The Restricted 3-body Problem (2D)

1. m_1 is large and fixed at the origin $(x_1 = 0, y_1 = 0)$
2. m_2 is next largest and, feeling a gravitational force due to m_1, orbits in a Keplerian ellipse. The force on 2 due to 1 $\vec{F}_{21}$ is

$$\vec{F}_{21} = -\frac{G m_1 m_2}{r_2^2}\hat{r}_2 = -G m_1 m_2 \frac{\vec{r}_2}{r_2^3}$$

where $\vec{r}_2 = x_2\hat{\imath} + y_2\hat{\jmath}$

$r_2^2 = x_2^2 + y_2^2$

The equation of motion for m_2 is $\vec{F}_{21} = m_2\vec{a}_2$

where $\vec{a}_2 = \frac{d\vec{v}_2}{dt} = \frac{d}{dt}v_{2x}\hat{\imath} + \frac{d}{dt}v_{2y}\hat{\jmath}$

OR

$$\left\{\begin{aligned} &(1) \quad \frac{d}{dt}v_{2x} = -\mu_1\frac{x_2}{r_2^3} \\ &(2) \quad \frac{d}{dt}v_{2y} = -\mu_1\frac{y_2}{r_2^3} \end{aligned}\right\} \quad \text{where } \mu_1 = Gm_1$$

In addition, we need to know how the position of m_2 (x_2, y_2) changes with time

$$\left\{\begin{aligned} &(3) \quad \frac{d}{dt}x_2 = v_{2x} \\ &(4) \quad \frac{d}{dt}y_2 = v_{2y} \end{aligned}\right\}$$

These are four (4) coupled, ordinary, first-order, differential equations — Easily solved numerically.

3. m_3 is even smaller ($m_3 < m_2 < m_1$) and feels a gravitational force due to both m_1 and m_2. {NOTE: m_3 is so small that it does not exert a gravitational force on m_1 or m_2.}

The equation of motion for m_3 is $\vec{F}_{31} + \vec{F}_{32} = m_3 \vec{a}_3$, where the definitions are as above. $\vec{F}_{31}$ is similar to $\vec{F}_{21}$, but $\vec{F}_{32}$ is

$$\vec{F}_{32} = - m_3 \mu_2 \frac{1}{r_{32}^3} \left\{ (x_3 - x_2)\hat{i} + (y_3 - y_2)\hat{j} \right\}$$

where $r_{32}^2 = (x_3 - x_2)^2 + (y_3 - y_2)^2$

Hence, the scalar components of the vector equation of motion are

$$\left\{ \begin{aligned} &\frac{d}{dt} v_{3x} = -\mu_1 \frac{x_3}{r_3^3} - \mu_2 \frac{x_3 - x_2}{r_{32}^3} && (5) \\ &\frac{d}{dt} v_{3y} = -\mu_1 \frac{y_3}{r_3^3} - \mu_2 \frac{y_3 - y_2}{r_{32}^3} && (6) \end{aligned} \right\}$$

where $\mu_2 = G m_2$

and, of course, we need to know how the position changes with time

$$\left\{ \begin{aligned} &\frac{d}{dt} x_3 = v_{3x} && (7) \\ &\frac{d}{dt} y_3 = v_{3y} && (8) \end{aligned} \right\}$$

We now have 8 coupled differential equations.

Unfortunately, computers do not keep track of units, so before we feed these equations to the machine, we need to make them dimensionless and get rid of variables with units.

It is not clear what is the natural time scale or the natural length scale for this problem, so we choose arbitrary values at first.

Let

$$\bar{t} \equiv \frac{t}{T} \qquad \bar{x} = \frac{x}{X} \qquad \bar{v} = \frac{v}{(X/T)} = \frac{T}{X} v$$

where $\bar{t}$, $\bar{x}$, $\bar{v}$ are dimensionless time, position and velocity, respectively. Of course, y scales just as x does.

Substituting these into Eq (1) give

$$\frac{X}{T^2} \frac{d}{d\bar{t}} \bar{v}_{2x} = -\mu_1 \frac{1}{X^2} \frac{\bar{x}_2}{\bar{r}_2^{\,3}}$$

Rearranging give

(9) $$\frac{d}{d\bar{t}} \bar{v}_{2x} = -\left(\mu_1 \frac{T^2}{X^3}\right) \frac{\bar{x}_2}{\bar{r}_2^{\,3}}$$

All the dimensions in the problem are in the terms in parentheses – everything else is dimensionless – and therefore the entire parenthetical term must be dimensionless. It makes sense to set it equal to 1 for computation.

(10) $$\mu_1 \frac{T^2}{X^3} = 1$$

So it is clear that there is not one length or time scale, but, once we choose a length scale $\bar{X}$, the time scale is chosen by Eq (10). In fact Eq (10) is just Kepler's 3rd law: the square of the orbital period is equal to the cube of the orbital radius.

Eq (2) results in the same eqn, and Eqs (3)(4) are automatically satisfied. If we try the same technique on Eq (5), we get

$$\frac{d}{d\bar{t}}\bar{v}_{3x} = -\left(\mu_1 \frac{T^2}{\bar{X}^3}\right)\frac{\bar{x}_3}{\bar{r}_3^{\,3}} - \left(\mu_2 \frac{T^2}{\bar{X}^3}\right)\frac{\bar{x}_3 - \bar{x}_2}{\bar{r}_{32}^{\,3}}$$

We can make the same choice as in Eq (10), but that only gets rid of one coefficient. The other becomes a ratio of the gravitational parameters:

(11) $$\frac{d}{d\bar{t}}\bar{v}_{3x} = -\frac{\bar{x}_3}{\bar{r}_3^{\,3}} - \frac{\mu_2}{\mu_1}\frac{\bar{x}_3 - \bar{x}_2}{\bar{r}_{32}^{\,3}}$$

where $\dfrac{\mu_2}{\mu_1} = \dfrac{G m_2}{G m_1} = \dfrac{m_2}{m_1}$ a dimensionless mass ratio

Of course, Eq (6) is similar, and Eqs (7)(8) automatically become dimensionless.