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Pre-Calculus Mathematics

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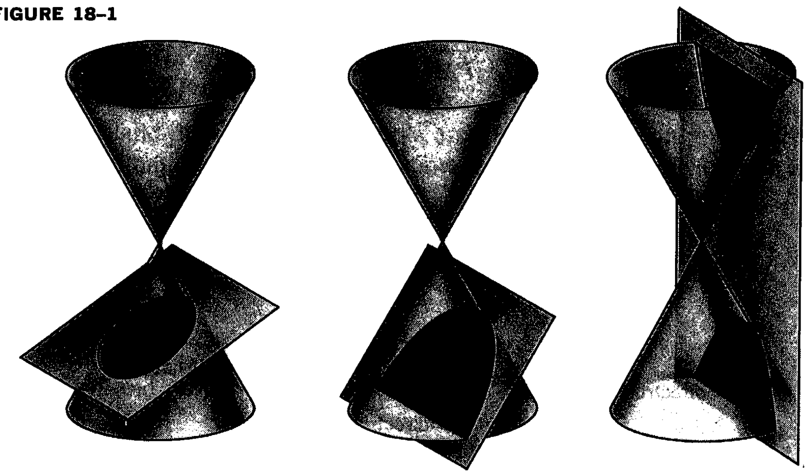
18–1 INTRODUCTION

We have seen that a first-degree (linear) equation in x and y has a graph that is a straight line; and conversely that every straight line in the plane has such an equation. It is reasonable, therefore, to ask, "What kind of graphs do second-degree equations have?" We shall see, in this chapter, that graphs of second-degree equations are the curves called *conic sections*.

These curves, the conic sections, were extensively studied by the ancient Greeks, and especially by Apollonius (third century B.C.). The Greeks considered the conics to be the curves of intersection of a plane and a right circular cone (see Fig. 18–1), and studied them in a purely geometric way.

FIGURE 18–1



Ellipse Parabola Hyperbola

However, in this chapter, we shall study them by means of their equations. We shall give different definitions of these curves. From our definitions, it would be possible to show that these curves are actually sections of a cone, but we shall not do so.

We shall see that all these conic sections have equations that are of the second degree in the rectangular coordinates x and y. And we shall see how to recognize the curve quickly from its equation. Finally, we shall see that every second-degree equation that has a graph consisting of more than one point and that cannot be factored into a product of two linear equations, must be an equation of one of the conic sections.

18–2 THE CONIC SECTIONS

There are two common definitions of these curves. We shall choose the one that clearly includes all three types (ellipse, parabola, hyperbola) in a single family of curves.

Definition 18–1

Let l be a given line, called the *directrix*, and F be a given point, not on l, called the *focus*. Let e be a given positive number, called the *eccentricity*. Let $\mathscr{C}$ be the set of all points P such that the ratio of the distance $|PF|$ (from P to the focus F) to the distance $|PM|$ (from P to the directrix l) is e. The point set $\mathscr{C}$ is called a *conic*.

$$\text{The conic } \mathscr{C} \text{ is} \begin{cases} \text{an } \textit{ellipse} & \text{if } e < 1, \\ \text{a } \textit{parabola} & \text{if } e = 1, \\ \text{a } \textit{hyperbola} & \text{if } e > 1. \end{cases}$$

Thus P is on the conic (see Fig. 18–2) if and only if

$$\frac{|PF|}{|PM|} = e. \tag{1}$$

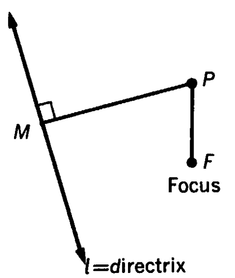


FIGURE 18–2

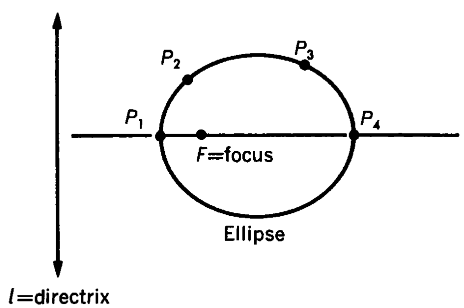


FIGURE 18–3

Example

Let us suppose that F and l are as shown in Fig. 18–3 and that $e = \frac{1}{2}$.

By trial, one can approximately locate points P_1, P_2, P_3, and P_4 on the conic. The conic is an ellipse. It appears that the ellipse has the same shape at both ends. We shall see later that this is so.

Problems

1. Draw a directrix l and choose a point F as a focus. Plot many points on the ellipse with eccentricity $\frac{1}{2}$. Then draw circles C_r of different radii r with center F. For each radius r draw a line l_r parallel to l and at a distance $2r$ from l. Points where the circle C_r meets l_r will be on the ellipse. Why? How large must r be for this construction to yield points of the conic? Can r be too large?
2. Draw a directrix and choose a point as a focus. Draw several points on the parabola with eccentricity 1 and your focus and directrix. Draw circles C_r with radius r and center at the focus. Draw lines l_r parallel to the directrix and at a distance r from it. What are points of intersection of C_r and l_r? How large must r be for C_r and l_r to intersect? Can r be too large?
3. Draw a directrix and choose a point as a focus. Draw several points on the hyperbola with eccentricity $\frac{3}{2}$ and your focus and directrix. Draw circles C_r with radius r and center at the focus. Draw lines l_r parallel to the directrix and at distance $\frac{2}{3}r$ from it. What are points of intersection of C_r and l_r? How large must r be for C_r and l_r to intersect? Can r be too large? Can lines l_r that are on the side of the directrix opposite to the focus meet C_r?
4. A parabola has focus at $(1, 0)$ and as directrix the line with the equation $x = -1$. Find an equation satisfied by the coordinates of all points (x, y) on the parabola.

*5. Show that any conic has an equation of the second degree in x and y. [*Hint:* Suppose that the directrix is the line

$$ax + by + c = 0$$

and that the focus is at (h, k). Then use equation (1) of Definition 18–1.]

18–3 THE PARABOLA

If the focus and directrix of a parabola are placed conveniently with respect to the coordinate axes, then the equation of the parabola will be especially simple.

Let us suppose that the focus is at $(c, 0)$, and that the directrix has an equation $x = -c$, as in Fig. 18–4.

Then the point $P(x, y)$ is on the parabola if and only if

$$|PF| = |PM|e, \quad \text{but} \quad e = 1,$$

or

$$\sqrt{(x - c)^2 + y^2} = |x + c|. \tag{1}$$

Equation (1) is equivalent to the following equations:

$$(x - c)^2 + y^2 = (x + c)^2,$$
$$y^2 = 4cx. \tag{2}$$

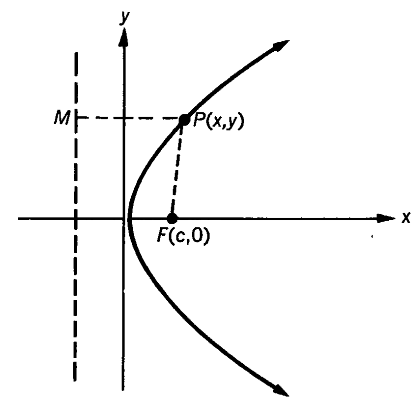


FIGURE 18–4

Equation (2) is an equation to remember. It is the *standard form* for a parabola with its vertex at the origin and its axis along the x-axis. (The *axis* of a parabola is the line through the focus perpendicular to the directrix. The *vertex* is the point midway between the focus and directrix.) *It is to be emphasized that all points* (x, y) *on the parabola satisfy* (2), *and conversely if* x *and* y *satisfy* (2), *then the point* (x, y) *is on the parabola.*

Several observations can be made from equation (2).

(a) The parabola is *symmetric* with respect to its axis. If (x, y) is on the parabola, so is $(x, -y)$, because then $y^2 = (-y)^2 = 4cx$.

(b) The derivation of equation (2) is valid when c is either positive or negative. Figure 18–4 shows the parabola for $c > 0$. For $c < 0$, the focus would be to the left of the origin and the parabola would "open" to the left.

(c) If the parabola has its focus on the y-axis and its directrix parallel to the x-axis, then it appears as shown in Fig. 18–5 and has the standard equation

$$x^2 = 4cy.$$

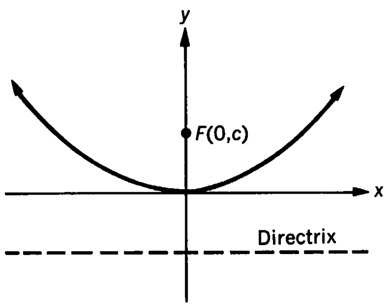


FIGURE 18–5

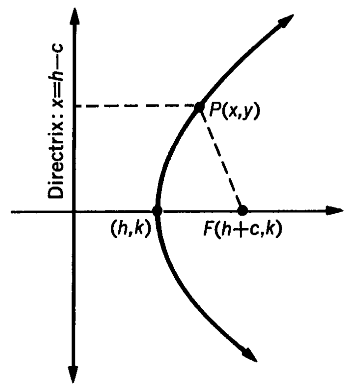


FIGURE 18–6

(d) If the vertex is at a point (h, k) other than the origin and the directrix is parallel to the y-axis, then the focus is at $(h + c, k)$ and the directrix is $x = h - c$. (See Fig. 18–6.)

Just as before, an equation of the parabola is

$$\sqrt{(x - h - c)^2 + (y - k)^2} = |x - h + c|,$$

which is easily seen to be equivalent to

$$(y - k)^2 = 4c(x - h). \tag{3}$$

Equation (3) is the standard form for a parabola with vertex at (h, k) and axis parallel to the x-axis. If the axis were parallel to the y-axis, the standard form would be

$$(x - h)^2 = 4c(y - k). \tag{4}$$

Examples

1. Sketch the parabola whose equation is $y^2 = -6x$. Find its focus and directrix.

Comparing $y^2 = -6x$ with the standard form $y^2 = 4cx$, we see that

$$4c = -6 \quad \text{and} \quad c = -\tfrac{3}{2}.$$

The focus is at $(-\frac{3}{2}, 0)$ and the directrix is the line $x = \frac{3}{2}$. The parabola is sketched in Fig. 18–7. Observe that the segment $\overline{AB}$ through the focus has a length $|AB| = 6 = |4c|$. Note that the segment corresponding to $\overline{AB}$ in a parabola always has a length of $4|c|$.

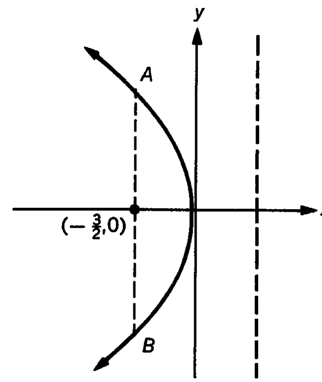


FIGURE 18–7

2. Find the standard form for the parabola with the equation $x^2 + 2x + 6y - 11 = 0$. Find its vertex, focus, and equation of directrix, and sketch the figure.

The equation is equivalent to

$$x^2 + 2x = -6y + 11, \qquad (x + 1)^2 = -6(y - 2).$$

This last is in the standard form (4). Therefore the vertex is at $(-1, 2)$, with the parabola opening down. We have $4c = -6$, where $c = -\frac{3}{2}$. The focus is at $(-1, \frac{1}{2})$, and an equation of the directrix is $y = \frac{7}{2}$. (See Fig. 18–8.)

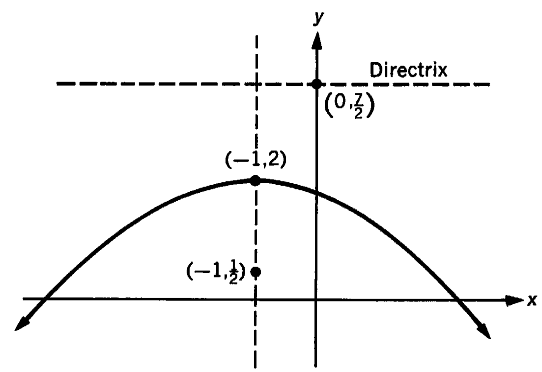


FIGURE 18–8

Problems

1. Find the focus and equation of the directrix, and sketch each of the following parabolas.

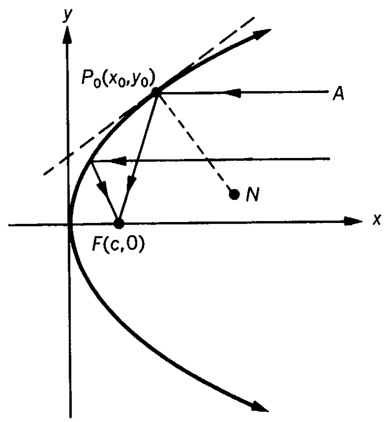
   (a) $y^2 = 6x$ (b) $y^2 = -6x$ (c) $x^2 = \frac{1}{2}y$ (d) $x^2 = -2y$

2. A parabola has its vertex at the origin, its axis parallel to the y-axis, and passes through $(-1, 3)$. Find its equation and sketch the parabola.

3. Sketch, on the same coordinate system, the parabolas $x^2 = 4cy$ for $c = \frac{1}{4}$, $\frac{1}{2}$, 1, 2, and 4.

4. Find vertex, focus, and equation of directrix, and sketch each of the following parabolas.
   (a) $3x^2 = 8y - 16$ (b) $y^2 - 4y - 2x - 8 = 0$
   (c) $y^2 + 4y = 4x$ (d) $3x^2 + 4y = 12$
5. Use the definition of a parabola to find equations of the following parabolas.
   (a) Focus at (4, 3), directrix $x + 2 = 0$
   (b) Focus at $(-1, 1)$, directrix $2y = 5$
6. Find the points of intersection of each of the following pairs of curves and sketch the curves.
   (a) $x = 3y - y^2$, $x - 2y + 2 = 0$ (b) $x^2 - y - 2 = 0$, $y - x = 0$
   (c) $y^2 - 2y + 3x = 3$, $y^2 = 3x + 1$ (d) $y^2 - 4y - 6x + 24 = 0$, $x^2 - 4x - 2y = 0$

*7. A parabola has a focusing property that has many practical applications. Imagine light rays parallel to the axis of a parabolic surface, say a parabolic mirror. (See figure.) All these rays are reflected from the parabola toward the focus. Show that this is the case by proceeding as follows.
   (a) First show that the line $yy_0 = 2cx + 2cx_0$ is tangent to the parabola at a point (x_0, y_0) on it. This may be proved by showing that the line and the parabola intersect at one point only, namely (x_0, y_0).
   (b) Then prove that $\angle AP_0N \cong \angle NP_0F$. [*Hint:* Show that the angles have the same cosines.]



18–4 THE ELLIPSE

Like the parabola, the ellipse also has a very simple equation if the focus and directrix are suitably placed with respect to the axes. However, for the ellipse, this "conveniently chosen" coordinate system is much less obvious. Let us anticipate how things will work out in order to justify our choice of the coordinate system. From the example of Section 18–2 and Fig. 18–3 we may expect the ellipse to have "the same shape at both ends." So there should be two foci F_1 and F_2, and two points V_1 and V_2 on the ellipse and on the line through the foci. Let us choose the origin midway between the foci, and the line through the foci as the x-axis. Then we may give coordinates to F_1, F_2, V_1, and V_2, as shown in Fig. 18–9. If the directrix is at a distance d from the origin, then it has an equation $x = d$. Consider the relation of V_1 and V_2 of the ellipse, to F_1 and the directrix; by Definition 18–1 we have

$$\frac{|V_1F_1|}{|V_1M|} = e = \frac{|V_2F_1|}{|V_2M|},$$
$$\frac{|c - a|}{|d - a|} = e = \frac{|c - (-a)|}{|d - (-a)|}.$$

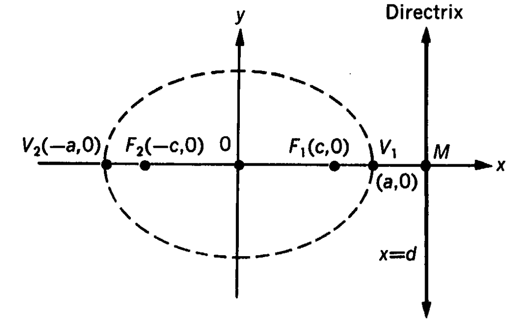


FIGURE 18–9

Since $-a < -c < 0 < c < a < d$, it follows that

$$\frac{a - c}{d - a} = e = \frac{a + c}{d + a},$$

and hence

$$a - c = de - ae,$$
$$a + c = de + ae.$$

Solving these simultaneous equations, we obtain

$$c = ae \quad \text{and} \quad a = de \quad \text{or} \quad d = \frac{c}{e^2}.$$

These last equations show where we should locate the focus and directrix in order to get a simple equation for the ellipse.

We have now arrived at a choice for the location of the ellipse with respect to the axes. Let us see how this works out. Accordingly, suppose that the focus is at $(c, 0)$, $c > 0$, and that the directrix has the equation

$$x = \frac{c}{e^2},$$

(see Fig. 18–10) where e is the eccentricity and, of course, $e < 1$. Then $P(x, y)$ is on the ellipse if and only if $|PF| = |PM|e$:

$$\sqrt{(x - c)^2 + y^2} = \left|\frac{c}{e^2} - x\right| e. \quad (1)$$

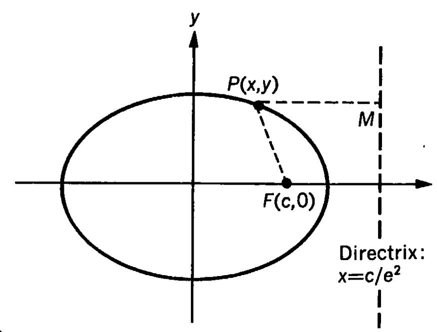


FIGURE 18–10

This equation is equivalent to the following equations:

$$(x - c)^2 + y^2 = \left(\frac{c}{e^2} - x\right)^2 e^2,$$

$$x^2 - 2cx + c^2 + y^2 = \frac{c^2}{e^2} - 2cx + e^2x^2,$$

$$x^2(1 - e^2) + y^2 = \frac{c^2(1 - e^2)}{e^2},$$

$$\frac{x^2}{\frac{c^2}{e^2}} + \frac{y^2}{\frac{c^2(1 - e^2)}{e^2}} = 1.$$

Let us now introduce new positive constants a and b such that

$$a^2 = \frac{c^2}{e^2}, \qquad b^2 = \frac{c^2(1 - e^2)}{e^2}. \tag{2}$$

Then the equation of the ellipse assumes the *standard form*,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{3}$$

Solving the first equation of (2) for c^2 and substituting in the second, we get $b^2 = a^2 - a^2e^2$. From a second substitution in the last equation, we find that $a^2 = b^2 + c^2$.

If a point (x, y) is on the ellipse, then x and y satisfy equation (3), and conversely, if x and y satisfy (3), then the point (x, y) is on the ellipse.

The points $(-a, 0)$ and $(a, 0)$ are *vertices* of the ellipse and the ends of the *major axis*, so that a is the length of the semimajor axis. Likewise, the points $(0, b)$ and $(0, -b)$ are *vertices* and the ends of the *minor axis* of the ellipse, so that b is the length of the semiminor axis. We observe that $b < a$. The origin is the *center* of the ellipse.

A number of observations can be made from equation (3).

(a) As we surmised earlier, the ellipse is *symmetric* with respect to both coordinate axes. If (x, y) is on the ellipse, so are $(x, -y)$, $(-x, y)$, and $(-x, -y)$. The two "ends" of the ellipse look alike.

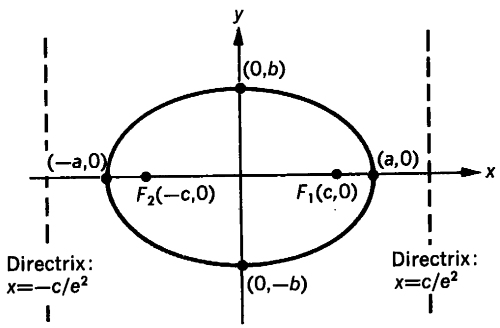


FIGURE 18–11

(b) As a consequence of this symmetry, there must be another directrix at the other end of the ellipse. The equation of this second directrix would be $x = d' = -c/e^2$. Equation (3) could also be derived by choosing F_2 (or $c < 0$) and the directrix at $-c/e^2$. The ellipse with its two foci and directrices is shown in Fig. 18–11.

(c) If the ellipse has its foci on the y-axis, the standard equation of the ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1,$$

where a and b are the semiaxes.

(d) If the center of the ellipse is at the point (h, k) and the two foci are at $(h + c, k)$ and $(h - c, k)$, then an equation of the ellipse is

$$\sqrt{(x - h - c)^2 + (y - k)^2} = \left|h + \frac{c}{e^2} - x\right| e,$$

which can be shown to be equivalent to

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1. \tag{4}$$

Equation (4) is the standard form for an ellipse with center at (h, k) and major axis parallel to the x-axis. If the major axis were parallel to the y-axis, a and b would be interchanged in the equation.

Examples

1. We graph the ellipse having an equation $4x^2 + 3y^2 = 48$, and find its foci, eccentricity, and the equations of its directrices.

Dividing both sides of the equation by 48, we get the standard form

$$\frac{x^2}{12} + \frac{y^2}{16} = 1.$$

Then $a = 4$, $b = \sqrt{12} = 2\sqrt{3}$, and the major axis is along the y-axis. (See Fig. 18–12.)

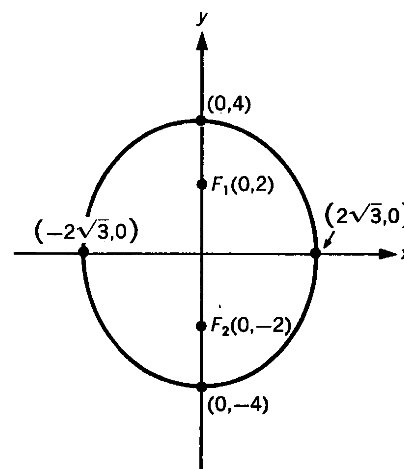


FIGURE 18–12

From equations (2), $a^2 = b^2 + c^2$, or $16 = 12 + c^2$, and $c = 2$. Therefore the foci are at $(0, \pm 2)$. The eccentricity is obtained from equations (2):

$$e = \frac{c}{a} = \frac{2}{4} = \frac{1}{2}.$$

Therefore,

$$\frac{c}{e^2} = \frac{2}{(\frac{1}{2})^2} = 8$$

and the directrices have equations $y = \pm 8$.

2. An ellipse has its center at $(-2, 3)$ and semiaxes of lengths 2 and 1, with the major axis parallel to the x-axis. We wish to find the standard form of the equation of the ellipse, its foci, and directrices, and sketch the curve.

In this case, $a = 2$ and $b = 1$; therefore, the standard form is

$$\frac{(x+2)^2}{4} + \frac{(y-3)^2}{1} = 1. \tag{5}$$

From $a^2 = b^2 + c^2$ we obtain $4 = 1 + c^2$ and $c = \sqrt{3}$. The foci are $(-2 \pm \sqrt{3}, 3)$. (See Fig. 18–13.)

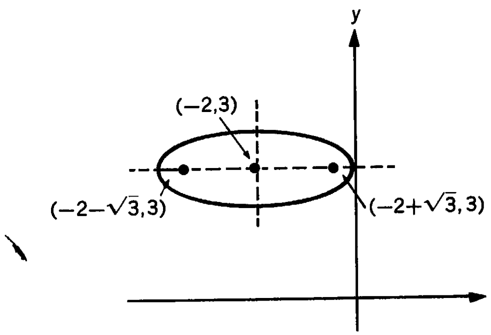


FIGURE 18–13

Remark

Another form of an equation of the ellipse is obtained if we expand $(x + 2)^2$ and $(y - 3)^2$ in (5) and clear of fractions. We then have

$$x^2 + 4y^2 + 4x - 24y + 36 = 0, \tag{6}$$

which is also an equation of the ellipse. Had we been given (6) we would have completed the squares on the x- and y-terms to obtain the standard form (5). (See the next example.)

Example 3

Let us show that the graph of $3x^2 + y^2 - 12x + 2y + 4 = 0$ is an ellipse. We will find its center, foci, eccentricity, and sketch the ellipse.

We rewrite the equation as

$$3(x^2 - 4x \quad) + (y^2 + 2y \quad) = -4,$$

and complete the squares of the terms in parentheses to obtain

$$3(x-2)^2 + (y+1)^2 = 9,$$

which is equivalent to

$$\frac{(x-2)^2}{3} + \frac{(y+1)^2}{9} = 1.$$

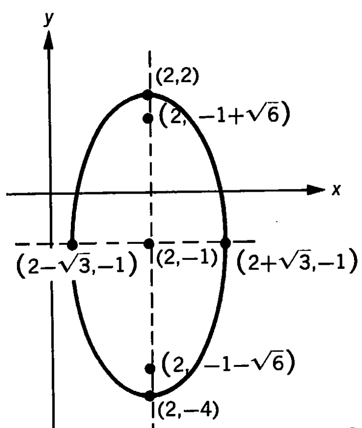


FIGURE 18–14

This last equation is the standard form for the ellipse with semiaxes $a = 3$, $b = \sqrt{3}$, and center at $(2, -1)$. (See Fig. 18–14.) Then $c^2 = 9 - 3$ and $c = \sqrt{6}$. The eccentricity is $e = c/a = \sqrt{6}/3$.

Remark on ellipses and circles

An ellipse, as we have defined it, is never a circle. Yet it is clear that when b is close to a, the ellipse is very nearly a circle. If b is close to a, then $c = \sqrt{a^2 - b^2}$ is very nearly 0, and the eccentricity $e = c/a$ is also nearly 0. The directrices $x = \pm c/e^2 = \pm a^2/c$ recede farther and farther from the center as b approaches a. (See Fig. 18–15.)

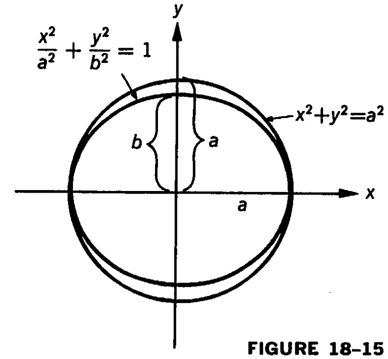


FIGURE 18–15

By agreement we shall say that a circle is a special ellipse with *eccentricity zero*, even though the circle does not satisfy the definition of an ellipse. The two foci then coincide with the center of the circle and there are no directrices.

Problems

1. Sketch the following ellipses. Find the eccentricity, foci, and equations of directrices of each.

(a) $x^2 + 4y^2 = 25$ (b) $5x^2 + y^2 = 80$
(c) $4x^2 + 4y^2 = 25$ (d) $2x^2 + y^2 = 16$

2. Write the equations of each of the following ellipses and sketch the curves.

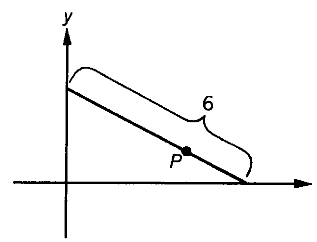
(a) Foci at $(\pm\sqrt{3}, 0)$ and semimajor axis $\sqrt{6}$
(b) Vertices at $(\pm 3, 0)$ and $(0, \pm 2)$
(c) Center at $(0, 0)$, eccentricity $1/\sqrt{2}$, and semiminor axis along the x-axis and 2 units long
(d) Foci at $(0, \pm 5)$ and eccentricity $\sqrt{5}/4$.

3. Find the points of intersection of the following pairs of curves and sketch each pair of curves.

(a) $x^2 + 3y^2 = 52$, $x^2 = 3y + 16$ (b) $3x^2 + y^2 = 16$, $3x + y = 1$
(c) $2x^2 + y^2 = 18$, $x^2 + 5y^2 = 45$ (d) $x^2 + 2y^2 = 9$, $x^2 + y^2 - 4x = 1$

4. Find an equation of the ellipse that has foci at $(\pm 2\sqrt{5}, 0)$ and passes through $(-3\sqrt{2}, 2\sqrt{2})$.

5. A rod 6 units long moves so that its ends are on two perpendicular lines. (See figure.) Show that a point two units from one end describes an ellipse.



6. Find an equation of a parabola with vertex at the origin and focus at a focus of the ellipse

$$x^2 + 5y^2 = 10.$$

7. Sketch the following ellipses. Find center and foci of each.

(a) $9x^2 + 16y^2 - 18x - 64y = 71$
(b) $9x^2 + 5y^2 + 36x - 30y + 36 = 0$
(c) $25x^2 + 9y^2 + 150x + 18y + 9 = 0$
(d) $9x^2 + y^2 - 8y + 7 = 0$
(e) $2x^2 + y^2 + 12x + 4y = 0$

8. Show that the graphs of the following equations either consist of one point or are the empty set.

(a) $3x^2 + y^2 = 0$
(b) $4x^2 + y^2 - 16x - 2y + 17 = 0$
(c) $4x^2 + y^2 - 16x - 2y + 18 = 0$

9. Write equations of the following ellipses and sketch the curves.

(a) Major axis 8 units long and foci at (0, 2) and (6, 2)
(b) Eccentricity $1/\sqrt{2}$ and foci at $(-1, -2 \pm 2\sqrt{2})$
(c) Minor axis 10 units long and parallel to the y-axis, with eccentricity $\frac{2}{3}$ and center (1, 1)
(d) Foci at (4, 1) and (0, 1) and eccentricity $\frac{2}{3}$
(e) Major axis 10 units long and foci at (1, 4) and (1, 0)
(f) Minor axis 8 units long and parallel to the x-axis, with center at $(-2, 3)$ and eccentricity $\frac{3}{4}$

10. Find the points of intersection of the graphs of the following pairs of equations and sketch the graphs.

(a) $9x^2 + y^2 - 18x - 4y - 72 = 0$, $y - 3x = 6$
(b) $x^2 + 2y^2 = 4$, $x^2 + 3y^2 - 2x = 0$
(c) $25x^2 + 12y^2 + 50x - 48y = 27$, $4y^2 - 4y = 5x + 3$
(d) $x^2 + y^2 + 6x + 4y = 0$, $2y^2 + 8y - 9x = 19$
(e) $x^2 + y^2 - 6x + 4 = 0$, $2x - y = 1$
(f) $y^2 - 5x + 10 = 0$, $y^2 + 5x - 10 = 0$
(g) $x^2 - 4x - 8y + 12 = 0$, $x^2 - 4x + 8y - 20 = 0$
(h) $3x^2 + 4y^2 - 48 = 0$, $3x^2 + 4y^2 - 8y - 44 = 0$

***11.** The ellipse has several properties that serve to characterize it. One of these is the following, which is sometimes used as the definition: *If P is on an ellipse with foci F_1 and F_2, then*

$$|PF_1| + |PF_2| \text{ is a constant, and conversely.}$$

Prove that if $P = (x, y)$ is on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ with foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$, then

$$|PF_1| + |PF_2| = 2a.$$

12. Show that the ellipse as given by equation (4) is symmetrical with respect to the lines $x = h$ and $y = k$.

18-5 THE HYPERBOLA

We choose the focus and directrix such that they are conveniently located with respect to the axes. Our choice of location is formally identical with the one we made for the ellipse.

Accordingly, let us suppose that the focus is at $(c, 0)$, where $c \neq 0$ $(c > 0$ in Fig. 18-16). Let us further suppose that the directrix is

$$x = \frac{c}{e^2},$$

where e is the eccentricity and, of course, $e > 1$.

Then $P(x, y)$ is on the hyperbola if and only if $|PF| = |PM|e$, or

$$\sqrt{(x - c)^2 + y^2} = \left|\frac{c}{e^2} - x\right| e. \quad (1)$$

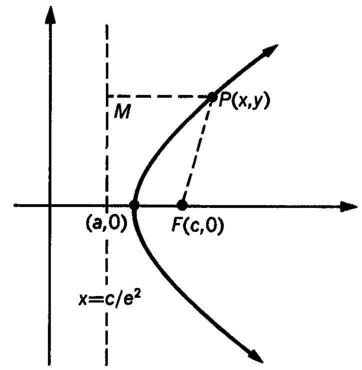


FIGURE 18-16

This equation is formally the same as equation (1) of Section 18-4, and so by the same algebraic steps is equivalent to

$$\frac{x^2}{\frac{c^2}{e^2}} + \frac{y^2}{\frac{c^2}{e^2}(1 - e^2)} = 1.$$

But for the hyperbola $e > 1$, so the factor $(1 - e^2)$ in the denominator of the second fraction is a negative number. Therefore we substitute $-(e^2 - 1)$ for $(1 - e^2)$ and rewrite the equation above in the more useful form

$$\frac{x^2}{\frac{c^2}{e^2}} - \frac{y^2}{\frac{c^2}{e^2}(e^2 - 1)} = 1.$$

As for the ellipse, we introduce new positive constants a and b according to

$$a^2 = \frac{c^2}{e^2}, \qquad b^2 = \frac{c^2}{e^2}(e^2 - 1) = c^2 - a^2,$$

so that

$$c^2 = a^2 + b^2. \quad (2)$$

Unlike the ellipse, the hyperbola may have $a > b$ or $b > a$, or even $a = b$, since $c^2 = a^2 + b^2$. The equation of the hyperbola then assumes the *standard form*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \quad (3)$$

If a point (x, y) is on the hyperbola, then x and y satisfy (3), and conversely, if x and y satisfy (3), then the point (x, y) is on the hyperbola. The points

$(a, 0)$ and $(-a, 0)$ are the *vertices* of the hyperbola. The number a is the length of the *semitransverse axis*. The number b is the length of the *semiconjugate axis*. (Geometric interpretations for a and b are given below.) The *center* of the hyperbola is at $(0, 0)$.

A number of observations can be made from equation (3).

(a) The hyperbola is *symmetric* with respect to the coordinate axes, which are called the *axes* of the hyperbola, and has its center at $(0, 0)$.

(b) From observation (a) we see that there must be a second focus and directrix. This fact may also be seen by observing that equations (1), (2), and (3) are not dependent on the sign of c. Figure 18–16 shows the focus and directrix when $c > 0$.

(c) If the hyperbola has its foci on the y-axis, the standard form of the equation is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

(d) If we solve (3) for y, we obtain

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}, \tag{4}$$

from which we see that x^2 must be greater than or equal to a^2, and that the hyperbola does not meet the y-axis.

From equation (4) we can also see that there are two straight lines, called *asymptotes*, that are intimately related to the hyperbola. From (4) we have

$$y = \pm \frac{b|x|}{a}\sqrt{1 - \frac{a^2}{x^2}} \quad \text{if} \quad |x| > a.$$

This last equation suggests that for large $|x|$, and therefore small a^2/x^2, the hyperbola should be very near the straight lines

$$y = \pm \frac{bx}{a}.$$

That this is indeed the case can be shown as follows (where, because of symmetry, we may restrict ourselves to the first quadrant):

$$\begin{aligned} y_{\text{line}} - y_{\text{hyperbola}} &= \frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2} \\ &= \frac{b}{a}(x - \sqrt{x^2 - a^2}) \\ &= \frac{b}{a}\frac{(x - \sqrt{x^2 - a^2})(x + \sqrt{x^2 - a^2})}{x + \sqrt{x^2 - a^2}} \\ &= \frac{ab}{x + \sqrt{x^2 - a^2}}. \end{aligned}$$

When x is very large, this last fraction represents a small number. Therefore the ordinate on the line is very near the corresponding ordinate on the hyperbola if x is very large.

Such a line is called an *asymptote*. That is, *an asymptote is a line that a curve approaches as a point on the curve "recedes to infinity."* An asymptote is *not* a part of the hyperbola.

A hyperbola with both foci, both directrices, and asymptotes is shown in Fig. 18–17.

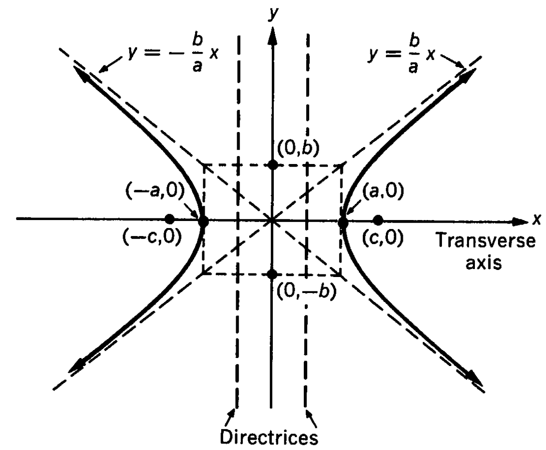


FIGURE 18-17

We note that the asymptotes are very helpful in drawing the hyperbola. The asymptotes are easy to draw if one constructs the dotted rectangle of dimensions $2a$ and $2b$ shown in Fig. 18–17. We see that, geometrically, a and b are equal to half the respective measures of this rectangle. The axis of the hyperbola containing the two foci is called the *transverse* axis.

(e) If the center of the hyperbola is at (h, k) and the two foci are at $(h \pm c, k)$, then an equation of the hyperbola is

$$\sqrt{(x - h - c)^2 + (y - k)^2} = \left| h + \frac{c}{e^2} - x \right| e,$$

which is equivalent to

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1. \tag{5}$$

Equation (5) is the standard form for a hyperbola with center at (h, k) and transverse axis parallel to the x-axis. If the transverse axis were parallel to the y-axis, the equation would be

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1.$$

Examples

1. We wish to graph the hyperbola having an equation $8x^2 - 3y^2 = 48$, and find its foci, the equations of its directrices and asymptotes, and its eccentricity.

Dividing both sides of the equation by 48, we get the standard form

$$\frac{x^2}{6} - \frac{y^2}{16} = 1.$$

Therefore (see Fig. 18–18),

$$a = \sqrt{6},$$

$$b = 4,$$

$$c = \sqrt{6 + 16} = \sqrt{22}.$$

The foci are at $(\pm\sqrt{22}, 0)$. The eccentricity is

$$e = \sqrt{\tfrac{22}{6}} = \sqrt{\tfrac{11}{3}}.$$

The directrices are

$$x = \pm\frac{c}{e^2} = \pm\frac{3\sqrt{22}}{11}.$$

The asymptotes are

$$y = \pm\frac{4}{\sqrt{6}}x.$$

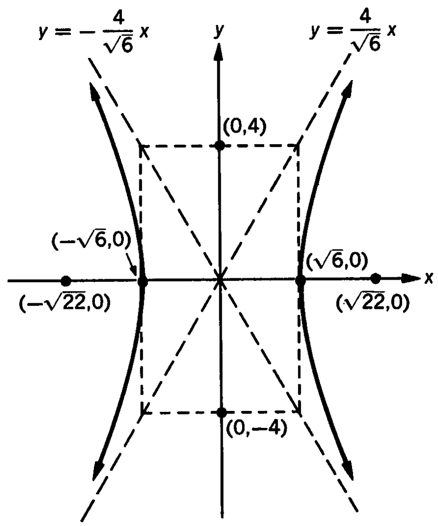


FIGURE 18–18

2. We wish to find the standard form for the equation of the hyperbola with center at $(-2, 3)$, transverse axis parallel to the y-axis, and semiaxes of lengths a and b equal to 4 and 2, respectively. What are the equations of the asymptotes? We shall sketch the hyperbola and its asymptotes.

Since $c^2 = a^2 + b^2$, we have $c = \sqrt{20} = 2\sqrt{5}$. The foci are at $(-2, 3 \pm 2\sqrt{5})$. (See Fig. 18–19.) The vertices are at $(-2, 7)$ and $(-2, -1)$. The asymptotes have slopes ± 2 and pass through the center. Their equations are

$$y - 3 = \pm 2(x + 2).$$

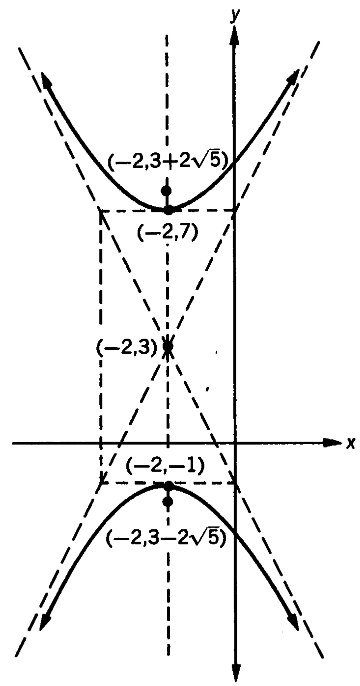


FIGURE 18–19

The equation of the hyperbola is

$$\frac{(y - 3)^2}{16} - \frac{(x + 2)^2}{4} = 1,$$

which is equivalent to $y^2 - 4x^2 - 16x - 6y = 23$.

3. We wish to show that the graph of $-4x^2 + 9y^2 + 36y + 8x + 68 = 0$ is a hyperbola, find its center, the equations of its asymptotes, and make a sketch.

Completing the squares on the x- and y-terms, we have

$$-4(x^2 - 2x + 1) + 9(y^2 + 4y + 4) = -68 - 4 + 36 = -36.$$

Dividing through by -36, we obtain the equivalent equation and standard form

$$\frac{(x - 1)^2}{9} - \frac{(y + 2)^2}{4} = 1,$$

which is recognized as the equation of a hyperbola with center at $(1, -2)$ and axis parallel to the x-axis. We have $a = 3$, $b = 2$, and asymptotes with equations $y + 2 = \pm\frac{2}{3}(x - 1)$. (See Fig. 18–20.)

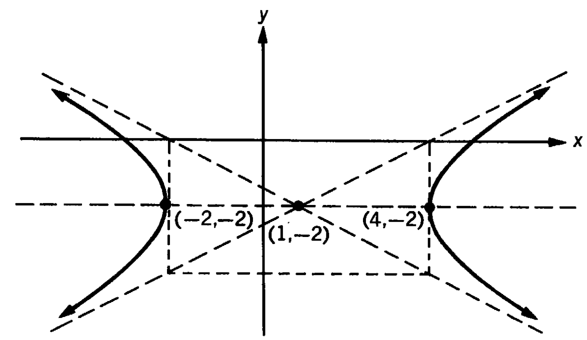


FIGURE 18–20

Problems

1. Sketch the following hyperbolas. Find the eccentricity, foci, and the equations of the asymptotes of each.
   (a) $16x^2 - 9y^2 = 144$ (b) $4x^2 - 5y^2 + 20 = 0$
   (c) $2x^2 - y^2 = 8$ (d) $10x^2 - 16y^2 + 25 = 0$
2. Write equations of each of the following hyperbolas and sketch them.
   (a) Foci at $(\pm\sqrt{3}, 0)$ and semitransverse axis $\sqrt{2}$
   (b) Center at (0, 0), eccentricity $\sqrt{2}$, and semiconjugate axis 2 units long along the x-axis
   (c) Vertices at $(\pm 3, 0)$ and asymptotes with slopes $\pm\frac{1}{3}$
   (d) Foci at $(0, \pm 5)$ and eccentricity $\frac{5}{4}$
   (e) Asymptotes with slope $\pm\frac{5}{2}$, center at $(-2, 3)$, and transverse axis parallel to the x-axis
   (f) Vertices at $(-3, 1)$ and $(-3, -3)$, asymptotes with slopes ± 1

3. Find the points of intersection of the following pairs of curves and sketch each pair.

(a) $3x^2 + 2y^2 = 8$, $x^2 - y^2 = 1$

(b) $x^2 - y^2 = 1$, $y = 2x + 1$

(c) $x^2 - 2y^2 = 2$, $y^2 = x$

(d) $3y^2 - 4x^2 = 12$, $2x = (2\sqrt{6} + 3)y - 8\sqrt{6} - 18$

(e) $x^2 + 10y^2 - 16x - 20y - 18 = 0$, $x^2 - 4y^2 - 16x - 20y - 4 = 0$

(f) $y^2 - 4x^2 + 2y - 8x - 6 = 0$, $3x - y + 2 = 0$

4. Find an equation of the hyperbola that has foci at $(0, \pm 2\sqrt{5})$ and passes through $(-2\sqrt{6}, 4\sqrt{3})$.

5. Sketch the following hyperbolas. For each find its center, foci, eccentricity, and the equations of its asymptotes.

(a) $9x^2 - 16y^2 - 18x - 64y = 19$
(b) $9x^2 - 5y^2 + 36x + 30y + 36 = 0$
(c) $25x^2 - 9y^2 + 150x + 18y = 9$
(d) $9x^2 - y^2 + 8y = 7$
(e) $y^2 - 2x^2 + 12x + 4y = -4$
(f) $9x^2 - 4y^2 + 72x + 24y + 72 = 0$

6. Write equations of the following hyperbolas and sketch them.

(a) Transverse axis 8 units long and foci at $(-2, 2)$ and $(8, 2)$
(b) Eccentricity $\sqrt{2}$ and foci at $(-1, -2 \pm 2\sqrt{2})$
(c) Semiconjugate axis 5, with eccentricity $\frac{3}{2}$, center at $(1, 1)$, and transverse axis parallel to the x-axis

7. The hyperbola has several properties that serve to characterize it. One of these is sometimes used as the definition, namely: *If P is on a hyperbola with foci F_1 and F_2, then*

$$|PF_2| - |PF_1| = \pm 2a$$

(depending on which branch P lies on), and conversely.

Show that if $P(x, y)$ is on the right-hand branch of the hyperbola of Fig. 18–17, then $|PF_2| - |PF_1| = 2a$ if $F_1 = (c, 0)$ and $F_2 = (-c, 0)$.

18–6 DEGENERATE CONICS

In the preceding sections we have seen that the conics, ellipses, parabolas, and hyperbolas (with axes parallel to the coordinate axes) have equations that are of the second degree in x and y. In this and the following section we shall complete our study of what the graphs of second-degree equations in x and y can be. We shall see that except for *degenerate cases*, these graphs are always conics. In this section we investigate these degenerate cases by a study of examples.

From Fig. 18–1 we see that there are intersections of a plane with a cone that are not ellipses, parabolas, or hyperbolas. If the plane passes through the vertex of the cone, it can intersect the cone in two intersecting straight lines. The plane can also intersect the cone in a single point, the vertex. Thus there are other sections of a cone than the ones we have studied so far.

If a plane intersects a cone at a point or in two intersecting straight lines, then we usually call these cases *degenerate conics.*

Examples

1. The equation $x^2 + 2y^2 = 0$ has a graph consisting of only one point.

2. The equation $x^2 + 2y^2 + 1 = 0$ has an empty graph.

3. Sometimes it is not easy to recognize the types shown in Examples 1 and 2; they may be disguised as shown here. The equation

$$x^2 + 2y^2 - 4x + 8y + 13 = 0$$

becomes, on completion of the squares on x and y,

$$(x - 2)^2 + 2(y + 2)^2 + 1 = 0.$$

The graph of this equation is empty. Sometimes the graph is described as being *imaginary.* This means that the only numbers x and y that satisfy the equation are complex numbers with at least one of them having a nonzero imaginary part.

These examples illustrate one type of degeneracy of second-degree equations. The examples below illustrate the other type of degeneracy.

4. We wish to sketch the graph of $x^2 - xy - 2y^2 - x + 11y - 12 = 0$. The polynomial happens to factor, so the equation is

$$x^2 - xy - 2y^2 - x + 11y - 12 = (x - 2y + 3)(x + y - 4) = 0.$$

Therefore its graph is the union of the graphs of $x - 2y + 3 = 0$ and $x + y - 4 = 0$. It is shown in Fig. 18–21.

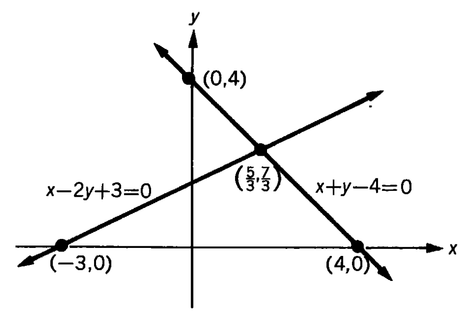
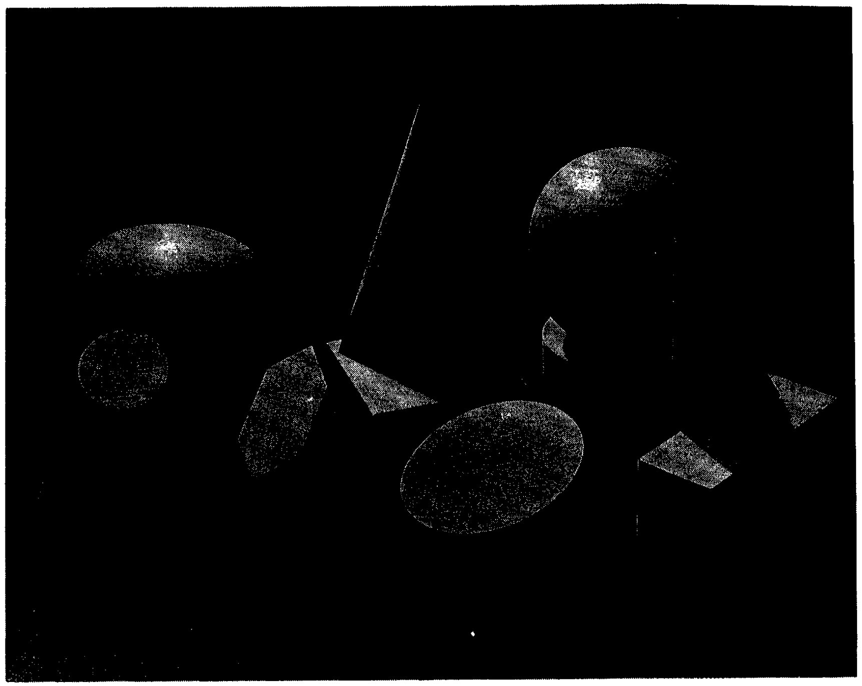


FIGURE 18–21

The student may very well wonder how one is to determine whether or not a given equation factors. Inspection of the quadratic terms will usually show how to proceed. Thus in the above example, the quadratic terms alone

HISTORICAL NOTE



Why do we study the conics? There are several reasons. First, these curves are encountered with astonishing frequency in all sorts of real problems. For example, because of gravitational attraction, described in Newton's law, the path of a planet around the sun is an ellipse, except for the slight disturbing effect of the other planets. Second, next to straight lines, the conics are the simplest curves in the plane; they have equations of the second degree in x and y. Third, the conics are interesting in themselves for the variety of surprising properties that they possess.

Apart from these mathematical reasons, there is also the influence of the past. The conics have attracted mathematicians for 2000 years. As in so many fields, we are indebted to the ancient Greeks, who, with a sure intuition as to what is elegant and important, developed an extensive theory of the conics. A great part of that theory is the work of one man, *Apollonius of Perga* (262?–200? B.C.), one of the three great geometers of the third century B.C., the other two being Euclid and Archimedes. In his remarkable book *Conic Sections*, Apollonius systematized and vastly extended the work of earlier writers. The 400 (approximately) propositions of the book include much more material than has been mentioned in this brief chapter.

Before Apollonius, the definitions of the conics were based on the requirement that the sectioning plane be perpendicular to a generator of the cone. Hence one obtained the three kinds of conics by making the vertex angle of the cone acute, right, or obtuse (see Fig. 18–30). Note

that since this construction used only one nappe, or half of the complete cone, only half of the hyperbola was obtained. Apollonius obtained all the conics from a single, complete cone (using both nappes) by permitting the sectioning plane to cut a generator at any angle.

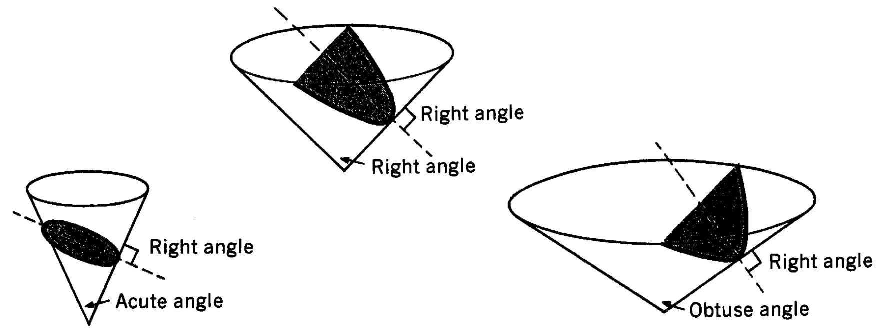


FIGURE 18–30

Apollonius also gave us the names *ellipse*, *parabola*, and *hyperbola*. To catch the spirit of his writing we shall paraphrase his terminology with respect to the eccentricity. Thus he said that a conic suffers *ellipsis* if the eccentricity *falls short* of 1, *parabole* if the eccentricity is *precisely* 1, or *hyperbole* if the eccentricity *exceeds* 1.

Up to the time of Descartes, all treatises on the conics used the synthetic methods familiar to the student of plane geometry. With the advent of algebraic methods the view of geometry was vastly expanded, all functions were available for geometric exploitation, and synthetic methods fell into disfavor. This happened in spite of the significant developments in synthetic geometry due to the mark of Pascal, Desargues, and La Hire. One hundred years, and more, were to pass before the development of projective geometry brought about the revival of synthetic methods.