^*,$$
$$\frac{d\hat{y}^*}{dt} = a_{21}\hat{x}^* + a_{22}\hat{y}^* + a_{23}\hat{z}^*, \tag{7.24}$$
$$\frac{d\hat{z}^*}{dt} = a_{31}\hat{x}^* + a_{32}\hat{y}^* + a_{33}\hat{z}^*.$$

By differentiating the equation

$$\hat{x}^* \cdot \hat{x}^* = 1, \tag{7.25}$$

we obtain

$$\frac{d\hat{x}^*}{dt} \cdot \hat{x}^* = 0. \tag{7.26}$$

From this and the corresponding equations for $\hat{y}^*$ and $\hat{z}^*$, we have

$$a_{11} = a_{22} = a_{33} = 0. \tag{7.27}$$

By differentiating the equation

$$\hat{x}^* \cdot \hat{z}^* = 0, \tag{7.28}$$

we obtain

$$\frac{d\hat{x}^*}{dt} \cdot \hat{z}^* = -\hat{x}^* \cdot \frac{d\hat{z}^*}{dt}. \tag{7.29}$$

From this and the other two analogous equations, we have

$$a_{31} = -a_{13}, \quad a_{12} = -a_{21}, \quad a_{23} = -a_{32}. \tag{7.30}$$

It is clear from Eqs. (7.27) and (7.30) that if the three coefficients a_{12}, a_{23}, a_{31} are specified, the other six are determined. Let us define a vector ω whose components in the x^*, y^*, z^* coordinate system are

$$\omega_x^* = a_{23}, \quad \omega_y^* = a_{31}, \quad \omega_z^* = a_{12}. \tag{7.31}$$

At this point, this is simply an arbitrary definition. We may always define a vector by giving its components in some coordinate system. We call this vector ω because we are going to show that it is in fact the angular velocity of the starred coordinate system. Equations (7.24) can now be rewritten, with the help of Eqs. (7.27), (7.30), and (7.31), in the form

$$\frac{d\hat{x}^*}{dt} = \omega \times \hat{x}^*,$$
$$\frac{d\hat{y}^*}{dt} = \omega \times \hat{y}^*, \tag{7.32}$$
$$\frac{d\hat{z}^*}{dt} = \omega \times \hat{z}^*.$$

According to Eq. (7.20), these time derivatives of $\hat{x}^*$, $\hat{y}^*$, $\hat{z}^*$ are just those to be expected if the starred unit vectors are rotating with an angular velocity ω. Thus no matter how the starred coordinate axes may be moving, we can define at any instant an angular velocity vector ω, given by Eq. (7.31), such that the time derivatives of any vector relative to the starred and unstarred coordinate systems are related by Eqs. (7.22) and (7.23).

Let us now suppose that the starred coordinate system is moving in such a way that its origin O^* remains fixed at the origin O of the fixed coordinate system. Then any point in space is located by the same position vector $\boldsymbol{r}$ in both coordinate systems [Eqs. (7.11) and (7.12)]. By applying Eqs. (7.22) and (7.23) to the position vector $\boldsymbol{r}$, we obtain formulas for the relation between velocities and accelerations in the two coordinate systems:

$$\frac{d\boldsymbol{r}}{dt} = \frac{d^*\boldsymbol{r}}{dt} + \boldsymbol{\omega} \times \boldsymbol{r}, \tag{7.33}$$

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{d^{*2}\boldsymbol{r}}{dt^2} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + 2\boldsymbol{\omega} \times \frac{d^*\boldsymbol{r}}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r}. \tag{7.34}$$

Formula (7.34) is called *Coriolis' theorem.* The first term on the right is the acceleration relative to the starred system. The second term is called the *centripetal acceleration* of a point in rotation about an axis (*centripetal* means "toward the center"). Using the notation in Fig. 7.4, we readily verify that $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$ points directly toward and perpendicular to the axis of rotation, and that its magnitude is

$$|\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})| = \omega^2 r \sin\theta$$

$$= \frac{v^2}{r \sin\theta}, \tag{7.35}$$

where $v = \omega r \sin\theta$ is the speed of circular motion and $(r \sin\theta)$ is the distance from the axis. The third term is present only when the point $\boldsymbol{r}$ is moving in the starred system, and is called the *coriolis acceleration.* The last term vanishes for a constant angular velocity of rotation about a fixed axis.

If we suppose that Newton's law of motion (7.7) holds in the unstarred coordinate system, we shall have in the starred system:

$$m\frac{d^{*2}\boldsymbol{r}}{dt^2} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + 2m\boldsymbol{\omega} \times \frac{d^*\boldsymbol{r}}{dt} + m\frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r} = \boldsymbol{F}. \tag{7.36}$$

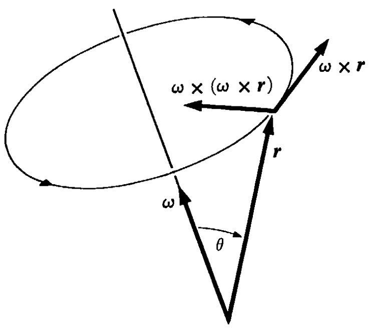

Fig. 7.4 Centripetal acceleration.

Transposing the second, third, and fourth terms to the right side, we obtain an equation of motion similar in form to Newton's equation of motion:

$$m\frac{d^{*2}\boldsymbol{r}}{dt^2} = \boldsymbol{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) - 2m\boldsymbol{\omega} \times \frac{d^*\boldsymbol{r}}{dt} - m\frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r}. \tag{7.37}$$

The second term on the right is called the *centrifugal force* (*centrifugal* means "away from the center"); the third term is called the *coriolis force.* The last term has no special name, and appears only for the case of non-uniform rotation. If we introduce the fictitious centrifugal and coriolis forces, the laws of motion relative to a rotating coordinate system are the same as for fixed coordinates. A great deal of confusion has arisen regarding the term "centrifugal force." This force is not a real force, at least in classical mechanics, and is not present if we refer to a fixed coordinate system in space. We can, however, treat a rotating coordinate system as if it were fixed by introducing the centrifugal and coriolis forces. Thus a particle moving in a circle has no centrifugal force acting on it, but only a force toward the center which produces its centripetal acceleration. However, if we consider a coordinate system rotating with the particle, in this system the particle is at rest, and the force toward the center is balanced by the centrifugal force. It is very often useful to adopt a rotating coordinate system. In studying the action of a cream separator, for example, it is far more convenient to choose a coordinate system in which the liquid is at rest, and use the laws of diffusion to study the diffusion of cream toward the axis under the action of the centrifugal force field, than to try to study the motion from the point of view of a fixed observer watching the whirling liquid.

We can treat coordinate systems in simultaneous translation and rotation relative to each other by using Eq. (7.1) to represent the relation between the coordinate vectors $\boldsymbol{r}$ and $\boldsymbol{r}^*$ relative to origins O, O^* not necessarily coincident. In the derivation of Eqs. (7.32), no assumption was made about the origin of the starred coordinates, and therefore Eqs. (7.22) and (7.23) may still be used to express the time derivatives of any vector with respect to the unstarred coordinate system in terms of its time derivatives with respect to the starred system. Replacing $d\boldsymbol{r}^*/dt$, $d^2\boldsymbol{r}^*/dt$ in Eqs. (7.5) and (7.6) by their expressions in terms of the starred derivatives relative to the starred system as given by Eqs. (7.33) and (7.34), we obtain for the position, velocity, and acceleration of a point with respect to coordinate systems in relative translation and rotation:

$$\boldsymbol{r} = \boldsymbol{r}^* + \boldsymbol{h}, \tag{7.38}$$

$$\frac{d\boldsymbol{r}}{dt} = \frac{d^*\boldsymbol{r}^*}{dt} + \boldsymbol{\omega} \times \boldsymbol{r}^* + \frac{d\boldsymbol{h}}{dt}, \tag{7.39}$$

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{d^{*2}\boldsymbol{r}^*}{dt^2} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}^*) + 2\boldsymbol{\omega} \times \frac{d^*\boldsymbol{r}^*}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r}^* + \frac{d^2\boldsymbol{h}}{dt^2}. \tag{7.40}$$

7.3 LAWS OF MOTION ON THE ROTATING EARTH

We write the equation of motion, relative to a coordinate system fixed in space, for a particle of mass m subject to a gravitational force $m\boldsymbol{g}$ and any other non-gravitational forces $\boldsymbol{F}$:

$$m\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{F}+m\boldsymbol{g}. \tag{7.41}$$

Now if we refer the motion of the particle to a coordinate system at rest relative to the earth, which rotates with constant angular velocity $\boldsymbol{\omega}$, and if we measure the position vector $\boldsymbol{r}$ from the center of the earth, we have, by Eq. (7.34):

$$m\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{F}+m\boldsymbol{g}$$

$$= m\frac{d^{*2}\boldsymbol{r}}{dt^2}+m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})+2m\boldsymbol{\omega}\times\frac{d^*\boldsymbol{r}}{dt}, \tag{7.42}$$

which can be rearranged in the form

$$m\frac{d^{*2}\boldsymbol{r}}{dt^2} = \boldsymbol{F}+m[\boldsymbol{g}-\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})]-2m\boldsymbol{\omega}\times\frac{d^*\boldsymbol{r}}{dt}. \tag{7.43}$$

This equation has the same form as Newton's equation of motion. We have combined the gravitational and centrifugal force terms because both are proportional to the mass of the particle and both depend only on the position of the particle; in their mechanical effects these two forces are indistinguishable. We may define the effective gravitational acceleration $\boldsymbol{g}_e$ at any point on the earth's surface by:

$$\boldsymbol{g}_e(\boldsymbol{r}) = \boldsymbol{g}(\boldsymbol{r})-\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}). \tag{7.44}$$

The gravitational force which we measure experimentally on a body of mass m at rest† on the earth's surface is $m\boldsymbol{g}_e$. Since $-\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})$ points radially outward from the earth's axis, $\boldsymbol{g}_e$ at every point north of the equator will point slightly to the south of the earth's center, as can be seen from Fig. 7.5. A body released near the earth's surface will begin to fall with acceleration $\boldsymbol{g}_e$, the direction determined by a plumb line is that of $\boldsymbol{g}_e$, and a liquid will come to equilibrium with its surface perpendicular to $\boldsymbol{g}_e$. This is why the earth has settled into equilibrium in the form of an oblate ellipsoid, flattened at the poles. The degree of flattening is just such as to make the earth's surface at every point perpendicular to $\boldsymbol{g}_e$ (ignoring local irregularities).

Equation (7.43) can now be written

$$m\frac{d^{*2}\boldsymbol{r}}{dt^2} = \boldsymbol{F}+m\boldsymbol{g}_e-2m\boldsymbol{\omega}\times\frac{d^*\boldsymbol{r}}{dt}. \tag{7.45}$$

†A body in motion is subject also to the coriolis force.

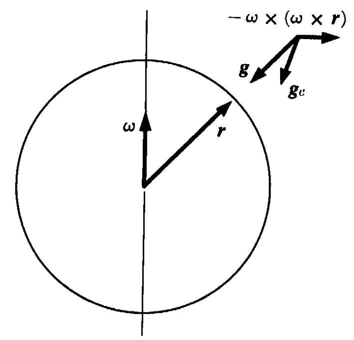

Fig. 7.5 Effective acceleration of gravity on the rotating earth.

The velocity and acceleration which appear in this equation are unaffected if we relocate our origin of coordinates at any convenient point at the surface of the earth; hence this equation applies to the motion of a particle of mass m at the surface of the earth relative to a local coordinate system at rest on the earth's surface. The only unfamiliar term is the coriolis force which acts on a moving particle. The reader can convince himself by a few calculations that this force is comparatively small at ordinary velocities $d^*\boldsymbol{r}/dt$. It will be instructive to try working out the direction of the coriolis force for various directions of motion at various places on the earth's surface. The coriolis force is of major importance in the motion of large air masses, and is responsible for the fact that in the northern hemisphere tornados and cyclones circle in the direction south to east to north to west. In the northern hemisphere, the coriolis force acts to deflect a moving object toward the right. As the winds blow toward a low pressure area, they are deflected to the right, so that they circle the low pressure area in a counterclockwise direction. An air mass circling in this way will have a low pressure on its left, and a higher pressure on its right. This is just what is needed to balance the coriolis force urging it to the right. An air mass can move steadily in one direction only if there is a high pressure to the right of it to balance the coriolis force. Conversely, a pressure gradient over the surface of the earth tends to develop winds moving at right angles to it. The prevailing westerly winds in the northern temperate zone indicate that the atmospheric pressure toward the equator is greater than toward the poles, at least near the earth's surface. The easterly trade winds in the equatorial zone are due to the fact that any air mass moving toward the equator will acquire a velocity toward the west due to the coriolis force acting on it. The trade winds are maintained by high pressure areas on either side of the equatorial zone.

7.4 THE FOUCAULT PENDULUM

An interesting application of the theory of rotating coordinate systems is the problem of the Foucault pendulum. The Foucault pendulum has a bob hanging from a string arranged to swing freely in any vertical plane. The pendulum is

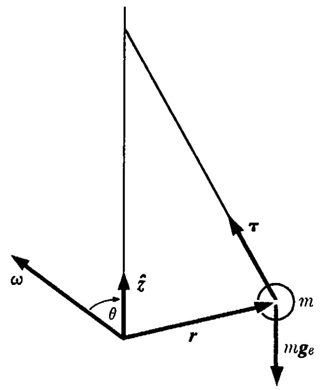

Fig. 7.6 The Foucault pendulum.

started swinging in a definite vertical plane and it is observed that the plane of swinging gradually precesses about the vertical axis during a period of several hours. The bob must be made heavy, the string very long, and the support nearly frictionless, in order that the pendulum can continue to swing freely for long periods of time. If we choose the origin of coordinates directly below the point of support, at the point of equilibrium of the pendulum bob of mass m, then the vector $\boldsymbol{r}$ will be nearly horizontal, for small amplitudes of oscillation of the pendulum. In the northern hemisphere, $\boldsymbol{\omega}$ makes an acute angle with the vertical, as in Fig. 7.6. Writing $\boldsymbol{\tau}$ for the tension in the string, we have as the equation of motion of the bob, according to Eq. (7.45):

$$m\frac{d^{*2}\boldsymbol{r}}{dt^2} = \boldsymbol{\tau}+m\boldsymbol{g}_e-2m\boldsymbol{\omega}\times\frac{d^*\boldsymbol{r}}{dt}. \tag{7.46}$$

If the coriolis force were not present, this would be the equation for a simple pendulum on a nonrotating earth. The coriolis force is very small, less than 0.1% of the gravitational force if the velocity is 5 mph or less, and its vertical component is therefore negligible in comparison with the gravitational force. (It is the vertical force which determines the magnitude of the tension in the string.) However, the horizontal component of the coriolis force is perpendicular to the velocity $d^*\boldsymbol{r}/dt$, and as there are no other forces in this direction when the pendulum swings to and fro, it can change the nature of the motion. Any force with a horizontal component perpendicular to $d^*\boldsymbol{r}/dt$ will make it impossible for the pendulum to continue to swing in a fixed vertical plane. In order to solve the problem including the coriolis term, we use the experimental result as a clue, and try to find a new coordinate system rotating about the vertical axis through the point of support at such an angular velocity that in this system the coriolis terms, or at least their horizontal components, are missing. Let us introduce a new coordinate system rotating about the vertical axis with constant angular velocity $\hat{\boldsymbol{z}}\Omega$ (relative to the earth), where $\hat{\boldsymbol{z}}$ is a vertical unit vector. We shall call this precessing coordinate

system the primed coordinate system, and denote the time derivative with respect to this system by d'/dt. Then we shall have, by Eqs. (7.33) and (7.34):

$$\frac{d^*\boldsymbol{r}}{dt} = \frac{d'\boldsymbol{r}}{dt}+\Omega\hat{\boldsymbol{z}}\times\boldsymbol{r}, \tag{7.47}$$

$$\frac{d^{*2}\boldsymbol{r}}{dt^2} = \frac{d'^2\boldsymbol{r}}{dt^2}+\Omega^2\hat{\boldsymbol{z}}\times(\hat{\boldsymbol{z}}\times\boldsymbol{r})+2\Omega\hat{\boldsymbol{z}}\times\frac{d'\boldsymbol{r}}{dt}. \tag{7.48}$$

Equation (7.46) becomes

$$\begin{aligned} m\frac{d'^2\boldsymbol{r}}{dt^2} &= \boldsymbol{\tau}+m\boldsymbol{g}_e-2m\boldsymbol{\omega}\times\left(\frac{d'\boldsymbol{r}}{dt}+\Omega\hat{\boldsymbol{z}}\times\boldsymbol{r}\right) \\ &\quad + m\Omega^2\hat{\boldsymbol{z}}\times(\hat{\boldsymbol{z}}\times\boldsymbol{r})-2m\Omega\hat{\boldsymbol{z}}\times\frac{d'\boldsymbol{r}}{dt} \\ &= \boldsymbol{\tau}+m\boldsymbol{g}_e-2m\Omega\boldsymbol{\omega}\times(\hat{\boldsymbol{z}}\times\boldsymbol{r})-m\Omega^2\hat{\boldsymbol{z}}\times(\hat{\boldsymbol{z}}\times\boldsymbol{r}) \\ &\quad -2m(\boldsymbol{\omega}+\hat{\boldsymbol{z}}\Omega)\times\frac{d'\boldsymbol{r}}{dt}. \end{aligned} \tag{7.49}$$

We expand the triple products by means of Eq. (3.35):

$$\begin{aligned} m\frac{d'^2\boldsymbol{r}}{dt^2} &= \boldsymbol{\tau}+m\boldsymbol{g}_e-m(2\Omega\boldsymbol{\omega}\cdot\boldsymbol{r}+\Omega^2\hat{\boldsymbol{z}}\cdot\boldsymbol{r})\hat{\boldsymbol{z}} \\ &\quad +m(2\Omega\hat{\boldsymbol{z}}\cdot\boldsymbol{\omega}+\Omega^2)\boldsymbol{r}-2m(\boldsymbol{\omega}+\hat{\boldsymbol{z}}\Omega)\times\frac{d'\boldsymbol{r}}{dt}. \end{aligned} \tag{7.50}$$

Every vector on the right side of Eq. (7.50) lies in the vertical plane containing the pendulum, except the last term. Since, for small oscillations, $d'\boldsymbol{r}/dt$ is practically horizontal, we can make the last term lie in this vertical plane also by making $(\boldsymbol{\omega}+\hat{\boldsymbol{z}}\Omega)$ horizontal. We therefore require that

$$\hat{\boldsymbol{z}}\cdot(\boldsymbol{\omega}+\hat{\boldsymbol{z}}\Omega) = 0. \tag{7.51}$$

This determines Ω:

$$\Omega = -\omega\cos\theta, \tag{7.52}$$

where ω is the angular velocity of the rotating earth, Ω is the angular velocity of the precessing coordinate system relative to the earth, and θ is the angle between the vertical and the earth's axis, as indicated in Fig. 7.6. The vertical is along the direction of $-\boldsymbol{g}_e$, and since this is very nearly the same as the direction of $-\boldsymbol{g}$ (see Fig. 7.5), θ will be practically equal to the colatitude, that is, the angle between $\boldsymbol{r}$ and $\boldsymbol{\omega}$ in Fig. 7.5. For small oscillations, if Ω is determined by Eq. (7.52), the cross product in the last term of Eq. (7.50) is vertical. Since all terms on the right of Eq. (7.50) now lie in a vertical plane containing the pendulum, the acceleration $d'^2\boldsymbol{r}/dt^2$ of the bob in the precessing system is always toward the vertical axis, and if the pendulum is initially swinging to and fro, it will continue to swing to and fro in the same vertical plane in the precessing coordinate system. Relative to the earth, the plane of the motion precesses with angular velocity Ω of magnitude and sense given by Eq. (7.52). In the northern hemisphere, the precession is clockwise looking down.

Since the last three terms on the right in Eq. (7.50) are much smaller than the first two, the actual motion in the precessing coordinate system is practically the same as for a pendulum on a nonrotating earth. Even at large amplitudes, where the velocity $d'\boldsymbol{r}/dt$ has a vertical component, careful study will show that the last term in Eq. (7.50), when Ω is chosen according to Eq. (7.52), does not cause any additional precession relative to the precessing coordinate system, but merely causes the bob to swing in an arc which passes slightly east of the vertical through the point of support. At the equator, Ω is zero, and the Foucault pendulum does not precess; by thinking about it a moment, perhaps you can see physically why this is so. At the north or south pole, $\Omega = \pm\omega$, and the pendulum merely swings in a fixed vertical plane in space while the earth turns beneath it.

Note that we have been able to give a fairly complete discussion of the Foucault pendulum, by using Coriolis' theorem twice, without actually solving the equations of motion at all.

7.5 LARMOR'S THEOREM

The coriolis force in Eq. (7.37) is of the same form as the magnetic force acting on a charged particle (Eq. 3.281), in that both are given by the cross product of the velocity of the particle with a vector representing a force field. Indeed, in the general theory of relativity, the coriolis forces on a particle in a rotating system can be regarded as due to the relative motion of other masses in the universe in a way somewhat analogous to the magnetic force acting on a charged particle which is due to the relative motion of other charges. The similarity in form of the two forces suggests that the effect of a magnetic field on a system of charged particles may be canceled by introducing a suitable rotating coordinate system. This idea leads to Larmor's theorem, which we state first, and then prove:

Larmor's Theorem. *If a system of charged particles, all having the same ratio q/m of charge to mass, acted on by their mutual (central) forces, and by a central force toward a common center, is subject in addition to a weak uniform magnetic field* $\boldsymbol{B}$, *its possible motions will be the same as the motions it could perform without the magnetic field, superposed upon a slow precession of the entire system about the center of force with angular velocity*

$$\boldsymbol{\omega} = -\frac{q}{2mc}\boldsymbol{B} \text{ (gaussian units).*} \tag{7.53}$$

The definition of a *weak* magnetic field will appear as the proof is developed. We shall assume that all the particles have the same charge q and the same mass

*In mks units, omit the c here and in the following equations.

m, although it will be apparent that the only thing that needs to be assumed is that the ratio q/m is constant. Practically the only important applications of Larmor's theorem are to the behavior of an atom in a magnetic field. The particles here are electrons of mass m, charge $q = -e$, acted upon by their mutual electrostatic repulsions and by the electrostatic attraction of the nucleus.

Let the central force acting on the kth particle be $\boldsymbol{F}_k^c$, and let the sum of the forces due to the other particles be $\boldsymbol{F}_k^i$. Then the equations of motion of the system of particles, in the absence of a magnetic field, are

$$m\frac{d^2\boldsymbol{r}_k}{dt^2} = \boldsymbol{F}_k^c + \boldsymbol{F}_k^i, \qquad k = 1, \ldots, N, \tag{7.54}$$

where N is the total number of particles. The force $\boldsymbol{F}_k^c$ depends only on the distance of particle k from the center of force, which we shall take as origin, and the forces $\boldsymbol{F}_k^i$ depend only on the distances of the particles from one another. When the magnetic field is applied, the equations of motion become, by Eq. (3.281):

$$m\frac{d^2\boldsymbol{r}_k}{dt^2} = \boldsymbol{F}_k^c + \boldsymbol{F}_k^i + \frac{q}{c}\frac{d\boldsymbol{r}_k}{dt}\times\boldsymbol{B}, \qquad k = 1, \ldots, N. \tag{7.55}$$

In order to eliminate the last term, we introduce a starred coordinate system with the same origin, rotating about this origin with angular velocity $\boldsymbol{\omega}$. Making use of Eqs. (7.33) and (7.34), we can write the equations of motion in the starred coordinate system:

$$m\frac{d^{*2}\boldsymbol{r}_k}{dt^2} = \boldsymbol{F}_k^c + \boldsymbol{F}_k^i - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_k) + \frac{q}{c}(\boldsymbol{\omega}\times\boldsymbol{r}_k)\times\boldsymbol{B} + \frac{d^*\boldsymbol{r}_k}{dt}\times\left(\frac{q\boldsymbol{B}}{c} + 2m\boldsymbol{\omega}\right). \tag{7.56}$$

We can make the last term vanish by setting

$$\boldsymbol{\omega} = -\frac{q}{2mc}\boldsymbol{B}. \tag{7.57}$$

Equation (7.56) then becomes

$$m\frac{d^{*2}\boldsymbol{r}_k}{dt^2} = \boldsymbol{F}_k^c + \boldsymbol{F}_k^i + \frac{q^2}{4mc^2}\boldsymbol{B}\times(\boldsymbol{B}\times\boldsymbol{r}_k), \qquad k = 1, \ldots, N. \tag{7.58}$$

The forces $\boldsymbol{F}_k^c$ and $\boldsymbol{F}_k^i$ depend only on the distances of the particles from the origin and on their distances from one another, and these distances will be the same in the starred and unstarred coordinate systems. Therefore, if we neglect the last term, Eqs. (7.58) have exactly the same form in terms of starred coordinates as Eqs. (7.54) have in unstarred coordinates. Consequently, their solutions will then be the same, and the motions of the system expressed in starred coordinates will be the same as the motions of the system expressed in unstarred coordinates in the absence of a magnetic field. This is Larmor's theorem.

The condition that the magnetic field be weak means that the last term in Eq. (7.58) must be negligible in comparison with the first two terms. Notice that the term we are neglecting is proportional to B^2, whereas the term in Eq. (7.55) which we have eliminated is proportional to B. Hence, for sufficiently weak fields, the former may be negligible even though the latter is not. The last term in Eq. (7.58) may be written in the form

$$\frac{q^2}{4mc^2}\boldsymbol{B}\times(\boldsymbol{B}\times\boldsymbol{r}_k) = m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_k). \tag{7.59}$$

Another way of formulating the condition for a weak magnetic field is to say that the Larmor frequency ω, given by Eq. (7.57), must be small compared with the frequencies of the motion in the absence of a magnetic field.

The reader who has understood clearly the above derivation should be able to answer the following two questions. The cyclotron frequency, given by Eq. (3.299), for the motion of a charged particle in a magnetic field is twice the Larmor frequency, given by Eq. (7.57). Why does not Larmor's theorem apply to the charged particles in a cyclotron? Equation (7.58) can be derived without any assumption as to the origin of coordinates in the starred system. Why is it necessary that the axis of rotation of the starred coordinate system pass through the center of force of the system of particles?

7.6 THE RESTRICTED THREE-BODY PROBLEM

We pointed out in Section 4.9 that the three-body problem, in which three masses move under their mutual gravitational forces, cannot be solved in any general way. In this section we will consider a simplified problem, the restricted problem of three bodies, which retains many features of the more general problem, among them the fact that there is no general method of solving it. In the restricted problem, we are given two bodies of masses M_1 and M_2 that revolve in circles under their mutual gravitational attraction and around their common center of mass. The third body of very small mass m moves in the gravitational field of M_1 and M_2. We are to assume that m is so small that the resulting disturbance of the motions of M_1 and M_2 can be neglected. We will further simplify the problem by assuming that m remains in the plane in which M_1 and M_2 revolve. The problem thus reduces to a one-body problem in which we must find the motion of m in the given (moving) gravitational field of the other two. An obvious example would be a rocket moving in the gravitational fields of the earth and the moon, which revolve very nearly in circles about their common center of mass.

If M_1 and M_2 are separated by a distance a, then according to the results of Section 4.7, their angular velocity is determined by equating the gravitational force to mass times acceleration in the reduced problem, in which M_1 is at rest and

M_2 has mass μ as given by Eq. (4.98):

$$\mu\omega^2 a = \frac{M_1 M_2 G}{a^2}, \tag{7.60}$$

so that

$$\omega^2 = \frac{(M_1+M_2)G}{a^3}. \tag{7.61}$$

The center of mass divides the distance a into segments that are proportional to the masses.

We now introduce a coordinate system rotating with angular velocity ω about the center of mass of M_1 and M_2. In this system, M_1 and M_2 are at rest, and we will take them to be on the x-axis at the points

$$x_1 = \frac{M_2}{M_1+M_2}a, \qquad x_2 = -\frac{M_1}{M_1+M_2}a. \tag{7.62}$$

The angular velocity $\boldsymbol{\omega}$ is taken to be along the z-axis. Then m moves in the xy-plane, and its equation of motion is

$$m\frac{d^{*2}\boldsymbol{r}}{dt^2} = \boldsymbol{F}_1+\boldsymbol{F}_2-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})-2m\boldsymbol{\omega}\times\frac{d^*\boldsymbol{r}}{dt}, \tag{7.63}$$

where $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are the gravitational attraction of M_1 and M_2 on m. Written in terms of components, the two equations become

$$\ddot{x} = -\frac{M_1G(x-x_1)}{[(x-x_1)^2+y^2]^{3/2}}-\frac{M_2G(x-x_2)}{[(x-x_2)^2+y^2]^{3/2}}+\frac{(M_1+M_2)Gx}{a^3}+2\omega\dot{y},$$

$$\ddot{y} = -\frac{M_1Gy}{[(x-x_1)^2+y^2]^{3/2}}-\frac{M_2Gy}{[(x-x_2)^2+y^2]^{3/2}}+\frac{(M_1+M_2)Gy}{a^3}-2\omega\dot{x}. \tag{7.64}$$

Note that the mass m cancels in these equations.

Since the coriolis force is perpendicular to the velocity, it does no 'work' in this moving coordinate system. Moreover, the centrifugal force has zero curl and can be derived from the 'potential energy'

$$V_c = -\tfrac{1}{2}m\omega^2(x^2+y^2). \tag{7.65}$$

Therefore the total 'energy' in the moving coordinate system is a constant of the motion:

$$\text{'}E\text{'} = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2)+\text{'}V\text{'}, \tag{7.66}$$

where

$$\text{'}V\text{'} = -\frac{mM_1G}{[(x-x_1)^2+y^2]^{1/2}}-\frac{mM_2G}{[(x-x_2)^2+y^2]^{1/2}} -\frac{m(M_1+M_2)G(x^2+y^2)}{2a^3}. \tag{7.67}$$

The energy equation (7.66) enables us to make certain statements about the kinds of orbits that may be possible. In order to simplify the algebra, let us set

$$\xi = x/a, \qquad \eta = y/a, \tag{7.68}$$

$$\xi_1 = \frac{M_2}{M_1+M_2}, \qquad \xi_2 = -\frac{M_1}{M_1+M_2} = \xi_1 - 1. \tag{7.69}$$

Then Eq. (7.67) can be written as

$$\text{`}V\text{'} = \frac{m(M_1+M_2)G}{a}\left\{\frac{\xi_2}{[(\xi-\xi_1)^2+\eta^2]^{1/2}} - \frac{\xi_1}{[(\xi-\xi_2)^2+\eta^2]^{1/2}} - \frac{1}{2}(\xi^2+\eta^2)\right\}. \tag{7.70}$$

In order to see the nature of this function, let us first look for its singular points, where $\partial\text{`}V\text{'}/\partial\xi$ and $\partial\text{`}V\text{'}/\partial\eta$ both vanish:

$$\begin{aligned} -\frac{\xi_2(\xi-\xi_1)}{[(\xi-\xi_1)^2+\eta^2]^{3/2}} + \frac{\xi_1(\xi-\xi_2)}{[(\xi-\xi_2)^2+\eta^2]^{3/2}} - \xi &= 0, \\ -\frac{\xi_2\eta}{[(\xi-\xi_1)^2+\eta^2]^{3/2}} + \frac{\xi_1\eta}{[(\xi-\xi_2)^2+\eta^2]^{3/2}} - \eta &= 0. \end{aligned} \tag{7.71}$$

A point (x, y) for which these equations are satisfied is an equilibrium point for the mass m (in the rotating coordinate system), since Eqs. (7.64) are evidently satisfied if m is at rest at this point. We first consider points on the $\eta = 0$ axis. The second equation is then satisfied, and the first becomes

$$-\frac{\xi_2(\xi-\xi_1)}{|\xi-\xi_1|^3} + \frac{\xi_1(\xi-\xi_2)}{|\xi-\xi_2|^3} - \xi = 0. \tag{7.72}$$

In Fig. 7.7, we plot the function 'V', as given by Eq. (7.70), along the $\eta = 0$ axis. The roots of Eq. (7.72) are the maxima of 'V' $(\xi, 0)$ in Fig. 7.7, where it can be seen that there are three such roots. Let us call them ξ_A, ξ_B, ξ_C as in the figure. Each is

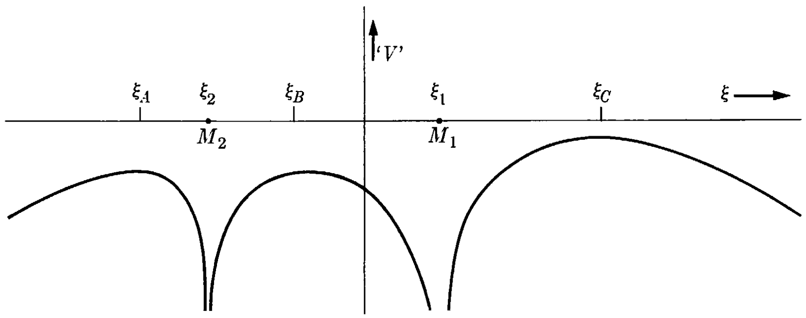

Fig. 7.7 A plot of 'V' $(\xi, 0)$.

the root of a quintic equation which may be derived from Eq. (7.72). It is possible to show that $\partial^2\text{`}V\text{'}/\partial\xi\,\partial\eta = 0$, $\partial^2\text{`}V\text{'}/\partial\xi^2 < 0$, and $\partial^2\text{`}V\text{'}/\partial\eta^2 > 0$ at these points A, B, and C. If we expand 'V' in a Taylor series about any one of these points, and consider only the quadratic terms, we see that the curves of constant 'V' are hyperbolas in the $\xi\eta$-plane in the neighborhood of points A, B, C, as shown in Fig. 7.8, where we plot the contours of constant 'V'. These points are *saddlepoints* of 'V'; that is, 'V' has a local maximum along the ξ-axis and a minimum along a line perpendicular to the ξ-axis at each of these points A, B, C. If $\eta \neq 0$, it can be factored from the second of Eqs. (7.71). We then multiply the second of Eqs. (7.71) by $(\xi - \xi_1)$ and subtract from the first of these equations. After some manipulation and using Eq. (7.69), we obtain

$$(\xi-\xi_2)^2+\eta^2 = 1, \tag{7.73}$$

and, similarly,

$$(\xi-\xi_1)^2+\eta^2 = 1. \tag{7.74}$$

These equations show that there are two singular points D, E, off the $\eta = 0$ axis,

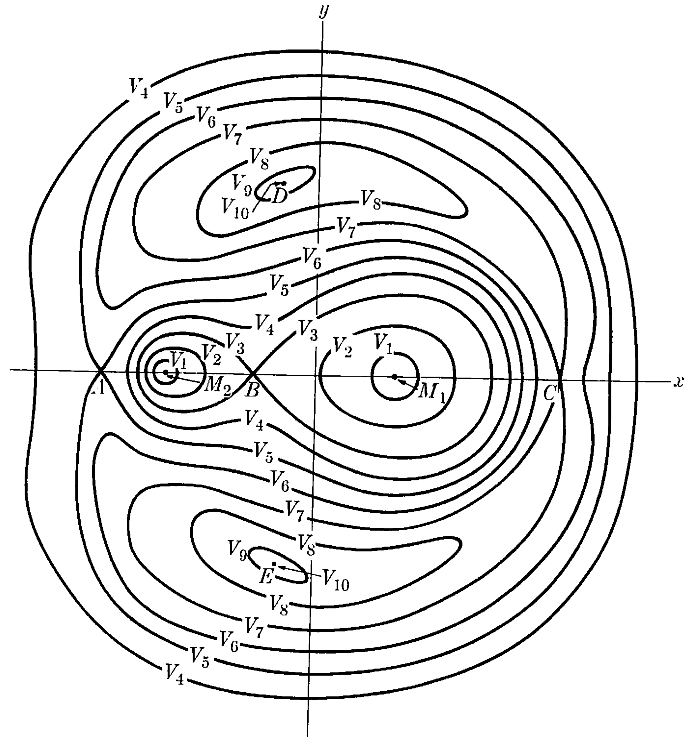

Fig. 7.8 Equipotential contours for 'V' (x, y).

which lie at unit distance from $(\xi_1, 0)$ and $(\xi_2, 0)$ which are themselves separated by a unit distance. By expanding 'V' in a Taylor series about point D or E, we can show that curves of constant 'V' are ellipses in the neighborhood of D or E, and that 'V' has a maximum at D and E. Knowing the behavior near the singular points, we can sketch the general appearance of the contours of constant 'V', as shown in Fig. 7.8. The curves are numbered in order of increasing 'V'.

If this were a fixed coordinate system, we could immediately conclude that equilibrium points A, B, C, D, E are all unstable, since the force $-\nabla$'V' is directed away from each equilibrium point when m is at some nearby points. However, this argument does not hold here because it neglects the coriolis force in Eqs. (7.64). If we expand the right members of Eqs. (7.64) in powers of the displacements (say $x - x_D$, $y - y_D$) from one of the equilibrium points (say D), and retain only linear terms, we may determine approximately the motion near the equilibrium point. If this is done near point D (or E), for example, we find that in linear approximation, the motion near D is stable if one of the masses M_1 or M_2 contains more than about 96% of the total mass $(M_1 + M_2)$. (See Problem 18.) For motions very near to point D, we may expect the linear approximation to yield a solution which is valid for very long times. Whether those motions which are stable in linear approximation are truly stable, in the sense that they remain near point D for all time, was until 1962 one of the unsolved problems of classical mechanics. This matter is discussed further at the end of Section 12.6.*

It is not difficult to show that, even in linear approximation, the equilibrium points A, B, C are unstable. If the motion in linear approximation is unstable, then the exact solution is certainly unstable. That is, regardless of how close m is initially to the equilibrium point (but not at it), it will not, in general, remain as close but will move exponentially away, at least at first. The neglected nonlinear terms may, of course, eventually prevent the solution from going more than some finite distance from the equilibrium point.

The only rigorous statements we can easily make about the motion of m, for very long periods of time, are those which can be derived from the energy equation (7.66). Given an initial position and velocity of m, we can calculate 'E'. The orbit then must remain in the region where 'V' $\leq$ 'E'. For example, motions which start near either mass M_1 or M_2, with 'E' $< V_3$, must remain confined to a region near that mass. Motions with 'E' $> V_5$ *may* go to arbitrarily large distances; whether they actually do, we cannot say from energy arguments.

If we could find another constant of the motion, say $F(x, y, \dot{x}, \dot{y})$, we could solve the problem by methods like those used in Chapter 3 for the central force

*A more complete discussion of the problem of three bodies, on a more advanced level than the present text, will be found in Aurel Wintner, *The Analytical Foundations of Celestial Mechanics*. Princeton: Princeton University Press, 1947. The proof that the motion near points D (or E) is often stable for all time was first given by A. M. Leontovitch in 1962 and is discussed in J. K. Moser, *Lectures on Hamiltonian Systems*, Memoirs of the Amer. Math. Soc., No. 81, 1968.

problem, where the angular momentum is also constant. Unfortunately, no other such constant is known, and it seems likely that none exists which could be used for this purpose. This problem has been studied very extensively.*

Faced with this situation, we may turn to the possibility of computing particular orbits from given initial conditions. This can be done either analytically, by approximation methods, or numerically, and in principle can be done to any desired accuracy and for any desired finite period of time.

In Chapter 12 we shall discuss a closely related special case of the three-body problem.

PROBLEMS

1. (a) Solve the problem of the freely falling body by introducing a translating coordinate system with an acceleration $\boldsymbol{g}$. Set up and solve the equations of motion in this accelerated coordinate system and transform the result back to a coordinate system fixed relative to the earth. (Neglect the earth's rotation.)

b) In the same accelerated coordinate system, set up the equations of motion for a falling body subject to an air resistance proportional to its velocity (relative to the fixed air).

2. A mass m is fastened by a spring (spring constant k) to a point of support which moves back and forth along the x-axis in simple harmonic motion at frequency ω, amplitude a. Assuming the mass moves only along the x-axis, set up and solve the equation of motion in a coordinate system whose origin is at the point of support.

3. Generalize Eq. (5.5) to the case when the origin of the coordinate system is moving, by adding fictitious torques due to the fictitious force on each particle. Express the fictitious torques in terms of the total mass M, the coordinate R^* of the center of mass, and a_h. Compare your result with Eq. (4.25).

4. Derive a formula for d^3A/dt^3 in terms of starred derivatives relative to a rotating coordinate system.

5. Westerly winds blow from west to east in the northern hemisphere with an average speed v. If the density of the air is ρ, what pressure gradient is required to maintain a steady flow of air from west to east with this speed? Make reasonable estimates of v and ρ, and estimate the pressure gradient in lb-in^{-2}-mile^{-1}. (You will need Eq. (5.172) from Chapter 5.)

6. (a) It has been suggested that birds may determine their latitude by sensing the coriolis force. Calculate the force a bird must exert in level flight at 30 mph against the sidewise component of coriolis force in order to fly in a straight line. Express your result in g's, that is, as a ratio of coriolis force to gravitational force, as a function of latitude and direction of flight.

b) If the bird's flight path is slightly circular, a centrifugal force will be present, which will add to the coriolis force and produce an error in estimated latitude. At 45° N latitude, how much may the flight path bend, in degrees per mile flown, if the latitude is to be determined within ±100 miles? (Assume the sidewise force is measured as precisely as necessary!)

*See A. Wintner, *op. cit.*, and J. K. Moser, *op. cit.*

7. A body is dropped from rest at a height h above the surface of the earth.

a) Calculate the coriolis force as a function of time, assuming as a first approximation that it has a negligible effect on the motion, and using the velocity of a freely falling body with acceleration $\boldsymbol{g}_e$. Neglect air resistance, and assume h is small so that $\boldsymbol{g}_e$ can be taken as constant.

b) Now as a second approximation, calculate the net displacement of the point of impact due to the coriolis force calculated in part (a).

***8.** Find the answer to Problem 7(b) by solving for the motion in a nonrotating coordinate system. What approximations are needed to arrive at the same result?

9. An airplane flies across the North pole at 500 mph, following a meridian of longitude (which rotates with the earth). Find the angle between the direction of a plumb line hanging freely in the airplane as it passes over the pole and one hanging freely at the surface of the earth over the pole.

10. Assume the earth is a uniform sphere of mass M, radius R. Imagine a pipe extending vertically from the North pole to the center of the earth and out again at right angles to the equator. The pipe is filled with a fluid so that the fluid level at the North pole is at the earth's surface. Find the fluid level relative to the surface of the sphere at the equator. Does it change the answer much if the pipe runs near the surface of the earth? Does it change the answer much if you use the actual shape of the earth? (You need to know the material in Section 5.11 to answer this problem.)

11. A gyroscope consists of a wheel of radius r, all of whose mass is located on the rim. The gyroscope is rotating with angular velocity $\dot{\theta}$ about its axis, which is horizontal and is fixed relative to the earth's surface. We choose a coordinate system at rest relative to the earth whose z-axis coincides with the gyroscope axis and whose origin lies at the center of the wheel. The angular velocity $\boldsymbol{\omega}$ of the earth lies in the xz-plane, making an angle α with the gyroscope axis.

Find the x-, y-, and z-components of the torque $\boldsymbol{N}$ about the origin, due to the coriolis force in the xyz-coordinate system, acting on a mass m on the rim of the gyroscope wheel whose polar coordinates in the xy-plane are r, θ. Use this result to show that the total coriolis torque on the gyroscope, if the wheel has a mass M, is

$$\boldsymbol{N} = -\tfrac{1}{2}\boldsymbol{j}Mr^2\omega\dot{\theta}\sin\alpha.$$

This equation is the basis for the operation of the gyrocompass.

***12.** A mass m of a perfect gas of molecular weight M, at temperature T, is placed in a cylinder of radius a, height h, and whirled rapidly with an angular velocity ω about the axis of the cylinder. By introducing a coordinate system rotating with the gas, and applying the laws of static equilibrium, assuming that all other body forces are negligible compared with the centrifugal force, show that

$$p = \frac{RT}{M}\rho_0 \exp\left(\frac{M\omega^2 r^2}{2RT}\right),$$

where p is the pressure, r is the distance from the axis, and

$$\rho_0 = \frac{mM\omega^2}{2\pi hRT[\exp(M\omega^2 a^2/2RT) - 1]}.$$

***13.** A particle moves in the xy-plane under the action of a force

$$\boldsymbol{F} = -k\boldsymbol{r},$$

directed toward the origin. Find its possible motions by introducing a coordinate system rotating about the z-axis with angular velocity ω chosen so that the centrifugal force just cancels the force $\boldsymbol{F}$, and solving the equations of motion in this coordinate system. Describe the resulting motions, and show that your result agrees with that of Problem 45, Chapter 3.

14. A ball of mass m slides without friction on a horizontal plane at the surface of the earth. Show that it moves like the bob of a Foucault pendulum provided it remains near the point of tangency. Find its frequency of oscillation. Assume the earth is a sphere.

15. The bob of a pendulum is started so as to swing in a circle. By substituting in Eq. (7.46), find the angular velocity and show that the contribution due to the coriolis force is given very nearly by Eq. (7.52). Neglect the vertical component of the coriolis force, after showing that it is zero on the average for the assumed motion.

16. An electron revolves about a fixed proton in an ellipse of semimajor axis 10^{-8} cm. If the corresponding motion occurs in a magnetic field of 10,000 gausses, show that Larmor's theorem is applicable, and calculate the angular velocity of precession of the ellipse.

17. Write down a potential energy for the last term in Eq. (7.58). If the plane of the orbit in Problem 16 is perpendicular to $\boldsymbol{B}$, and if the orbit is very nearly circular, calculate (by the methods of Chapter 3) the rate of precession of the ellipse due to the last term in Eq. (7.58) in the rotating coordinate system. Is this precession to be added to or subtracted from that calculated in Problem 16?

18. Find the three second derivatives of 'V' with respect to ξ, η for the point D in Fig. 7.7. Expand the equations of motion (7.64), keeping terms linear in $\xi' = \xi - \xi_0$ and $\eta' = \eta - \eta_0$. Using the method of Section 4.10, find the condition on M_1, M_2 in order that the normal modes of oscillation be stable. If $M_1 > M_2$, what is the minimum value of $M_1/(M_1 + M_2)$?

***19.** Prove the statements made in Section 7.6 regarding the second derivatives of 'V' at points A, B, and C in Fig. 7.7. Expand the equations of motion about points A and B, keeping terms linear in η and $\xi' = \xi - \xi_{A,B}$. Show by the method of Section 4.10 that some of the solutions are unstable for any values of the masses. (You cannot find the second derivatives explicitly, but the proof depends only on their signs.)

***20.** (a) Write out the quintic equation which must be solved for ξ_A in Fig. 7.7. Show that if $M_2 = 0$, the solution is $\xi_A = -1$.

b) Solve numerically for ξ_A to two decimal places for the earth-moon system.

c) Find the minimum launching velocity from the surface of the earth for which it is 'energetically' possible for a rocket to leave the earth-moon system. Compare with the escape velocity from the earth.

21. Two planets, each of mass M and radius R, revolve in circles about each other at a distance a apart, under their mutual gravitational attraction. Find the minimum (relative) velocity with which a rocket might leave one planet to arrive at the other. Show that the rocket must have a larger velocity than would be calculated if the motion of the planets were neglected.

22. (a) Locate all fixed points in the limiting case $M_2 \to 0$, and sketch Fig. 7.7 for this case. Show that the results in Section 7.6 applied to this case are consistent with the complete solution given in Section 3.14.

b) Show from this example for which the complete solution is known, that the minimum 'energetically' possible launching velocity for escape calculated as in Problem 18(c) is not necessarily the true minimum escape velocity.

CHAPTER 8

INTRODUCTION TO THE MECHANICS OF CONTINUOUS MEDIA

In this chapter we begin the study of the mechanics of continuous media, solids, fluids, strings, etc. In such problems, the number of particles is so large that it is not practical to study the motion of individual particles, and we instead regard matter as continuously distributed in space and characterized by its density. We are interested primarily in gaining an understanding of the concepts and methods of treatment which are useful, rather than in developing in detail methods of solving practical problems. In the first four sections, we shall treat the vibrating string, using concepts which are a direct generalization of particle mechanics. In the remainder of the chapter, the mechanics of fluids will be developed in a way less directly related to particle mechanics.

8.1 THE EQUATION OF MOTION FOR THE VIBRATING STRING

In this section we shall study the motion of a string of length l, stretched horizontally and fastened at each end, and set into vibration. In order to simplify the problem, we assume the string vibrates only in a vertical plane, and that the amplitude of vibration is small enough so that each point on the string moves only vertically, and so that the tension τ in the string does not change appreciably during the vibration.

We shall designate a point on the string by giving its horizontal distance x from the left-hand end (Fig. 8.1). The distance the point x has moved from the horizontal straight line representing the equilibrium position of the string will be designated by $u(x)$. Thus any position of the entire string is to be specified by specifying the function $u(x)$ for $0 \leq x \leq l$. This is precisely analogous, in the case of a system of N particles, to specifying the coordinates x_i, y_i, z_i, for $i = 1, \ldots, N$. In the case of the string, x is not a coordinate, but plays the same role as the subscript i; it designates a point on the string. Our idealized continuous string has infinitely many points, corresponding to the infinitely many values of x between 0 and l. For a given point x, it is $u(x)$ that plays the role of a coordinate locating

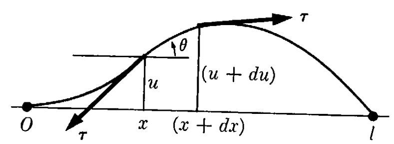

Fig. 8.1 The vibrating string.