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Simple models of complex chaotic systems

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Many phenomena in the real world are inherently complex and involve many dynamical variables interacting nonlinearly through feedback loops and exhibiting chaos, self-organization, and pattern formation. It is useful to ask if there are generic features of such systems, and if so, how simple can such systems be and still display these features. This paper describes several such systems that are accessible to undergraduates and might serve as useful examples of complexity.

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I. INTRODUCTION

The study of nonlinear dynamics has blossomed in the past few decades primarily as a result of readily available, powerful computers. Such topics provide the opportunity for students to perform original multidisciplinary research and offer motivation for them to hone their computational skills. Much of the past interest has centered on relatively simple, low-dimensional iterated maps such as the logistic map[1] and the Hénon map,[2] and systems of ordinary differential equations (ODEs) such as the Lorenz equations[3] and the Rössler equations.[4]

One line of research is to seek the algebraically simplest systems of various types that are capable of producing chaos.[5] For example, the simplest dissipative chaotic flow with a quadratic nonlinearity is[6]

$$\dddot{x} + a\ddot{x} - \dot{x}^2 + x = 0, \tag{1}$$

where $\dot{x} = dx/dt$. This system is chaotic over most of the range $2.0168 < a < 2.0577$ and has its maximum Lyapunov exponent (the exponential rate of separation of initially nearby orbits and a useful measure of chaos) of $\lambda = 0.0559$ at $a \simeq 2.0169$.

The simplest such system with a cubic nonlinearity is[7]

$$\dddot{x} + a\ddot{x} - x\dot{x}^2 + x = 0, \tag{2}$$

which is chaotic over most of the range $2.0277 < a < 2.0840$ and has its maximum Lyapunov exponent $\lambda = 0.0852$ at $a \simeq 2.0278$. The simplest such system with an absolute-value nonlinearity is[8]

$$\dddot{x} + a\ddot{x} + \dot{x} - |x| + 1 = 0, \tag{3}$$

which is chaotic over most of the range $0.5463 < a < 0.6410$ and has its maximum Lyapunov exponent $\lambda = 0.0768$ at $a \simeq 0.5641$. The simplest periodically driven conservative chaotic flow with a cubic nonlinearity is[9]

$$\ddot{x} + x^3 = \sin \Omega t, \tag{4}$$

which is chaotic over most of the range $0 < \Omega < 2.8$ with its maximum Lyapunov exponent $\lambda = 0.0791$ at $\Omega \simeq 1.88$. Note that in Eqs. (1)–(3), the bifurcation parameter a is a damping rate and the chaos occurs over a narrow range of this parameter. This feature will also characterize the more complex systems considered here. Equation (4) also has a dissipative variant with an added $a\dot{x}$ term, called the Ueda oscillator,[10] which is a special case of Duffing's oscillator,[11] which also contains a term linear in x.

Systems such as these provide a wealth of opportunity for undergraduates to perform original research on systems with complicated dynamics including bifurcations, multistability, hysteresis, calculation of Lyapunov exponents and fractal dimensions, basins of attraction, synchronization, chaotic electrical circuit implementation, and much more. These topics are described in most of the standard chaos texts.[12–14] However, their low dimension makes them suitable models only for relatively simple systems with a small number of dynamical variables. Much of the current interest in nonlinear dynamics is in complex high-dimensional systems involving large networks of many nonlinearly interacting variables.[15] Such networks arise often in ecology, economics, sociology, epidemiology, meteorology, and neurology, among others.[16] The physics laboratory is a highly atypical environment where phenomena are well described by low-dimensional models.

High-dimensional nonlinear systems can exhibit all the complexity of their low-dimensional counterparts, and can also exhibit new phenomena such as self-organization, evolution, learning, and adaptation,[17–19] phenomena that we normally ascribe to living systems. In some ways their behavior is simpler, much as a large collection of molecules in a gas can be described statistically without concern for the details of the interactions. Whereas chaotic systems are readily identified by their sensitive dependence on initial conditions and quantified by their Lyapunov exponents, complex systems are less well characterized. They contain many nonlinearly interacting parts with positive and negative feedback loops and are driven out of equilibrium by the flow of energy or other resource through the system. They are usually (perhaps weakly) chaotic in both space and time. Complex systems may be complicated in the sense of requiring many parameters, but they can also be relatively simple, as this paper will show.

II. FULLY CONNECTED NEURAL NETWORKS

How does one model a complex system without being an expert in the area the model is attempting to describe? One approach is to construct a model of sufficient generality that it can represent almost any complex system with an appropriate choice of parameters and then see if there are universal or at least generic behaviors when those parameters are chosen at random. There are many candidate models, including cellular automata, coupled map lattices, and partial differential equations. As a natural extension of current chaos research and because continuous-time systems are more familiar to physicists, we consider a system of coupled ODEs with a sigmoidal nonlinearity such as the hyperbolic tangent:

$$\dot{x}_i = - b_i x_i + \tanh\left(\sum_{j=1, j \neq i}^{N} a_{ij} x_j\right), \tag{5}$$

where N is the dimension of the system (the number of variables). The particular nonlinearity in Eq. (5) is appropriate because it mimics the common situation in nature where a small stimulus produces a linear response, but the response saturates when the stimulus is large, thereby avoiding unbounded and hence unphysical solutions.

With an appropriate choice of the vector b_i and the matrix a_{ij}, Eq. (5) can model a wide range of dynamics, including chaos for N as small as 4. In what follows, it will be convenient to take $b_i = b$ for all i as the bifurcation parameter in analogy with the low-dimensional systems described in Sec. I. This system is equivalent to an alternate form[20] in which the hyperbolic tangent of x_j is inside the summation. The $-bx_i$ term is analogous to frictional damping and guarantees that the solutions are bounded.

Equation (5) can be considered as an artificial neural network in which the neurons accept weighted inputs from all the other neurons and nonlinearly squash their sum. It could equally well be considered as a collection of nonlinearly interacting agents corresponding to people, firms, animals, cells, molecules, or any number of other entities. The system has a static equilibrium with all $x_i = 0$, and can be driven away from equilibrium by the positive feedback among the neurons, which implies an external energy source not explicit in the equations. Similar and more extensive studies have been done on discrete-time neural networks (iterated maps),[21,22] but such models are less common in physics and lead too easily to chaotic solutions that may not be physically realizable.

An interesting project would be to optimize the a_{ij} values to fit some observed time series, such as the stock market, but our interest here is to illustrate the generic behavior of such a network. For that purpose, we take a large but still tractable value of $N = 101$ (a prime number) and select the a_{ij} values from a random Gaussian distribution with mean zero and variance $1/(N-1)$. In contrast to other studies,[23] the matrix is asymmetrical ($a_{ji} \neq a_{ij}$), which complicates the theoretical analysis, but makes the model more general and enriches the dynamics. Initial conditions are taken randomly from a small (10^{-8}) Gaussian neighborhood of the origin, and the equations are iterated 10^5 or more times using a fourth-order Runge–Kutta integrator with a fixed step size of 0.1. The results were compared to a fifth-order Runge–Kutta integrator with an adaptive step size and an absolute error bound of 10^{-6} at each step.

The largest Lyapunov exponent is calculated using the method of Sprott.[12] An N-dimensional system has N Lyapunov exponents, but we will be concerned only with the largest Lyapunov exponent because its sign indicates the nature of the dynamics, with a positive value signifying chaos. The calculation of the Lyapunov exponent is the most challenging task in this work, but the Lyapunov exponent is critical for studying chaos. Existing routines are available,[24] but much is gained by programming the method from scratch, and it is not conceptually difficult. It involves following two initially nearby orbits and averaging the logarithmic growth of their separation while normalizing the separation to the initial value at each time step but preserving the orientation. A practical problem is that the value often converges slowly with large fluctuations, especially in the vicinity of bifurcations. The orbit must be followed for a long enough time to reach and then sample all regions of the attractor, whose dimension can be 50 or more. Some measure of convergence is useful, such as requiring that the amplitude of the fluctuations over the previous thousand time steps be less than the resolution of the plot. Many of the figures in this paper required several days of computation using a compiled language.

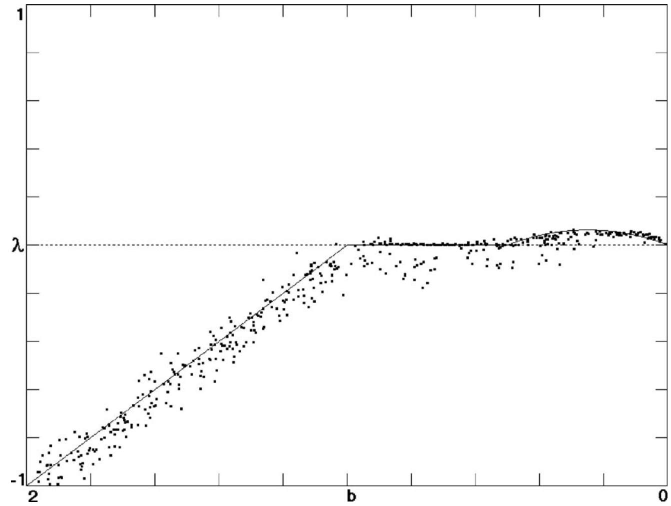

Fig. 1. Largest Lyapunov exponent λ for a collection of 472 fully connected artificial neural networks in Eq. (5) with $N = 101$ and random Gaussian weights.

Values of b were chosen randomly in the range $0 < b < 2$, and the Lyapunov exponents for 472 networks are plotted in Fig. 1. Note that the scale for b is plotted such that the network activity increases to the right as the damping is reduced. Three regimes are evident in the figure. For strong damping ($b > 1$), most values of the largest Lyapunov exponent λ are negative, implying a stable equilibrium. For $1 \geqslant b \geqslant 0.5$ most Lyapunov exponents are zero, implying a periodic (limit cycle) or quasiperiodic (attracting torus) solution. For weak damping ($0.5 > b > 0$) most Lyapunov exponents are positive, implying a chaotic solution and an accompanying strange attractor. The value of λ is reasonably well fit in the three regimes by the simple piecewise-continuous function

$$\lambda = \begin{cases} 1 - b & (b > 1) \\ 0 & (1 \geqslant b \geqslant 0.5) \\ b(0.5 - b) & (0.5 > b \geqslant 0) \end{cases} \tag{6}$$

shown as the solid line in Fig. 1. Note that the maximum value of λ from Eq. (6) is 1/16 and occurs at $b = 1/4$. This relatively small value is consistent with the appealing but controversial idea that complex adaptive systems evolve at the "edge of chaos."[25] Although it is not shown here, the fractal dimension of the attractor, which is a measure of its complexity, for the maximally chaotic systems with $b = 1/4$ is approximately $N/2$. These systems exhibit the quasiperiodic route to chaos that is thought to be generic for high-dimensional systems[26,27] with a variation of λ, which

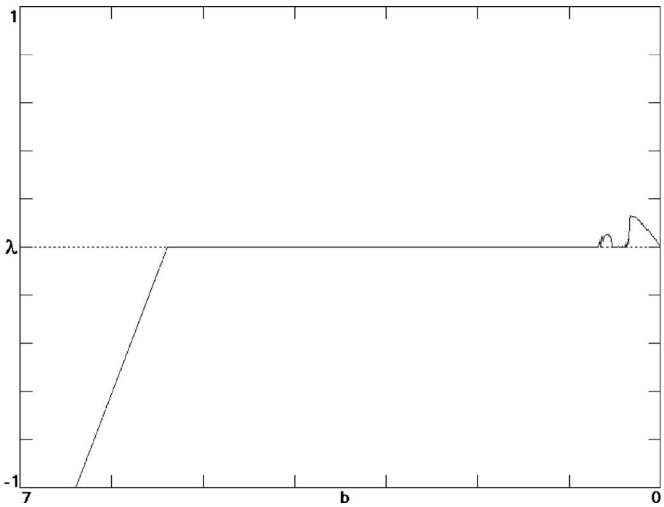

Fig. 2. Largest Lyapunov exponent λ for the sparse circulant neural network in Eq. (7) with $N=101$ and four-neighbor interactions.

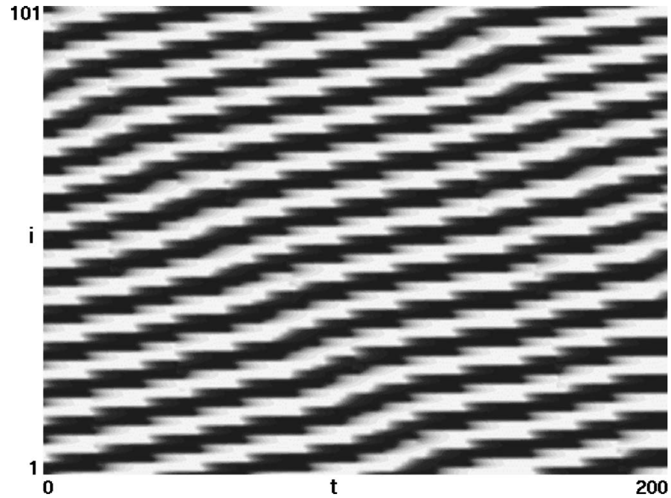

Fig. 3. Spatiotemporal plot of chaos in a sparse circulant neural network with $N=101$ and $b=0.25$.

depends only weakly on the details of the interactions and hence may be universal in the limit of infinite N. The plot in Fig. 1 will be used as the standard to which other simpler models will be compared.

III. SPARSE CIRCULANT NEURAL NETWORKS

Although the prototypical system studied in Sec. II is reasonably simple and ripe for student projects such as studying the effect of varying N, the distribution of weights (a_{ij} values), and the connectedness of the network on the likelihood that a solution is chaotic, its route to chaos, and the value of the largest Lyapunov exponent, our interest here is finding even simpler systems with similar behavior. In particular, we want to keep the dimension high, but reduce the number of parameters in order to speed computation and facilitate exploration. In particular, we desire interaction matrices that are circulant,[28] corresponding to neurons arranged on a ring, each interacting identically with its neighbors, and sparse (most interactions are zero), and also restricted to values $a_{ij}=\pm a$ (unweighted networks). These conditions lead naturally to networks described by

$$\dot{x}_i = -bx_i + \tanh\left(\sum_{j=1}^{N-1} a_j x_{i+j}\right), \tag{7}$$

where a_j is a vector with most components zero and $x_{i+j} = x_{i+j-N}$ for $i+j>N$ (periodic boundary conditions). Sparse networks (also called diluted networks),[29,30] even in a ring configuration,[31] have been studied using both discrete-time and continuous-time models,[32] but typically not with circulant matrices.

Although chaos is relatively rare in such networks, one example with $N=101$, found by trial and error, has $a_j=(4, -5, 1, -3, 0, 0, \ldots)$ and a route to chaos as shown in Fig. 2. No claim is made that this system is the simplest such example or that it is optimized in any way, but it does illustrate that the number of parameters can be reduced from $\sim 10^4$ to 4 and still achieve similar behavior, except for a somewhat broader periodic region. Furthermore, this example has relatively few of the periodic windows and other bifurcations that are so ubiquitous in low-dimensional chaotic systems, illustrating how high-dimensional chaotic systems generally have simpler behavior than their low-dimensional counterparts. Even more remarkable is that the dynamics in the periodic and chaotic regimes are asymmetric (the x_i values are not all equal) even though the equations are symmetric. This symmetry breaking is a common but counterintuitive feature of such circulant networks. It is necessary to use asymmetric initial conditions to avoid synchronization that would preclude the chaos, but the choice of initial conditions is otherwise not critical except when there are multiple coexisting attractors.

Circulant networks with near-neighbor interactions provide the opportunity to study spatiotemporal chaos. One way to illustrate this behavior is with a spatiotemporal plot in which the values of $x_i(t)$ are plotted in the $i-t$-plane, as shown in Fig. 3 for $b=0.25$. There is a dominant wavelength of about 8.5, which probably has to do with the size of the neighborhood (4 in this case), and a dominant period of about 60, implying that the structures propagate at a velocity of about 1/7. The behavior is not perfectly periodic, but shows signs of chaos as confirmed by $\lambda=0.1149$, which is nearly twice the value of 0.0625 predicted by Eq. (6).

Although Fig. 3 used initial conditions on the attractor (obtained by discarding the first 200 time units with a random initial condition), we could also observe the self-organization and pattern formation that occurs when the initial conditions are chosen randomly or highly ordered (but not uniform). Such plots offer one of the few ways to visualize chaos in systems whose attractor dimension is high, and they illustrate why circulant examples are desired. This system invites a study of the effect of sparseness and neighborhood size on the dynamics, with implications for social and other types of complex networks. An interesting challenge would be to find an even simpler and more elegant circulant chaotic system than the one in Fig. 3.

IV. HYPERLABYRINTH CHAOS

A particularly simple circulant complex system was suggested by Thomas *et al.*[33] and has been called hyperlabyrinth chaos[34] because the trajectory wanders throughout an N-dimensional, spatially periodic labyrinth. It is a simple

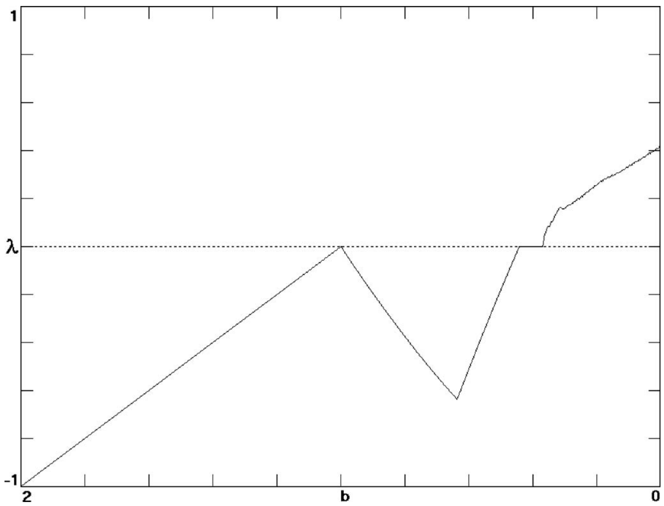

Fig. 4. Largest Lyapunov exponent λ for the hyperlabyrinth system in Eq. (8) with $N=101$.

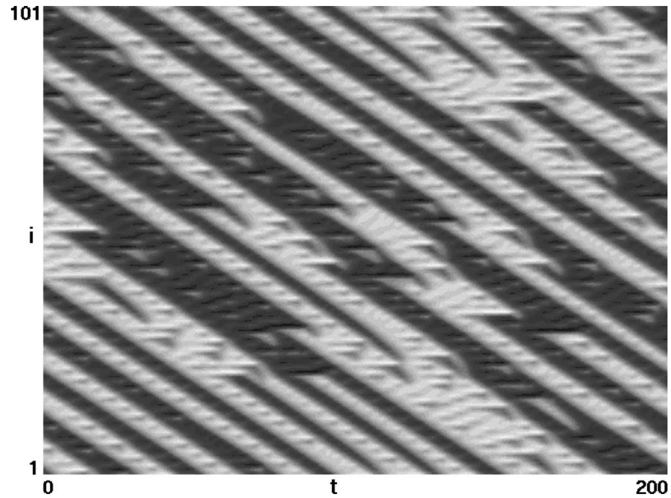

Fig. 5. Spatiotemporal plot of hyperlabyrinth chaos for $N=101$ and $b=0.25$.

variant of Eq. (7) in which the hyperbolic tangent is replaced by the sine, and each variable interacts with a single nearest neighbor according to

$$\dot{x}_i = -bx_i + \sin x_{i+1}. \tag{8}$$

Its route to chaos for $N=101$ as indicated in Fig. 4 resembles the previous cases except there is an additional pitchfork bifurcation at $b=1$ prior to the Hopf bifurcation at $b \simeq 0.4421$, a relatively narrow region of quasiperiodicity, and a positive Lyapunov exponent that increases all the way to the conservative limit of $b=0$, where its value is $\lambda=0.4188$. The complexity as measured by the Kaplan–Yorke dimension[35] D_{KY} increases linearly with N and decreases with b (for $b<0.34$) as $D_{KY} \approx (0.992-1.758b)N$. The spatiotemporal plot for $b=0.25$ in Fig. 5 shows self-organized propagating chaotic structures.

V. LOTKA–VOLTERRA MODEL

Another simple complex system is a variant of the Lotka–Volterra model,[36] which is popular among ecologists because it represents the interaction of different species competing for a set of common resources and can be considered as a low-order Taylor expansion of a wide range of more complicated models.[37] A particularly simple circulant example is given by[38]

$$\dot{x}_i = x_i(1 - x_{i-2} - bx_i - x_{i+1}). \tag{9}$$

The existence of a Lyapunov function[39] precludes periodicity and chaos in the more obvious symmetric case in which x_{i-2} is replaced by x_{i-1}. The route to chaos of Eq. (9) as indicated by Fig. 6 resembles the previous cases except that the periodic region is narrow and the solution becomes static for $b<0.62$, with some number of species surviving and all other x_i values equal to zero (the principle of competitive exclusion[40] or survival of the fittest). Because there are 2^N equilibrium points of which $N+1$ are nondegenerate, this region is very complicated and deserving of further study. The spatiotemporal plot for $b=0.8$ in Fig. 7 shows the now familiar self-organized propagating chaotic structures with a Lyapunov exponent of $\lambda=0.0227$.

VI. DELAY DIFFERENTIAL EQUATIONS

Other simple models with complex dynamics use delay differential equations,[41] the simplest form of which is the autonomous retarded functional differential equation $\dot{x}=f(x(t-\tau))$ in which the function f depends on the value of x at a single previous time $t-\tau$. Such equations have been used extensively to model population dynamics[42] with their inherent gestation and maturation time delays and to study epidemics,[43] tumor growth,[44] immune systems,[45] lossless electrical transmission lines,[46] and the electrodynamics of interacting charged particles (the Lorentz force with Liénard–Weichert potentials.)[47] Such systems are infinite-dimensional in the sense that infinitely many initial conditions over a continuous range $-\tau<t<0$ are required, but the system can be approximated by an $(N+1)$-dimensional system of ODEs such as

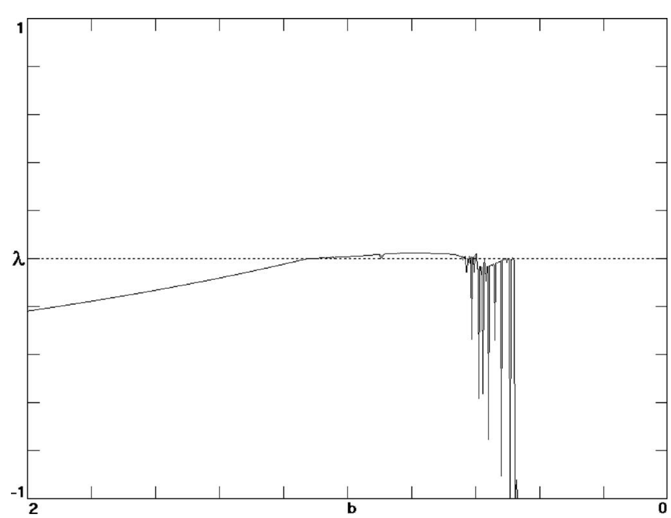

Fig. 6. Largest Lyapunov exponent λ for the Lotka–Volterra model in Eq. (9) with $N=101$.

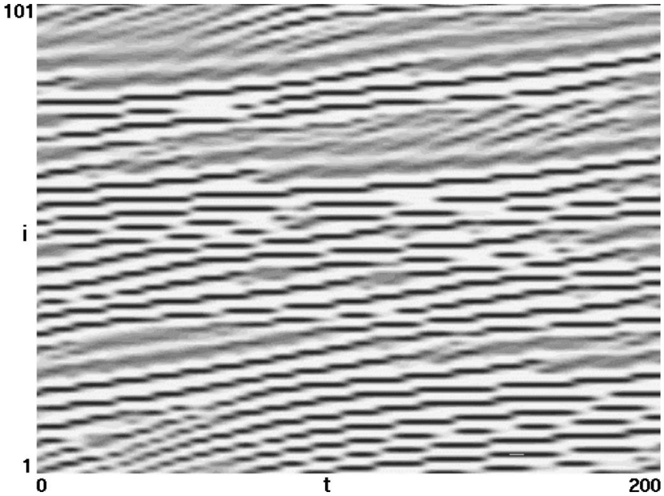

Fig. 7. Spatiotemporal plot of the Lotka–Volterra model for $N=101$ and $b=0.8$.

$$\dot{x}_0 = f(x_N) \tag{10a}$$

$$\dot{x}_i = N(x_{i-1} - x_i)/\tau. \tag{10b}$$

This representation illustrates that $1/\tau$ is a damping analogous to b in the previous examples.

Many functional forms of $f(x)$ are known to exhibit chaos; one of the simplest[48] is $f(x)=\sin x$, for which the route to chaos as shown in Fig. 8 with $N=100$ resembles the previous examples. This system is not an example of spatiotemporal chaos, but it does produce multiscroll strange attractors, as shown in Fig. 9 for $\tau=8$ for which $\lambda=0.0794$ and whose complexity increases linearly with τ as $D_{KY} \approx 0.437\tau + 0.406$. The attractor stretches from $-24 < x < 24$, and the trajectory exhibits Brownian motion despite its deterministic origin. Delay differential equations have been relatively little explored and offer many opportunities for student projects. For example, we could explore other nonlinear functions $f(x)$ in Eq. (10a) besides the sine and cosine.

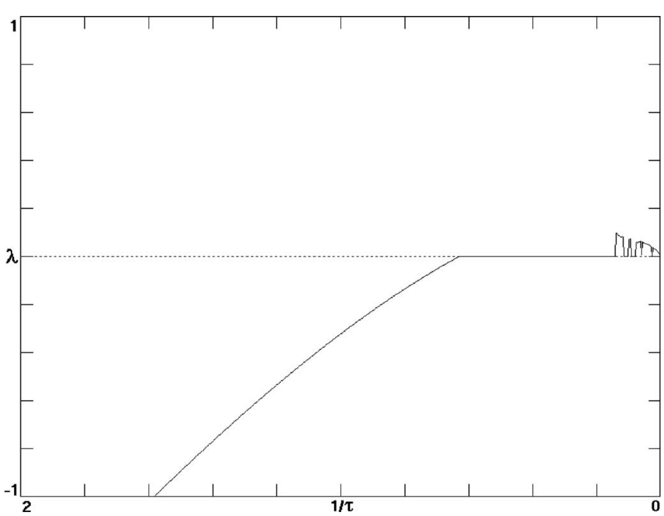

Fig. 8. Largest Lyapunov exponent λ for the delayed differential equation model in Eq. (10) with $f(x)=\sin x$ and $N=100$.

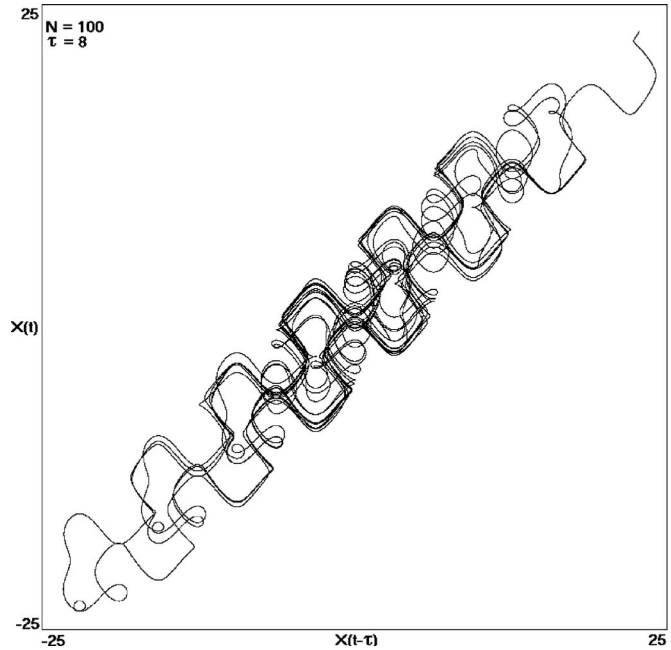

Fig. 9. Strange attractor for the delayed differential equation model in Eq. (10) with $f(x)=\sin x$, $\tau=8$, and $N=100$.

VII. PARTIAL DIFFERENTIAL EQUATIONS

Other infinite-dimensional systems are described by partial differential equations (PDEs) in which the temporal, spatial, and state variables are all continuous (in contrast to a cellular automaton in which they are all discrete). Many of the fundamental laws of physics are described by PDEs, including Maxwell's equations, the time-dependent Schrödinger equation, and the Navier–Stokes equation. One of the simplest PDEs that is known to exhibit chaos is the Kuramoto–Sivashinsky equation

$$\dot{u} = -uu' - u''/R - u'''', \tag{11}$$

where $u' = \partial u/\partial x$. Equation (11) consists of an antiviscosity term u'' which causes the long wavelength modes to grow and a hyperviscosity term u'''' which damps the short wavelength modes. The nonlinearity uu' transports energy from the growing modes to the damped modes. Equation (11) has been used to model waves in chemical reactions,[49] flame front modulations,[50] and a thin liquid film flowing down an inclined plane.[51] It is known to exhibit chaos for $R=2$.

One of the most straightforward ways to solve such a PDE numerically is to approximate it as a high-dimensional system of ODEs, in which case it resembles the previous examples. The derivatives can be represented by their lowest-order Taylor expansions as

$$u_i' = (u_{i+1} - u_{i-1})/2 \tag{12a}$$

$$u_i'' = u_{i+1} - 2u_i + u_{i-1} \tag{12b}$$

$$u_i'''' = u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}, \tag{12c}$$

which leads to the following system of ODEs for $R=2$,

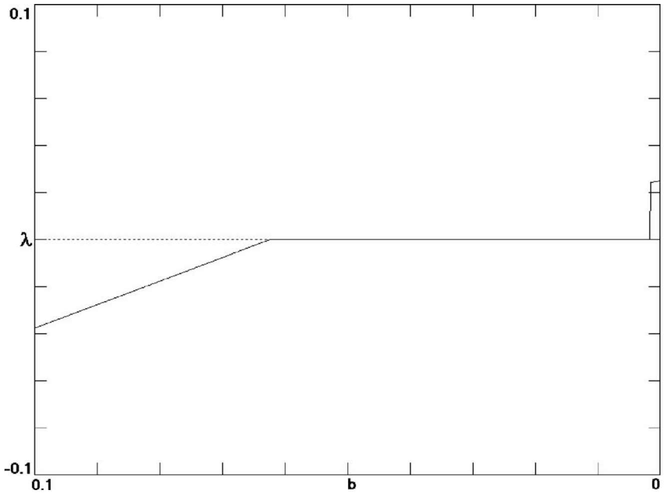

Fig. 10. Largest Lyapunov exponent λ for the PDE model in Eq. (13) with $N=101$.

$$\dot{u}_i = -0.5u_i(u_{i+1} - u_{i-1}) - u_{i+2} + 3.5u_{i+1} - 5u_i + 3.5u_{i-1} - u_{i-2} - bu_i. \tag{13}$$

An additional damping term $-bu$ has been added to serve as a bifurcation parameter, and the subscripts are periodic around a ring with N elements as in the previous cases.

Figure 10 shows the Lyapunov exponent as a function of b for $N=101$ with the now familiar behavior, albeit with a compressed scale on both axes. Figure 11 shows turbulent spatiotemporal behavior for $b=0$ where $\lambda=0.025$. Chaos in PDEs is an almost completely unexplored subject, and it is not even known whether Eq. (11) is the simplest such example, although it does serve as a proof of principle. A challenging student project would be to find a simpler example.

VIII. CONCLUSIONS

Nonlinear networks with many nodes and interconnections are very likely to behave chaotically provided their solutions are bounded. Thus we would expect chaos to be common in nature. Surprisingly, there are features of such networks, such as their route to chaos, that are relatively simple and do not depend strongly on the details of the interactions, and thus may be universal. Even more surprising is that there are very simple circulant networks with many nodes that have similar behavior, although such networks are not always easy to identify. These systems typically exhibit spatiotemporal chaos, self-organization, pattern formation, and symmetry breaking. This paper has described several such systems which can be used as a starting point for multidisciplinary research by undergraduates and others and which may have profound implications on our understanding of complex systems in nature.

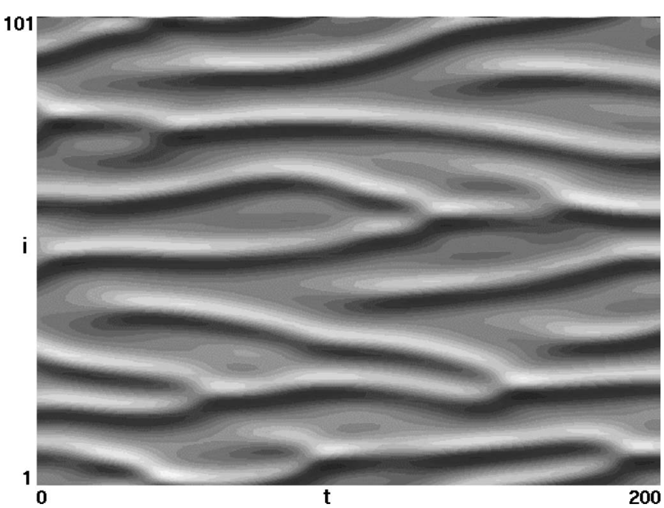

Fig. 11. Spatiotemporal plot of the PDE model in Eq. (12) for $N=101$ and $b=0$.

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