23
Analytic and Differential Geometry in the Eighteenth Century

> Geometry may sometimes appear to take the lead over analysis but in fact precedes it only as a servant goes before the master to clear the path and light him on his way.
>
> JAMES JOSEPH SYLVESTER

1. *Introduction*

The exploration of physical problems led inevitably to the search for greater knowledge of curves and surfaces, because the paths of moving objects are curves and the objects themselves are three-dimensional bodies bounded by surfaces. The mathematicians, already enthusiastic about the method of coordinate geometry and the power of the calculus, approached geometrical problems with these two major tools. The impressive results of the century were obtained in the already established area of coordinate geometry and the new field created by applying the calculus to geometrical problems, namely, differential geometry.

2. *Basic Analytic Geometry*

Two-dimensional analytic geometry was extensively explored in the eighteenth century. The improvements in elementary plane analytics are readily summarized. Whereas Newton and James Bernoulli had introduced and used what are essentially polar coordinates for special curves (Chap. 15, sec. 5), Jacob Hermann in 1729 not only proclaimed their general usefulness, but applied them freely to study curves. He also gave the transformation from rectangular to polar coordinates. Strictly Hermann used as variables p, $\cos\theta$, $\sin\theta$, which he designated by z, n, and m. Euler extended the use of polar coordinates and used trigonometric notation explicitly; with him the system is practically modern.

Though some seventeenth-century men—for example, Jan de Witt

(1625–72) in his *Elementa Curvarum Linearum* (1659)—did reduce some second degree equations in x and y to standard forms, James Stirling, in his *Lineae Tertii Ordinis Neutonianae* (1717), reduced the general second degree equation in x and y to the several standard forms.

In his *Introductio* (1748) Euler introduced the parametric representation of curves, wherein x and y are expressed in terms of a third variable. In this famous text Euler treated plane coordinate geometry systematically.

The suggestion of three-dimensional coordinate geometry can be found, as we know, in the work of Fermat, Descartes, and La Hire. The actual development was the work of the eighteenth century. Though some of the early work, Pitot's and Clairaut's for example, is tied up with the development of differential geometry, we shall consider only coordinate geometry proper at this point.

The first task was the improvement of La Hire's suggestion of a three-dimensional coordinate system. John Bernoulli, in a letter to Leibniz of 1715, introduced the three coordinate planes we use today. Through contributions too detailed to warrant space here, Antoine Parent (1666–1716), John Bernoulli, Clairaut, and Jacob Hermann clarified the notion that a surface can be represented by an equation in three coordinates. Clairaut, in his book *Recherche sur les courbes à double courbure* (Research on the Curves of Double Curvature, 1731), not only gave the equations of some surfaces but made clear that two such equations are needed to describe a curve in space. He also saw that certain combinations of the equations of two surfaces passing through a curve, the sum for example, give the equation of another surface passing through the curve. Using this fact, he explains how one can obtain the equations of the projections of these curves or, equivalently, the equations of the cylinders perpendicular to the planes of projection.

The quadric surfaces, e.g. sphere, cylinder, paraboloid, hyperboloid of two sheets, and ellipsoid, were of course known geometrically before 1700; in fact, some of them appear in Archimedes' work. Clairaut in his book of 1731 gave the equations of some of these surfaces. He also showed that an equation that is homogeneous in x, y, and z (all terms are of the same degree) represents a cone with vertex at the origin. To this result Jacob Hermann, in a paper of 1732,[1] added that the equation $x^2 + y^2 = f(z)$ is a surface of revolution about the z-axis. Both Clairaut and Hermann were primarily concerned with the shape of the earth, which by their time was believed to be some form of ellipsoid.

Though Euler had done some earlier work on the equations of surfaces, it is in Chapter 5 of the Appendix to the second volume of his *Introductio* (1748)[2] that he systematically takes up three-dimensional coordinate

1. *Comm. Acad. Sci. Petrop.*, 6, 1732/33, 36–67, pub. 1738.
2. *Opera*, (1), 9.

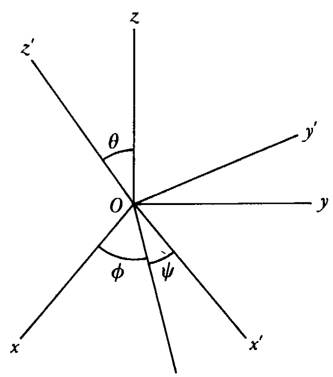

Figure 23.1

geometry. He presents much of what had already been done and then studies the general second degree equation in three variables

$$(1)\qquad ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + kz = l.$$

He now seeks to use change of axes to reduce this equation to the forms that result from having the principal axes of the quadric surfaces represented by (1) as the coordinate axes. He introduces the transformation from the xyz-system to the $x'y'z'$-system, whose equations are expressed (Fig. 23.1) in terms of the angles ϕ, ψ, and θ. The angle ϕ is measured in the xy-plane from the x-axis to the line of nodes, which is the line in which the $x'y'$-plane cuts the xy-plane. The angle ψ is measured in the $x'y'$-plane and locates x' with respect to the line of nodes. The angle shown is θ. Then the equations of transformation, including translation, are

$$\begin{aligned} x &= x_0 + x'(\cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi) \\ &\qquad - y'(\cos\psi\sin\phi + \cos\theta\sin\psi\sin\phi) + z'\sin\theta\sin\phi \\ (2)\quad y &= y_0 + x'(\sin\psi\cos\phi + \cos\theta\cos\psi\sin\phi) \\ &\qquad - y'(\sin\psi\sin\phi - \cos\theta\cos\psi\sin\phi) - z'\sin\theta\sin\phi \\ z &= z_0 + x'\sin\theta\sin\phi + y'\sin\theta\cos\phi + z'\cos\theta. \end{aligned}$$

Euler uses this transformation to reduce (1) to canonical forms and obtains six distinct cases: cone, cylinder, ellipsoid, hyperboloid of one and two sheets, hyperbolic paraboloid (which he discovered), and parabolic cylinder. Like Descartes, Euler maintained that classification by the degree of the equation was the correct principle; Euler's reason was that the degree is invariant under linear transformation.

After continuing work on this problem of change of axes, he wrote another paper,[3] in which he considers the transformation that will carry

3. *Novi Comm. Acad. Sci. Petrop.*, 15, 1770, 75–106, pub. 1771 = *Opera*, (1), 6, 287–315.

$x^2 + y^2 + z^2$ into $x'^2 + y'^2 + z'^2$. Here he—and Lagrange a little later, in a paper on the attraction of spheroids[4]—gave the symmetric form for the rotation of axes, the homogeneous linear orthogonal transformation

$$\begin{aligned} x &= \lambda x' + \mu y' + \nu z' \\ y &= \lambda' x' + \mu' y' + \nu' z' \\ z &= \lambda'' x' + \mu'' y' + \nu'' z', \end{aligned}$$

where

$$\begin{aligned} \lambda^2 + \lambda'^2 + \lambda''^2 &= 1 &\qquad \lambda\mu + \lambda'\mu' + \lambda''\mu'' &= 0 \\ \mu^2 + \mu'^2 + \mu''^2 &= 1 & \lambda\nu + \lambda'\nu' + \lambda''\nu'' &= 0 \\ \nu^2 + \nu'^2 + \nu''^2 &= 1 & \mu\nu + \mu'\nu' + \mu''\nu'' &= 0. \end{aligned}$$

The λ, μ, and ν, unprimed and primed, are of course direction cosines in today's terminology.

Gaspard Monge's writings contain a great deal of three-dimensional analytic geometry. His outstanding contribution to analytic geometry as such is to be found in the paper of 1802 written with his pupil Jean-Nicolas-Pierre Hachette (1769–1834), "Application de l'algèbre à la géométrie."[5] The authors show that every plane section of a second degree surface is a second degree curve, and that parallel planes cut out similar and similarly placed curves. These results parallel Archimedes' geometric theorems. The authors also show that the hyperboloid of one sheet and the hyperbolic paraboloid are ruled surfaces, that is, each can be generated in two different ways by the motion of a line or each surface is formed by two systems of lines. The result on the one-sheeted hyperboloid was known by 1669 to Christopher Wren, who said that this figure could be generated by revolving a line about another not in the same plane. With the work of Euler, Lagrange, and Monge, analytic geometry became an independent and full-fledged branch of mathematics.

3. *Higher Plane Curves*

The analytic geometry described thus far was devoted to curves and surfaces of the first and second degree. It was of course natural to investigate the curves of equations of higher degree. In fact, Descartes had already discussed such equations and their curves somewhat. The study of curves of degree higher than two became known as the theory of higher plane curves, though it is part of coordinate geometry. The curves studied in the eighteenth century were algebraic; that is, their equations are given by $f(x, y) = 0$ where f is a polynomial in x and y. The degree or order is the highest degree of the terms.

4. *Nouv. Mém. de l'Acad. de Berlin*, 1773, 85–120 = *Œuvres*, 3, 619–58.
5. *Jour. de l'Ecole Poly.*, 11 cahier, 1802, 143–69.

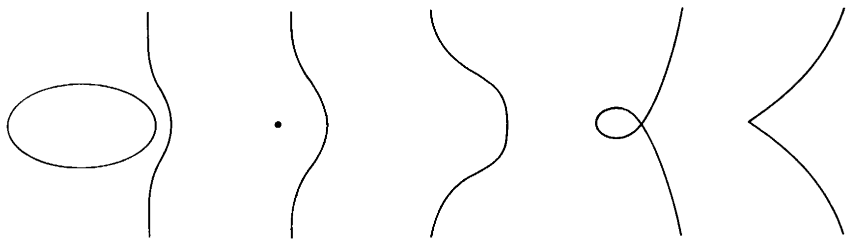

Figure 23.2

The first extensive study of higher plane curves was made by Newton. Impressed by Descartes's plan to classify curves according to the degree of their equations and then to study systematically each degree by methods suited to that degree, Newton undertook to study third degree curves. This work appeared in his *Enumeratio Linearum Tertii Ordinis*, which was published in 1704 as an appendix to the English edition of his *Opticks* but had been composed by 1676. Though the use of negative x- and y-values appears in works of La Hire and Wallis, Newton not only uses two axes and negative x- and y-values but plots in all four quadrants.

Newton showed how all curves comprised by the general third degree equation

$$(3) \qquad ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + jy + k = 0$$

can, by a change of axes, be reduced to one of the following four forms:

$$\begin{aligned} &(a) \quad xy^2 + ey = ax^3 + bx^2 + cx + d \\ &(b) \quad xy = ax^3 + bx^2 + cx + d \\ &(c) \quad y^2 = ax^3 + bx^2 + cx + d \\ &(d) \quad y = ax^3 + bx^2 + cx + d. \end{aligned}$$

The third class, which Newton called diverging parabolas, contains five species of curves whose types are shown in Figure 23.2. The species are distinguished by the nature of the roots of the cubic right-hand member, as follows: all real and distinct; two roots complex; all real but two equal and the double root greater or less than the simple root; and all three equal. Newton affirmed that every cubic curve can be obtained by projection of one of these five types from a point and then by a section of the projection.

Newton gave no proofs of many of the assertions in his *Enumeratio*. In his *Lineae*, James Stirling proved or reproved in other ways most of the assertions but not the projection theorem, which Clairaut[6] and François Nicole (1683–1758)[7] proved. Also, whereas Newton recognized seventy-two species of third degree curves, Stirling added four more and abbé Jean-Paul de Gua de Malves, in a little book of 1740 entitled *Usage de l'analyse de Descartes pour découvrir sans le secours du calcul différentiel . . .*, added two more.

6. *Mém. de l'Acad. des Sci., Paris*, 1731, 490–93, pub. 1733.
7. *Mém. de l'Acad. des Sci., Paris*, 1731, 494–510, pub. 1733.

Newton's work on third degree curves stimulated much other work on higher plane curves. The topic of classifying third and fourth degree curves in accordance with one or another principle continued to interest mathematicians of the eighteenth and nineteenth centuries. The number of classes found varied with the methods of classification.

As is evident from the figures of Newton's five species of cubic curves, the curves of higher-degree equations exhibit many peculiarities not found in first and second degree curves. The elementary peculiarities, called singular points, are inflection points and multiple points. Before proceeding, let us see what some of them look like.

Inflection points are familiar from the calculus. A point at which there are two or more tangents which may coincide, is called a multiple point. At such a point two or more branches of the curve intersect. If two branches intersect at the multiple point, it is called a double point. If three branches intersect then the point is called a triple point, and so on.

If we take the equation of an algebraic curve

$$f(x, y) = 0,$$

f being a polynomial in x and y, we can by a translation always remove the constant term. If this is done and if there are first degree terms in f, say $a_1x + b_1y$, then $a_1x + b_1y = 0$ gives the equation of the tangent to the curve at the origin. The origin is not in this case a multiple point. If there are no first degree terms, and if $a_2x^2 + b_2xy + c_2y^2$ are the second degree terms, then several cases arise. The equation $a_2x^2 + b_2xy + c_2y^2 = 0$ may represent two distinct lines. These lines are tangents at the origin (this can be proven), and since there are two distinct tangents the origin is a double point; it is called a node. Thus the equation of the lemniscate (Fig. 23.3) is

$$(4) \qquad a^2(y^2 - x^2) + (y^2 + x^2)^2 = 0$$

and the second degree terms yield $y^2 - x^2 = 0$. Then $y = x$ and $y = -x$ are the equations of the tangents. Likewise, the folium of Descartes (Fig. 23.4) has the equation

$$(5) \qquad x^3 + y^3 = 3axy$$

and the tangents at the origin, which is a node, are given by $x = 0$ and $y = 0$.

When the two tangent lines are coincident, the single line is considered as a double tangent and the two branches of the curve touch each other at the point of tangency, which is called a cusp. (Sometimes cusps are

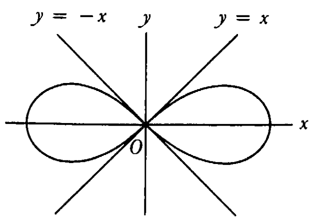

Figure 23.3. Lemniscate

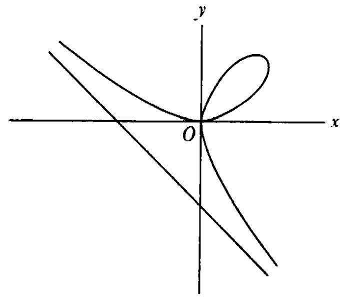

Figure 23.4. Folium of Descartes

included among the double points.) Thus the semicubical parabola (Fig. 23.5)

(6) $$ay^2 = x^3$$

has a cusp at the origin, and the equation of the two coincident tangents is $y^2 = 0$. On the curve $(y - x^2)^2 = x^5$ (Fig. 23.6) the origin is a cusp. Here both branches lie on the same side of the double tangent, which is $y = 0$. De Gua in his *Usage* had tried to show that this type of cusp could not occur, but Euler[8] gave many examples. A cusp is also called a stationary point or point of retrogression because a point moving along the curve must come to rest before continuing its motion at a cusp.

When the two tangent lines are imaginary, the double point is called a conjugate point. The coordinates of the point satisfy the equation of the curve but the point is isolated from the rest of the curve. Thus the curve (Fig. 23.7) of $y^2 = x^2(2x - 1)$ has a conjugate point at the origin. The equation of the double tangent there is $y^2 = -x^2$ and the tangents are imaginary.

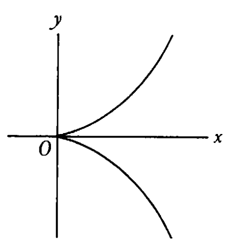

Figure 23.5. Semicubical parabola

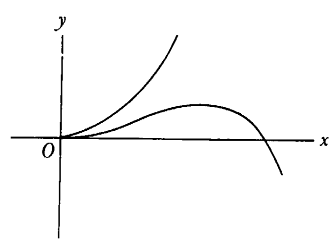

Figure 23.6

8. *Mém. de l'Acad. de Berlin*, 5, 1749, 203–21, pub. 1751 = *Opera*, (1), 27, 236–52.

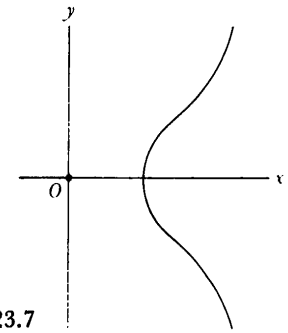

Figure 23.7

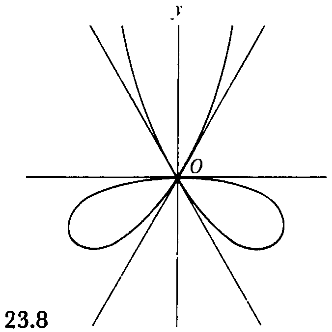

Figure 23.8

The curve of $ay^3 - 3ax^2y = x^4$ (Fig. 23.8) has a triple point at the origin. The equation of the three tangents is

$$ay^3 - 3ax^2y = 0$$

or $y = 0$ and $y = \pm x\sqrt{3}$.

The curve of $ay^4 - ax^2y^2 = x^5$ (Fig. 23.9) has a quadruple point at the origin. The origin is a combination of a node and a cusp. The tangents are $y = 0, y = 0, y = \pm x$.

Curves of the third degree (order) may have a double point (which may be a cusp) but no other multiple point. There are of course cubics with no double point.

To return to the history proper, many of these special or singular points on curves were studied by Leibniz and his successors. The analytical conditions for such points, such as that $\ddot{y} = 0$ at an inflection point and that $\dot{y}$ is indeterminate at a double point, were known even to the founders of the calculus.

Clairaut in the 1731 book referred to above assumed that a third degree curve cannot have more than three real inflection points and must have at least one. De Gua in *Usage* proved that if a third degree curve has three real inflection points, a line through two of them passes through the third

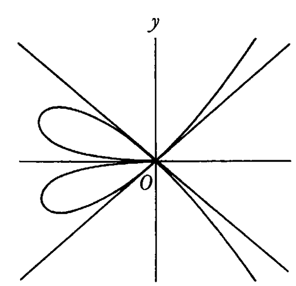

Figure 23.9

one. This theorem is often credited to Maclaurin. De Gua also investigated double points and gave the condition that if $f(x, y) = 0$ is the equation of the curve, then f_x and f_y must be 0 at a double point. A k-fold point is characterized by the vanishing of all partial derivatives through the $(k - 1)$st order. He showed that the singularities are compounded of cusps, ordinary points, and points of inflection. In addition, de Gua treated middle points of curves, the form of branches extending to infinity, and properties of such branches.

Maclaurin in his *Geometria Organica* (1720), written when he was nineteen, proved that the maximum number of double points of an irreducible curve of the nth degree is $(n - 1)(n - 2)/2$. For this purpose he counted a k-fold point as $k(k - 1)/2$ double points. He also gave upper bounds for the number of higher multiple points of each kind. He then introduced the notion of the deficiency (later called genus) of an algebraic curve as the maximum possible number of double points minus the actual number. Among curves those of deficiency 0 or possessing the maximum possible number of double points received a great deal of attention. Such curves are also called rational or unicursal. Geometrically a unicursal curve may be traversed by the continuous motion of a moving point (which may, however, pass through the point at infinity). Thus the conics, including the hyperbola, are unicursal curves.

In his *Method of Fluxions* Newton gave a method, commonly referred to as Newton's diagram or Newton's parallelogram, for determining series representations of the various branches of a curve at a multiple point (Chap. 20, sec. 2). De Gua in *Usage* replaced the Newton parallelogram by a *triangle algébrique*. Then if the origin is a singular point, for small x the equation of an algebraic curve breaks down into factors of the form $y^m - Ax^n$, where m is positive integral and n integral. The branches of the curve are given by those factors for which n is also positive. Euler noted (1749) that de Gua had neglected imaginary branches.

Gabriel Cramer (1704–52), in his *Introduction à l'analyse des lignes courbes algébriques* (1750), also tackled the expansion of y in terms of x when y and x are given in an implicit function, that is, $f(x, y) = 0$, in order to determine a series expression for each branch of the curve, particularly the branches that extend to infinity. He treated y as an ascending and as a descending series of powers of x. Like de Gua he used the triangle in place of Newton's parallelogram, and like others he neglected imaginary branches of the curves.

The conclusion that resulted from the work of obtaining series expansions for each branch of a curve issuing from a multiple point was drawn much later by Victor Puiseux (1820–83)[9] and is known as Puiseux's theorem:

9. *Jour. de Math.*, 13, 1850, 365–480.

The total neighborhood of a point (x_0, y_0) of an algebraic plane curve can be expressed by a finite number of developments

$$(7) \qquad y - y_0 = a_1(x - x_0)^{q_1/q_0} + a_2(x - x_0)^{q_2/q_0} + \cdots.$$

These developments converge in some interval about x_0 and all the q_i have no common factors. The points given by each development are called a branch of the algebraic curve.

The intersections of a curve and line and of two curves is another topic that received a great deal of attention. Stirling, in his *Lineae* of 1717, showed that an algebraic curve of the nth degree (in x and y) is determined by $n(n + 3)/2$ of its points because it has that number of essential coefficients. He also asserted that any two parallel lines cut a given curve in the same number of points, real or imaginary, and he showed that the number of branches of a curve that extend to infinity is even. Maclaurin's work, *Geometria Organica*, founded the theory of intersections of higher plane curves. He generalized on results for special cases and on this basis concluded that an equation of the mth degree and one of the nth degree intersect in mn points.

In 1748 Euler and Cramer sought to prove this result, but neither gave a correct proof. Euler[10] relied upon an argument by analogy; realizing that his argument was not complete, he said one should apply the method to particular examples. Cramer's "proof" in his book of 1750 relied entirely on examples and was certainly not acceptable. Both men took into account points of intersection with imaginary coordinates and infinitely distant common points and noted that the number mn will be attained only if both types of points are included and if any factor, such as $ax + by$, common to both curves is excluded. However, both failed to assign the proper multiplicity to several types of intersections. In 1764 Etienne Bezout (1730–83) gave a better proof of the theorem, but this was also incomplete in the count of the multiplicity assigned to points at infinity and multiple points. The proper count of the multiplicity was settled by Georges-Henri Halphen (1844–89) in 1873.[11]

In his book of 1750 Cramer took up a paradox noted by Maclaurin in his *Geometria* concerning the number of points common to two curves. A curve of degree n is determined by $n(n + 3)/2$ points. Two nth degree curves meet in n^2 points. Now if n is 3, the curve should, by the first statement, be determined by 9 points. But since two third degree curves meet in 9 points, these 9 points do not determine a unique third degree curve. A similar paradox arises when $n = 4$. Cramer's explanation of the paradox, now referred to as his, was that the n^2 equations that determine the n^2 points of intersection are not independent. All cubics that pass through 8 fixed

10. *Mém. de l'Acad. de Berlin*, 4, 1748, 234–48 = *Opera*, (1), 26, 46–59.
11. *Bull. Soc. Math. de France*, 1, 1873, 130–48; 2, 1873, 34–52; 3, 1875, 76–92 = *Œuvres*, 1, 98–157, 171–93, 337–57.

points on a given cubic must pass through the same ninth fixed point. That is, the ninth point is dependent on the first 8. Euler gave the same explanation in 1748.[12]

In 1756 Matthieu B. Goudin (1734–1817) and Achille-Pierre Dionis du Séjour (1734–94) wrote the *Traité des courbes algébriques*. Its new features are that a curve of order (degree) n cannot have more than $n(n-1)$ tangents with a given direction, nor more than n asymptotes. As had Maclaurin, they pointed out that an asymptote cannot cut the curve in more than $n-2$ points.

The two best eighteenth-century compendia of results on higher plane curves are the second volume of Euler's *Introductio* (1748) and Cramer's *Lignes courbes algébriques*. The latter book has a unity of viewpoint, is excellently set forth, and contains good examples. The work was often cited, even to the point of crediting Cramer with some results that were not original with him.

4. *The Beginnings of Differential Geometry*

Differential geometry was initiated while analytic geometry was being extended, and the two developments were often intertwined. Interest in the theory of algebraic curves waned during the latter part of the eighteenth century, and differential geometry became more important as far as geometry was concerned. This subject is the study of those properties of curves and surfaces that vary from point to point and therefore can be grasped only with the techniques of the calculus. The term "differential geometry" was first used by Luigi Bianchi (1856–1928) in 1894.

To a large extent, differential geometry was a natural outgrowth of problems of the calculus itself. Consideration of normals to curves, points of inflection, and curvature is actually the differential geometry of plane curves. However, many new problems of the late seventeenth and early eighteenth centuries, more knowledge about the curvature of plane and space curves, envelopes of families of curves, geodesics on surfaces, the study of rays of light and of wave surfaces of light, dynamical problems of motion along curves and constraints posed by surfaces, and, above all, map-making led to questions about curves and surfaces; it became evident that the calculus must be applied.

The eighteenth- and even early nineteenth-century workers in differential geometry used geometrical arguments along with analytic ones, although the latter dominated the picture. The analysis was still crude. An infinitesimal or differential of an independent variable was regarded as an extremely small constant. No real distinction was drawn between the increment of a

12. *Mém. de l'Acad. de Berlin*, 4, 1748, 219–33, pub. 1750 = *Opera*, (1), 26, 33–45.

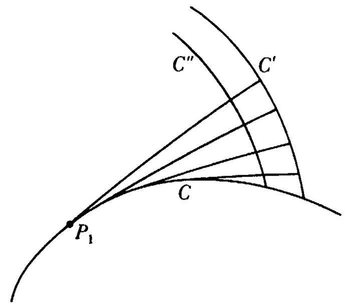

Figure 23.10

dependent variable and the differential. Differentials of higher orders were considered, but all were regarded as small and freely neglected. The mathematicians spoke of adjacent or next points on a curve as though there were no points between two adjacent points if the distance between two was sufficiently small; thus a tangent to a curve connected a point with the next one.

5. *Plane Curves*

The first applications of the calculus to curves dealt with plane curves. Some of the concepts subsequently treated by the calculus were introduced by Christian Huygens, who used purely geometrical methods. His work in this direction was motivated by his interest in light and in the design of pendulum clocks. In 1673, in the third chapter of his *Horologium Oscillatorium*, he introduced the involute of a plane curve C. Imagine a cord wrapped around C from P_1 to the right (Fig. 23.10). The end at P_1 on C is held fixed and the other unwound while the cord is kept taut. The locus C' of the free end is an involute of C. Huygens proved that at the free end the cord is perpendicular to the locus C'. Each point of the cord also describes an involute; thus C'' is also an involute, and Huygens proved that the involutes cannot touch one another. Since the cord is tangent to C at the point where it just leaves C, it follows that every orthogonal trajectory of the family of tangents to a curve is an involute of the curve.

Huygens then treated the evolute of a plane curve. Given a fixed normal at a point P on a curve, as an adjacent normal moves toward it the point of intersection of the two normals attains a limiting position on the fixed normal, which is called the center of curvature of the curve at P. The distance from the point on the curve along the fixed normal to the limiting position was shown by Huygens to be (in modern notation)

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

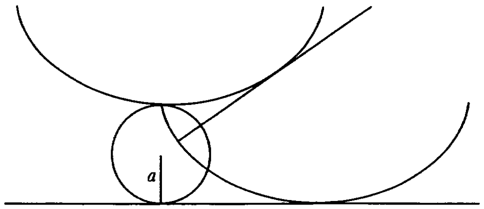

Figure 23.11

This length is the radius of the curvature of the curve at P. The locus of the centers of curvature, one on each normal, is called the evolute of the original curve. Thus the curve C (Fig. 23.10) is the evolute of any one of its involutes. In this work Huygens proved that the evolute of a cycloid is a cycloid or, more precisely, the evolute of the left half of the lower cycloid in Figure 23.11 is the right half of the upper cycloid. This theorem was proved analytically by Euler in 1764.[13] The significance of the cycloid for Huygens's work on pendulum clocks is that a pendulum bob swinging along a cycloidal arc takes exactly the same time to complete swings of large and of small amplitude. For this reason the cycloid is called the tautochrone.

Newton, too, in his *Geometria Analytica* (published in 1736 though most of it was written by 1671) introduces the center of curvature as the limiting point of intersection of a normal at P with an adjacent normal. He then states that the circle with center at the center of curvature and radius equal to the radius of curvature is the circle of closest contact with the curve at P; that is, no other circle tangent to the curve at P can come between the curve and the circle of closest contact. This circle of closest contact is called the osculating circle, the term "osculating" having been used by Leibniz in a paper of 1686.[14] The curvature of this circle is the reciprocal of its radius and is the curvature of the curve at P. Newton also gave the formula for the curvature and calculated the curvature of several curves, including the cycloid. He noted that at a point of inflection a curve has zero curvature. These results duplicate those of Huygens, but probably Newton wished to show that he could use analytical methods to establish them.

In 1691 John Bernoulli took up the subject of plane curves and produced some new results on envelopes. The caustic of a family of light rays, that is, the envelope of the family, had been introduced by Tschirnhausen in 1682. In the *Acta Eruditorum* of 1692 Bernoulli obtained the equations of some caustics, for example, the caustic of rays reflected from a spherical mirror when a beam of parallel rays strikes it.[15] Then he tackled the problem posed

13. *Novi Comm. Acad. Sci. Petrop.*, 10, 1764, 179–98, pub. 1766 = *Opera*, (1), 27, 384–400.
14. *Acta Erud.*, 1686, 289–92 = *Math. Schriften*, 7, 326–29.
15. *Opera*, 1, 52–59.

to him by Fatio de Duillier, to find the envelope of the family of parabolas that are the paths of cannon balls fired from a cannon with the same initial velocity but at various angles of elevation. Bernoulli showed that the envelope is a parabola with focus at the gun. This result had already been established geometrically by Torricelli. In the *Acta Eruditorum* of 1692 and 1694[16] Leibniz gave the general method of finding the envelope of a family of curves. If the family is given by (in our notation) $f(x, y, \alpha) = 0$, where α is the parameter of the family, the method calls for eliminating α between $f = 0$ and $\partial f/\partial\alpha = 0$. L'Hospital's text, *L'Analyse des infiniment petits* (1696), helped to perfect and spread the theory of plane curves.

6. *Space Curves*

Clairaut launched the theory of space curves, the first major development in three-dimensional differential geometry. Alexis-Claude Clairaut (1713–65) was precocious. At the age of twelve he had already written a good work on curves. In 1731 he published *Recherche sur les courbes à double courbure*, which was written in 1729 when he was but sixteen. In this book he treated the analytics of surfaces and space curves (sec. 2). Another paper by Clairaut led to his election to the Paris Academy of Sciences at the unprecedented age of seventeen. In 1743 he produced his classic work on the shape of the earth. Here he treated in more complete form than Newton or Maclaurin the shape a rotating body such as the earth assumes under the mutual gravitational attraction of its parts. He also worked on the problem of three bodies, primarily to study the moon's motion (Chap. 21, sec. 7), and wrote several papers on it, one of which won a prize from the St. Petersburg Academy in 1750. In 1763 he published his *Théorie de la lune*. Clairaut had great personal charm and was a figure in Paris society.

In his 1731 work he treated analytically fundamental problems of curves in space. He called space curves "curves of double curvature" because, following Descartes, he considered their projections on two perpendicular planes. The space curve then partakes of the curvatures of the two curves on the planes. Geometrically he thought of a space curve as the intersection of two surfaces; analytically the equation of each surface was expressed as an equation in three variables (sec. 2). Clairaut then studied tangents to curves of double curvature. He saw that a space curve can have an infinity of normals located in a plane perpendicular to the tangent. The expressions for the arc length of a space curve and the quadrature of certain areas on surfaces are also due to him.

Though Clairaut had taken a few steps in the theory of space curves, very little had been done in this subject or in the theory of surfaces by 1750.

16. Page 311; see also *Math. Schriften*, 2, 166; 3, 967, 969.

This is reflected in Euler's *Introductio* of 1748, where he presented the differential geometry of planar and spatial figures. The first was rather complete, but the second was scanty.

The next major step in the differential geometry of space curves was taken by Euler. A great deal of his work in differential geometry was motivated by his use of curves and surfaces in mechanics. His *Mechanica* (1736),[17] written when he was twenty-nine, is a major contribution to the analytical foundation of mechanics. He gave another treatment of the subject in his *Theoria Motus Corporum Solidorum seu Rigidorum* (1765).[18] In this book he derived the currently used polar coordinate formulas for the radial and normal components of acceleration of a particle moving along a plane curve, namely,

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2, \qquad a_\theta = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}.$$

He started to write on the theory of space curves in 1774. The particular problem that very likely motivated Euler to take up this theory was the study of the skew elastica, that is, the form assumed by an initially straight band when, under pressure at the ends, it is bent and twisted into the shape of a skew curve. To treat this problem he introduced some new concepts in 1774.[19] He then gave a full treatment of the theory of skew curves in a paper presented in 1775.[20]

Euler represented space curves by the parametric equations $x = x(s)$, $y = y(s)$, $z = z(s)$, where s is arc length, and like other writers of the eighteenth century he used spherical trigonometry to carry out the analysis. From the parametric equations he has

$$dx = p\,ds, \qquad dy = q\,ds, \qquad dz = r\,ds,$$

where p, q, and r are direction cosines, varying from point to point and, of course, with $p^2 + q^2 + r^2 = 1$. The quantity ds, the differential of the independent variable, he regarded as a constant.

To study the properties of the curve he introduced the spherical indicatrix. Around any point (x, y, z) of a curve Euler describes a sphere of radius 1. The spherical indicatrix may be defined as the locus on the unit sphere of the points whose position vectors emanating from the center O are equal to the unit tangent at (x, y, z) and the unit tangents at neighboring points. Thus the two radii in Figure 23.12 represent a unit tangent at (x, y, z) and at a neighboring point of the curve. Let ds' be the arc or the angle between the two neighboring tangents of the two points that are ds apart along the curve. Euler's definition of the radius of curvature of the curve is

17. *Opera*, (2), 1 and 2.
18. *Opera*, (2), 3 and 4.
19. *Novi Comm. Acad. Sci. Petrop.*, 19, 1774, 340–70, pub. 1775 = *Opera*, (2), 11, 158–79.
20. *Acta Acad. Sci. Petrop.*, 1, 1782, 19–57, pub. 1786 = *Opera*, (1), 28, 348–81.

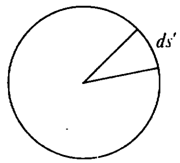

Figure 23.12

$$\frac{ds'}{ds}.$$

He then derives an analytical expression for the radius of curvature:

$$(8) \qquad \rho = \frac{ds^2}{\sqrt{(d^2x)^2 + (d^2y)^2 + (d^2z)^2}} = \frac{1}{\sqrt{x''^2 + y''^2 + z''^2}}.$$

The plane through ds' and the center O is Euler's definition of the osculating plane at (x, y, z). John Bernoulli, who introduced the term, regarded the plane as determined by three "coincident" points. Its equation, as given by Euler, is

$$x(r\,dq - q\,dr) + y(p\,dr - r\,dp) + z(q\,dp - p\,dq) = t,$$

where t is determined by the point (x, y, z) on the curve through which the osculating plane passes. This equation is equivalent to the one we write today in vector notation as

$$(\mathbf{R} - \mathbf{r})\cdot\mathbf{r}'\times\mathbf{r}'' = 0,$$

where $\mathbf{r}(s)$ is the position vector with respect to some point in space of the point on the curve at which the osculating plane is determined, and $\mathbf{R}$ is the position vector of any point in the osculating plane. In vector form $\mathbf{r}$ is given by

$$x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k}$$

and $\mathbf{R}$ has the form $X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$, where (X, Y, Z) are the coordinates of $\mathbf{R}$.

Clairaut had introduced the idea that a space curve has two curvatures. One of these was standardized by Euler in the manner just described. The other, now called "torsion" and representing geometrically the rate at which a curve departs from a plane at a point (x, y, z), was formulated explicitly and analytically by Michel-Ange Lancret (1774–1807), an engineer and mathematician who was a student of Monge and worked in his spirit. He singled out[21] three principal directions at any point of a

21. *Mém. divers Savans*, 1, 1806, 416–54.

curve. The first is that of the tangent. "Successive" tangents lie in a plane, the osculating plane. The normal to the curve that lies in the osculating plane is the principal normal, and the perpendicular to the osculating plane, the binormal, is the third principal direction. Torsion is the rate of change of the direction of the binormal with respect to arc length; Lancret used the terminology, flexion of successive osculating planes or successive binormals.

Lancret represented a curve by

$$x = \phi(z), \qquad y = \psi(z)$$

and called $d\mu$ the angle between successive normal planes and $d\nu$ that between successive osculating ones. Then, in modern notation,

$$\frac{d\mu}{ds} = \frac{1}{\rho}, \qquad \frac{d\nu}{ds} = \frac{1}{\tau},$$

where ρ is the radius of curvature and τ is the radius of torsion.

Cauchy improved the formulation of the concepts and clarified much of the theory of space curves in his famous *Leçons sur les applications du calcul infinitésimal à la géométrie* (1826).[22] He discarded constant infinitesimals, the *ds*'s, and straightened out the confusion between increments and differentials. He pointed out that when one writes

$$ds^2 = dx^2 + dy^2 + dz^2$$

one should mean

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2.$$

Cauchy preferred to write a surface as $w(x, y, z) = 0$ instead of the unsymmetric form $z = f(x, y)$, and he wrote the equation of a straight line through the point (ξ, η, ζ) as

$$\frac{\xi - x}{\cos\alpha} = \frac{\eta - y}{\cos\beta} = \frac{\zeta - z}{\cos\gamma},$$

where $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the direction cosines of the line, though more often he used direction numbers instead of direction cosines.

Cauchy's development of the geometry of curves is practically modern. He got rid of the spherical trigonometry in the proofs, but he, too, took the arc length as the independent variable. He obtains for the direction cosines of the tangent at any point

$$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}, \quad \text{or} \quad x'(s), y'(s), z'(s).$$

22. *Œuvres*, (2), 5.

The direction numbers of the principal normal are shown to be

$$\frac{d^2x}{ds^2}, \frac{d^2y}{ds^2}, \frac{d^2z}{ds^2}, \quad \text{or} \quad x''(s), y''(s), z''(s)$$

and the curvature k of the curve is

$$k = \frac{1}{\rho} = \sqrt{(x'')^2 + (y'')^2 + (z'')^2}.$$

Then he proves that, if the direction cosines of the tangent are $\cos\alpha$, $\cos\beta$, and $\cos\gamma$,

$$(9) \qquad x'' = \frac{d(\cos\alpha)}{ds} = \frac{\cos\lambda}{\rho}, \qquad y'' = \frac{d(\cos\beta)}{ds} = \frac{\cos\mu}{\rho}, \qquad z'' = \frac{d(\cos\gamma)}{ds} = \frac{\cos\nu}{\rho},$$

where ρ is the radius of curvature already introduced and $\cos\lambda$, $\cos\mu$, and $\cos\nu$ are the direction cosines of a normal, which he takes to be the principal one. He shows next that

$$\frac{1}{\rho} = \frac{d\omega}{ds}$$

where ω is the angle between adjacent tangents.

He introduces the osculating plane as the plane of the tangent and principal normal. The normal to this plane is the binormal, and its direction cosines $\cos L$, $\cos M$, and $\cos N$ are given by the formulas

$$\frac{\cos L}{dy\,d^2z - dz\,d^2y} = \frac{\cos M}{dz\,d^2x - dx\,d^2z} = \frac{\cos N}{dx\,d^2y - dy\,d^2x}.$$

He can then prove that

$$(10) \qquad \frac{d\cos L}{ds} = \frac{\cos\lambda}{\tau}, \qquad \frac{d\cos M}{ds} = \frac{\cos\mu}{\tau}, \qquad \frac{d\cos N}{ds} = \frac{\cos\nu}{\tau},$$

where $1/\tau$ is the torsion, and that the torsion equals $d\Omega/ds$, where Ω is the angle between osculating planes.

Formulas (9) and (10) are two of the three famous Serret-Frénet formulas, the third being

$$(11) \qquad \frac{d\cos\lambda}{ds} = -\frac{\cos\alpha}{\rho} - \frac{\cos L}{\tau}, \qquad \frac{d\cos\mu}{ds} = -\frac{\cos\beta}{\rho} - \frac{\cos M}{\tau}, \qquad \frac{d\cos\nu}{ds} = -\frac{\cos\gamma}{\rho} - \frac{\cos N}{\tau},$$

where $1/\tau$ is the torsion and $1/\rho$ is the curvature. These formulas (9), (10), and (11), which give the derivatives of the direction cosines of the tangent, binormal, and normal respectively, were published by Joseph Alfred Serret (1819–85) in 1851[23] and Fréderic-Jean Frénet (1816–1900) in 1852.[24] The significance of curvature and torsion is that they are the two essential properties of space curves. Given the curvature and torsion as functions of arc length along the curve, the curve is completely determined except for position in space. This theorem is readily proven on the basis of the Serret-Frénet formulas.

7. *The Theory of Surfaces*

Like the theory of space curves, the theory of surfaces made a slow start. It began with the subject of geodesics on surfaces, with geodesics on the earth as the main concern. In the *Journal des Sçavans* of 1697, John Bernoulli posed the problem of finding the shortest arc between two points on a convex surface.[25] He wrote to Leibniz in 1698 to point out that the osculating plane (the plane of the osculating circle) at any point of a geodesic is perpendicular to the surface at that point. In 1698 James Bernoulli solved the geodesic problem on cylinders, cones, and surfaces of revolution. The method was a limited one, though in 1728 John Bernoulli[26] did have some success with the method and found geodesics on other kinds of surfaces.

In 1728 Euler[27] gave differential equations for geodesics on surfaces. Euler used the method he introduced in the calculus of variations (see Chap. 24, sec. 2). In 1732 Jacob Hermann[28] also found geodesics on particular surfaces.

Clairaut in 1733 and again in 1739[29] in his work on the shape of the earth treated more fully geodesics on surfaces of revolution. He proved in the 1733 paper that for any surface of revolution, the sine of the angle made by a geodesic curve and any meridian (any position of the generating curve) it crosses varies inversely as the length of the perpendicular from the point of intersection to the axis. In another paper[30] he also proved the nice theorem that if at any point M of a surface of revolution a plane be passed normal to the surface and to the plane of the meridian through M, then the curve cut out on the surface has a radius of curvature at M equal to the

23. *Jour. de Math.*, 16, 1851, 193–207.
24. *Jour. de Math.*, 17, 1852, 437–47.
25. *Opera*, 1, 204–5.
26. *Opera*, 4, 108–28.
27. *Comm. Acad. Sci. Petrop.*, 3, 1728, 110–24, pub. 1732 = *Opera*, (1), 25, 1–12.
28. *Comm. Acad. Sci. Petrop.*, 6, 1732/3, 36–67.
29. *Hist. le l'Acad. des Sci., Paris*, 1733, 186–94, pub. 1735 and 1739, 83–96, pub. 1741.
30. *Mém. de l'Acad. des Sci., Paris*, 1735, 117–22, pub. 1738.

length of the normal between M and the axis of revolution. Clairaut's methods were analytical, but like most of his predecessors he did not employ the ideas we now associate with the calculus of variations.

In 1760, in his *Recherches sur la courbure des surfaces*[31] Euler established the theory of surfaces. This work is Euler's most important contribution to differential geometry and a landmark in the subject. He represents a surface by $z = f(x, y)$ and introduces the now standard symbolism

$$p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y}, \qquad r = \frac{\partial^2 z}{\partial x^2}, \qquad s = \frac{\partial^2 z}{\partial x\, \partial y}, \qquad t = \frac{\partial^2 z}{\partial y^2}.$$

He then says, "I begin by determining the radius of curvature of any plane section of a surface; then I apply this solution to sections which are perpendicular to the surface at any given point; and finally I compare the radii of curvature of these sections with respect to their mutual inclination, which puts us in a position to establish a proper idea of the curvature of surfaces."

He obtains first a rather complex expression for the radius of curvature of any curve made by cutting the surface with a plane. He then particularizes the result by applying it to normal sections (sections containing a normal to the surface). For normal sections the general expression for the radius of curvature simplifies a little. Next he defines the principal normal section as that normal section perpendicular to the xy-plane. (This use of "principal" is not followed today.) The radius of curvature of a normal section whose plane makes an angle ϕ with the plane of the principal normal section has the form

$$\frac{1}{L + M\cos 2\phi + N\sin 2\phi}, \tag{12}$$

where L, M, and N are functions of x and y. To obtain the greatest and least curvature of all normal sections through one point on the surface (or, when the form of the denominator in [12] is indefinite, to get the two greatest curvatures), he sets the derivative with respect to ϕ of the denominator equal to zero and (in both cases) obtains $\tan 2\phi = N/M$. There are two roots differing by 90° so that there are two mutually perpendicular normal planes. We call the corresponding curvatures the principal curvatures κ_1 and κ_2.

It follows from Euler's results that the curvature κ of any other normal section making an angle α with one of the sections with principal curvature is

$$\kappa = \kappa_1 \cos^2 \alpha + \kappa_2 \sin^2 \alpha. \tag{13}$$

This result is called Euler's theorem.

The same results were obtained in 1776 in a more elegant manner by a student of Monge, Jean-Baptiste-Marie-Charles Meusnier de La Place (1754–93), who also worked in hydrostatics and in chemistry with Lavoisier.

31. *Mém. de l'Acad. de Berlin*, 16, 1760, 119–43, pub. 1767 = *Opera*, (1), 28, 1–22.