Self-Reconfigurable Control for Dual-Quaternion/Dual-Vector Systems

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***Abstract*— In this paper we suggest *self-reconfigurable* control for dual-quaternion systems with unknown control direction . The technique is based on the creation of multiple equilibrium surfaces for the system in the extended state space. We describe the mathematical tools of dual quaternions and technique required to design such system. Examples are presented to illustrate the proposed method.**

I. Introduction

As it is well known, in application to mechanical systems, namely in robotics, spacecraft control and others the quaternion formulation of the rotational kinematics has certain advantages. For example, it allows easily design the rotational kinematic part of the control by using the quaternion error. Further extension of this technique known as dual-quaternions makes it possible to represent by one dual-quaternion variable both rotational and translational spatial rigid body displacements. The dynamics in this case should be represented via dual-numbers and/or dual-vectors by introducing into the real numbers the dual unit ε satisfying the property $\varepsilon^2 = 0$. The space of dual-quaternions is actually a Clifford algebra. The models of mechanical systems that include many rotational and translational parts as well as actuator and other dynamics in this case have the multidimensional state space such that each of the dimensions is represented by dual quaternion. By combining a dual quaternion-based dynamic representation with sliding-mode control approach, simultaneous rotation and translation control can be achieved for spatial rigid body systems, where the dynamics contain multiple sources of uncertainty and unmodeled effects. In this work we use dual quaternion models in combination with self-configurable variable structure/sliding-mode control first suggested in the work [1].

Classical variable structure approach has been widely used for the problems of dynamic systems control and observation due to finite time convergence, robustness to uncertainties, and insensitivity to external disturbances especially in sliding mode (for basic ideas see, for example, [2]). The main thrust of the sliding mode control research for many years has been in designing an appropriate one-component sliding manifold described by the equations

$$\sigma_i(t,x) = 0 \tag{1}$$

where $\sigma = col(\sigma_1, ..., \sigma_m)$ and the goal of the design was to make the system reach their intersection

$$\{\sigma(t,x) = 0\} = \bigcap_{i=1}^{m} \{\sigma_i(t,x)\} = 0 \tag{2}$$

The present work concentrates, instead, on the design of equilibrium sets in the state space with more complicated structure than just the intersection of several one-component manifolds.

Robust control of various classes of uncertain nonlinear systems has been widely researched in controls literature .

A particularly challenging class of uncertain systems are those containing uncertainty in the control sign . The control sign in this context represents the control direction - the control force or torque direction, for example, under any given control command. While many of the systems addressed in previous controls literature concern systems with known time-varying control direction, control design for systems with unknown control direction is a much more challenging task.

The technique [1] was later extended to a larger class in [3]. The application of this method is especially useful for the mechanical systems acting in unpredictable uncertain environment. One of the successful examples is the design of the ABS system for the ground vehicles and landing aircrafts [4].

The method that was presented in [1], uses a purely robust feedback technique to achieve finite-time convergence to a sliding manifold in the presence of unknown control direction. This robust feedback control design is much less computationally intensive due to the fact that it requires no monitoring functions, function approximators, or online adaptive laws. By applying this computationally efficient control scheme with a compact dual-quaternion-based dynamic paramerization, an effective and versatile control method can be developed to achieve simultaneous translational and rotational control of a spatial rigid body.

To achieve computationally efficient six degree of freedom (DOF) control (i.e., simultaneous translational and rotational control) of a spatial rigid-body system, proper choice of the position and orientation vector parameterization is critical.

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Three-element orientation vectors such as Euler angles can provide a unified representation of position and orientation; however, the Euler parameterization has inherent singularities in the parametrization. The unit quaternion has the benefit that it does not suffer from singularities. However, it has been shown that quaternion-based controller design can be complicated by the fact that the position and orientation errors are calculated separately [5], [6]. To eliminate the need for control designs using two separate loops for controlling rotation and translation, dual quaternions are widely regarded as the most compact and efficient means for simultaneously representing translational and rotational motion [5], [7], [8].

By utilizing multi component sliding surfaces based on the dual quaternion norms sliding mode control algorithms developed in this paper allow to achieve finite-time convergence to a sliding manifold for a class of dual-quaternion-based systems with unknown control input direction. The result is a robust control law, which is rigorously proven to achieve finite-time convergence to a sliding manifold in the presence of unknown control direction, without the use of function approximators or online parameter adaptation.

II. Mathematical Preliminaries

In this section, we present the mathematical concepts of dual vectors and dual quaternions required to formulate proper the class of systems and the suggested control algorithm.

As it is known, one can formally extend the set of real numbers $\mathbb{R}$ by adding a *dual factor* ϵ with nilpotent property $\epsilon^2 = 0$ and considering dual numbers of the form $\hat{x} = x+\epsilon x'$. The set of such numbers $\mathbb{DR}$ is called a set of dual numbers. In a similar way, we can extend $\mathbb{R}^n$ and $\mathbb{C}^n$, thus obtaining the sets of dual vectors $\mathbb{DR}^n$ and $\mathbb{DC}^n$ and dual quaternions $\mathbb{DH}^n$. Namely, dual quaternions are defined as

$$\hat{q} = q + \epsilon q' \tag{3}$$

where $q \in \mathbb{H}$ and $q' \in \mathbb{H}$ are quaternions [9]. The product of dual quaternions is defined as

$$\hat{q}_1 \circ \hat{q}_2 = q_1 \circ q_2 + \epsilon(q_1' \circ q_2 + q_1 \circ q_2'), \tag{4}$$

where in the right hand side the symbol $\circ$ is a standard product of quaternions.

A 6-DOF transformation consisting of a rotation q followed by a translation $\mathbf{p} \in \mathbb{R}^3$, expressed in the body frame, can be represented by a dual quaternion by setting

$$q' = \frac{1}{2} q \circ [0, \mathbf{p}]. \tag{5}$$

A dual quaternion defined in this matter has the characteristic $q \cdot q' = 0$ and is referred to as normalized. A normalized dual quaternion can also be defined as

$$\hat{q} = q + \epsilon q'^{\epsilon\gamma} \tag{6}$$

where $\gamma = q'/q$. Based on (5), $\gamma = \frac{1}{2}\mathbf{p}$. The logarithm of a dual quaternion can then be defined as

$$\log \hat{q} = \log(q e^{\epsilon \frac{1}{2}\mathbf{p}}) = \frac{1}{2}(\theta \mathbf{n} + \epsilon p \mathbf{s}) \tag{7}$$

where $\theta \in \mathbb{R}$ and $\mathbf{n} \in \mathbb{R}^3$ are respectively the rotational angle and eigenaxis, and $p \in \mathbb{R}$ and $\mathbf{s} \in \mathbb{R}^3$ are respectively the norm and unit direction of $\mathbf{p}$. The dual quaternion $\hat{O} \in \mathbb{DH}$ is defined as $\hat{O} = (1,0,0,0) + \epsilon(0,0,0,0)$ and the $\log(\pm\hat{O})$ are dull null vectors [10].

Another useful function is $\mathbf{sgn}^\rho(\mathbf{x}) \in \mathbb{R}^n$ defined as [11]:

$$\mathbf{sgn}^\rho(\mathbf{x}) = [|x_1|^\rho \mathrm{sgn}(x_1) ... |x_n|^\rho \mathrm{sgn}(x_n)]^T \tag{8}$$

where $\mathbf{x} \in \mathbb{R}^n$ is an arbitrary vector and $\rho \in \mathbb{R}$. For a dual vector $\hat{\mathbf{x}} = \mathbf{x}_1 + \epsilon \mathbf{x}_2 \in \mathbb{DR}^n, \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ [11]:

$$\mathbf{sgn}^\rho(\hat{\mathbf{x}}) = \mathbf{sgn}^\rho(\mathbf{x}_1) + \epsilon \mathbf{sgn}^\rho(\mathbf{x}_2). \tag{9}$$

III. Model

For a single rigid body the kinematic equation describing simultaneously rotation and translation is

$$\dot{\hat{q}} = \frac{1}{2}\hat{q} \circ [\hat{0}, \hat{\omega}] \tag{10}$$

where the dual vector $\hat{\omega} \in \mathbb{HR}^3$, called a twist, is defined as

$$\hat{\omega} = \omega + \epsilon \mathbf{v} = \omega + \epsilon(\dot{\mathbf{p}} + \omega \times \mathbf{p}). \tag{11}$$

where $\omega \in \mathbb{R}^3$ is the angular velocity in the body frame. The vectors $\mathbf{p}$ and $\dot{\mathbf{p}}$ refer to the position and body-frame time derivatives in the body frame while the vector $\mathbf{v} \in \mathbb{R}^3$ refers to the translation velocity in the body frame.

In order to describe the dynamics of dual quaternions, the dual inertia matrix is defined as

$$\begin{aligned} \hat{\mathbf{M}} &= m\frac{d}{d\epsilon}\mathbf{I} + \epsilon \mathbf{J} \\ &= \begin{bmatrix} m\frac{d}{d\epsilon} + \epsilon J_{xx} & \epsilon J_{xy} & \epsilon J_{xz} \\ \epsilon J_{xy} & m\frac{d}{d\epsilon} + \epsilon J_{yy} & \epsilon J_{yz} \\ \epsilon J_{xz} & \epsilon J_{yz} & m\frac{d}{d\epsilon} + \epsilon J_{zz} \end{bmatrix} \end{aligned} \tag{12}$$

where m is mass, $\mathbf{J} \in \mathbb{R}^{3\times3}$ is the inertia matrix, and $\mathbf{I} \in \mathbb{R}^{3\times3}$ the real identity matrix. The operator $\frac{d}{d\epsilon}$ is complementary to the element ϵ. The operations ϵ and $\frac{d}{d\epsilon}$ are defined as follows

$$\begin{aligned} \epsilon\hat{\mathbf{v}} &= \epsilon(\mathbf{v} + \epsilon\mathbf{v}') = \epsilon\mathbf{v} \\ \frac{d}{d\epsilon}\hat{\mathbf{v}} &= \frac{d}{d\epsilon}(\mathbf{v} + \epsilon\mathbf{v}') = \mathbf{v}'. \end{aligned} \tag{13}$$

The inverse of the dual inertia matrix is defined as $\hat{\mathbf{M}}^{\mathbf{-1}} = \mathbf{J}^{\mathbf{-1}}\frac{d}{d\epsilon} + \epsilon\frac{1}{m}\mathbf{I}$.

The dynamics of a rigid body is then defined as

$$\dot{\hat{\omega}} = -\hat{\mathbf{M}}^{\mathbf{-1}}(\hat{\omega} \times \hat{\mathbf{M}}\hat{\omega}) + \hat{\mathbf{M}}^{\mathbf{-1}}\hat{\mathbf{f}} \tag{14}$$

where $\hat{\mathbf{f}} = \mathbf{f} + \epsilon\,\tau \in \mathbb{DR}^3$ is a dual vector called the force motor with $\mathbf{f} \in \mathbb{R}^3$ and $\tau \in \mathbb{R}^3$ being the force and torque vectors in the body frame. The kinematics and dynamics of a single rigid body can be expanded on and generalized to include multiple bodies:

$$\dot{\hat{q}}^i = \frac{1}{2}\hat{q}^i \circ [\hat{0}, \hat{\omega}^i] \tag{15}$$

$$\begin{aligned}\dot{\hat{\omega}}^i = &-\hat{\mathbf{M}}_i^{-1}\hat{\mathbf{g}}_i\left(\hat{q}^1, \hat{q}^2, ..., \hat{q}^n, \hat{w}^1, \hat{w}^2, ..., \hat{w}^n, t\right) \\ &+ \hat{\mathbf{M}}_i^{-1}\hat{\mathbf{h}}_i(\hat{q}^1, \hat{q}^2, ..., \hat{q}^n, \hat{w}^1, \hat{w}^2, ..., \hat{w}^n, t)\hat{\mathbf{f}}^i\end{aligned} \tag{16}$$

where the integer i represents a single rigid body and n represents the total number of rigid bodies. In these equations the functions $\hat{\mathbf{g}}_i$ and $\hat{\mathbf{h}}_i$ represent the internal forces/torques and the direction of the control force $\hat{\mathbf{f}}^i = \hat{\mathbf{u}}_i$, respectively.

Introducing the generalized position dual quaternion vector $\hat{\mathbf{Q}} = [\hat{q}^1, \hat{q}^2, \ldots, \hat{q}^n]^T \in \mathbb{DH}^n$ and generalized dual velocities vector $\hat{\mathbf{\Omega}} = [\hat{\omega}^1, \hat{\omega}^2, \ldots, \hat{\omega}^n]^T \in \mathbb{DR}^n$ the model can be written as

$$\dot{\hat{\mathbf{Q}}} = \frac{1}{2}\hat{\mathbf{Q}} \circ [\hat{0}, \hat{\mathbf{\Omega}}] \tag{17}$$

$$\dot{\hat{\mathbf{\Omega}}} = -\hat{\mathbf{M}}^{-1}\hat{\mathbf{g}}(\hat{\mathbf{Q}}, \hat{\mathbf{\Omega}}, t) + \hat{\mathbf{M}}^{-1}\hat{\mathbf{h}}(\hat{\mathbf{Q}}, \hat{\mathbf{\Omega}}, t)\hat{\mathbf{u}}. \tag{18}$$

If in the equations (15),(17) $\hat{\mathbf{\Omega}} = [\hat{\omega}^1, \hat{\omega}^2, \ldots, \hat{\omega}^n]^T$ is considered as control the logarithmic feedback law can be used to solve the kinematic regulation control problem [10] $\hat{\omega}^i = -2k \log \lambda \hat{q}^i,\ k > 0$ or

$$\hat{\mathbf{\Omega}} = -2k \log \lambda \hat{\mathbf{Q}}, \tag{19}$$

where the log is understood componentwise. The parameter λ is used to have the controller take the shorter path for the identical equilibrium positions $\hat{O}$ and $-\hat{O}$, where [10]

$$\lambda = \begin{cases} 1, & \text{if } \hat{q}(0) \cdot \hat{O} \geq 0 \\ -1, & \text{otherwise.} \end{cases} \tag{20}$$

Using (19), the sliding surface is described as

$$\hat{\sigma} = \hat{\mathbf{\Omega}} + 2k \log \lambda \hat{\mathbf{Q}} = 0. \tag{21}$$

IV. Self-Reconfigurable Control

In this section, we present control algorithm to solve stabilization problem to the sliding manifold

$$\mathcal{M} = \{[\hat{\mathbf{Q}}, \hat{\mathbf{\Omega}}] \in \mathbb{DH}^n \times \mathbb{DR}^n | \hat{\sigma}(\hat{\mathbf{Q}}, \hat{\mathbf{\Omega}}) = 0\} \tag{22}$$

introduced in the previous section. Let us note here, that $\hat{\sigma}$ is an n-dimensional dual vector belonging to $\mathbb{DR}^n$ which is a linear vector space[1]. It is assumed that the dynamic model (17), (18) contains an unknown, state- and time-varying input gain matrix, which causes unmodeled variations that manifest themselves as a priori unknown changes in the commanded control direction. Once the dual-quaternion-based dynamic model is expressed in the general form, a robust sliding mode controller will be presented, which will be proven to mitigate the unknown control direction based on the approach suggested in [1], [4] and achieve finite-time convergence to a sliding surface.

Differentiating (21) we obtain

$$\dot{\hat{\sigma}} = \hat{\mathbf{B}}(\hat{\mathbf{Q}}, \hat{\mathbf{\Omega}}, t)\hat{\mathbf{u}} + \hat{\mathbf{F}}(\hat{\mathbf{Q}}, \hat{\mathbf{\Omega}}, t) \tag{23}$$

where $\hat{\mathbf{B}} = \hat{\mathbf{M}}^{-1}\hat{\mathbf{h}}(\hat{\mathbf{Q}}, \hat{\mathbf{\Omega}}, t) \in \mathbb{DR}^{n \times n}$ is a matrix defining direction of the control action in σ-space. Our goal is to develop a control algorithm that does not require knowledge of $\hat{\mathbf{B}}$, but we assume that this dual matrix satisfy natural conditions that follow from mechanical properties of the controlled system. $\hat{\mathbf{B}}$ is such that (i) it is nonsingular almost everywhere, and (ii) there exists matrix $\hat{\mathbf{U}}_0$ such that the corresponding quadratic form $\xi^T\hat{\mathbf{B}}\hat{\mathbf{U}}_0\xi$ is sign definite. The matrix $\hat{\mathbf{B}}\hat{\mathbf{U}}_0$ can be state dependent. In this case a manifold (if such exists) in $\mathbb{DH}^n \times \mathbb{DR}^n$ space where this matrix can become singular does not intersect with the desired sliding manifold $\mathcal{M}$ at least in some area of the state space $\mathbb{DH}^n \times \mathbb{DR}^n$ where the system trajectories evolve.

The main idea behind our control is in partitioning the $\hat{\sigma}$-subspace onto a grid comprised of concentric manifolds that are spheres defined by $\|\hat{\sigma}\|_p^p = \Delta(t)k$, where $\Delta(t) > 0$ is the variable grid step, k is a nonnegative integer and $\|\cdot\|_p$ is a p-norm. Inside each layer $\mathcal{L}_k$ between these manifolds $\mathcal{L}_k = \{\Delta(t)k \leq \|\hat{\sigma}\|_p^p \leq \Delta(t)(k+1)\}$ the control may be constant, but its sign alternates from one layer to another. We show that this control structure under a nonsingularity condition results in a set of stable equilibrium spheres in $\hat{\sigma}$-subspace. Then we choose the dynamics of $\Delta(t)$ so that eventually all spheres radii converge to zero, thus, stabilizing $\hat{\sigma}$ to the origin of the corresponding dual vector space.

The union of the concentric manifolds forms a switching manifold:

$$\mathcal{G} = \bigcup_{k=0,\pm1,\ldots}^{r} \mathcal{G}_k = \bigcup_{k=0,\pm1,\ldots}^{r} \left\{x : \|\hat{\sigma}(x)\|_p^p = \Delta(t)k\right\}. \tag{24}$$

We pick $\hat{\mathbf{u}}$ as

$$\hat{\mathbf{u}} = \hat{\mathbf{U}}_0\mathbf{sgn}\left[\sin\left(\pi\frac{\|\hat{\sigma}\|_p^p}{\Delta(t)}\right)\right]\mathbf{sgn}(\hat{\sigma}), \tag{25}$$

where $\hat{\mathbf{U}}_0$ is a dual matrix control gain of the form:

$$\hat{\mathbf{U}}_0 = \mathbf{K_f}\frac{d}{d\epsilon} + \epsilon\mathbf{K}_\tau \tag{26}$$

where $\mathbf{K_f}$ and $\mathbf{K}_\tau$ may be constant diagonal matrixes or state dependent diagonal matrixes related to the gains for translation and rotation respectively. The operators $\frac{d}{d\epsilon}$ and ϵ are included in the dual matrix control gain so that the real (force) and dual (torque) components of the force motor $\hat{\mathbf{f}}$ are

[1]It is also a Banach space with corresponding norms. So that, for example, if $\hat{\sigma} = \sigma + \epsilon\sigma'$, then this dual vector p-norm is $\|\hat{\sigma}\|_p = \left[\|\sigma\|_p^p + \|\sigma'\|_p^p\right]^{\frac{1}{p}} = \left[\sum_{k=1}^n (|\sigma_k|^p + |\sigma'{}_k|^p)\right]^{\frac{1}{p}}$, $(p \geq 1)$.

associated with the dual (displacement) and real (rotation) components of the sliding surface $\hat{\sigma}$. Let us note that the term $\mathbf{sgn}\left[\sin\left(\pi\frac{\|\hat{\sigma}\|_p^p}{\Delta(t)}\right)\right]$ is a changing sign scalar and the last term $\mathbf{sgn}(\hat{\sigma})$ in (25) is a dual vector part of the control that alternate signs in the quadrants of the corresponding dual vector space $\mathbb{DR}^n$. It is needed to guarantee stability on one of the sliding manifolds $\mathcal{G}_k$.

The function $\Delta(t)$ is the following:

$$\Delta(t) = C - \mu\int_0^t \|\hat{\sigma}(\tau)\|_p^p d\tau, \tag{27}$$

where $C > 0$ is chosen from the area of initial conditions and $\mu > 0$ is a control parameter regulating spheres' radii convergence rate.

In the Fig 1 we demonstrate the vector field of velocities and one possible scenario of convergence toward the sliding manifold (red line) using control (25). In this simulation experiment the matrix B and the initial conditions were chosen randomly.

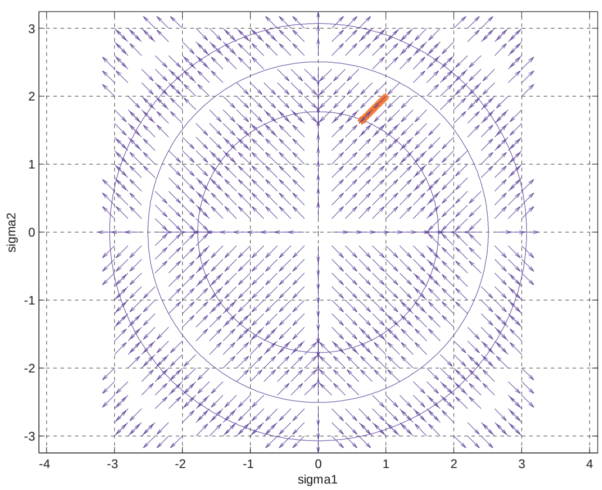

Fig. 1. Equilibrium manifolds in $\hat{\sigma}$-space ($p = 2$). Red line shows one of the actual trajectories converging to the manifold $\|\hat{\sigma}\|_p^p = const$.

Let us also note, that in (25) we used $\sin(\pi x)$ function only for convenience and relate this control algorithm to the one described in [3]. In fact, the main property that is required from this function is the alternating sign. It does not even have to be periodic. So the more general form of the control algorithm is

$$\hat{\mathbf{u}} = \hat{\mathbf{U}}_0\psi\left(\frac{\|\hat{\sigma}\|_p^p}{\Delta(t)}\right)\mathbf{sgn}(\hat{\sigma}),, \tag{28}$$

where the function $\psi(x)$ is such that, for example, $\psi(x) = sign(x)$ if $|x| \leq 1$, and $\psi(x) = -sign(x)$ otherwise.

Proof: Here we provide the sketch of the proof just to demonstrate our technique by considering the case of real $\hat{\mathbf{B}}$. The general situation is treated similarly by considering separately the real and dual part of the control (25). We will also take $p = 1$, and real scalar $\hat{\mathbf{U}}_0$. The idea behind the convergence proof is the following: we consider a Lyapunov function with multiple zeroes:

$$V = \left|\sin\left(\pi\frac{\|\hat{\sigma}\|_1}{\Delta(t)}\right)\right|. \tag{29}$$

V is positive everywhere except it is zero at the points where

$$\|\hat{\sigma}\|_1 = \Delta(t)k \tag{30}$$

$k = 0, \pm 1, \pm 2, \ldots$.

The derivative of V along the system trajectories is

$$\dot{V} = \mathbf{sgn}\left[\sin\left(\pi\frac{\|\hat{\sigma}\|_1}{\Delta(t)}\right)\right]\cos\left(\pi\frac{\|\hat{\sigma}\|_1}{\Delta(t)}\right)\pi\frac{d}{dt}\left[\frac{\|\hat{\sigma}\|_1}{\Delta(t)}\right]. \tag{31}$$

Since the 1-norm can be represented as $\|\sigma\|_1 = (\mathbf{sgn}\sigma)^T\sigma$ the derivative of $\|\sigma\|_1$ using (23) can be written as

$$\frac{d\|\hat{\sigma}\|_1}{dt} = (\mathbf{sgn}\sigma)^T\dot{\sigma} = (\mathbf{sgn}\sigma)^T\hat{\mathbf{B}}\hat{\mathbf{u}} + (\mathbf{sgn}\sigma)^T\hat{\mathbf{F}}, \tag{32}$$

or using (25) we have

$$\begin{aligned}\frac{d\|\hat{\sigma}\|_1}{dt} &= (\mathbf{sgn}\sigma)^T\hat{\mathbf{B}}\hat{\mathbf{U}}_0(\mathbf{sgn}\sigma)\mathbf{sgn}\left[\sin\left(\pi\frac{\|\hat{\sigma}\|_1}{\Delta(t)}\right)\right] \\ &+ (\mathbf{sgn}\sigma)^T\hat{\mathbf{F}}.\end{aligned} \tag{33}$$

Combining this with (31) the Lyapunov function derivative can be written as

$$\dot{V} = \frac{\pi}{\Delta(t)}(\mathbf{sgn}\sigma)^T\hat{\mathbf{B}}\hat{\mathbf{U}}_0(\mathbf{sgn}\sigma)\cos\left(\pi\frac{\|\hat{\sigma}\|_1}{\Delta(t)}\right) + \mathbf{G}, \tag{34}$$

where we combined all terms that don't depend on the control in the variable $\mathbf{G}$. By our assumption the quadratic form with the matrix $\hat{\mathbf{B}}\hat{\mathbf{U}}_0$ in this expression is sign definite. On the other hand at the points where $V = 0$ ($\|\hat{\sigma}\|_1 = \Delta(t)k$) the cos is $+1$ or -1, so at every other point it is guaranteed that $\dot{V} < 0$ if the norm of $\hat{\mathbf{U}}_0$ is big enough. This proves the stability some of the points (30). In fact, sliding mode will start at one of these points and $\|\hat{\sigma}\|_1 = \Delta(t)k$ will be true after some moment of time.

Now using the expression for Δ (27) and (30) we have

$$\|\hat{\sigma}\|_1 = \left[C - \mu\int_0^t \|\hat{\sigma}(\tau)\|_1 d\tau\right]k. \tag{35}$$

The latter is stable equation that guarantees $\|\hat{\sigma}\|_1 \to 0$ exponentially as $t \to \infty$. ∎

V. Application Case

A. Case I:

In our first numerical example to test the control law (25), (27) we consider a planar motion of the rigid body:

$$
\begin{aligned}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{\theta} &= \omega \\
\dot{v_x} &= f_x \\
\dot{v_y} &= f_y \\
\dot{\omega} &= \tau
\end{aligned} \tag{36}
$$

where v_x, v_y are the velocity in the x and y direction, ω is the angular velocity, and $\hat{f} = [f_x, f_y, \tau]^T$ is the generalized force vector that depends on the control $u \in \mathbb{R}^3$. $\hat{f}$ and u are related via an unknown 3×3 possibly state dependent matrix $B = B(X)$ ($X = [x, y, v_x, v_y, \theta, \omega]^T$) satisfying conditions listed on the previous page.

$$\hat{f} = B(X)u. \tag{37}$$

Let our sliding surface be $\sigma = [\sigma_1 \ \sigma_2 \ \sigma_3]^T$, where

$$
\begin{aligned}
\sigma_1 &= k_x x + v_x \\
\sigma_2 &= k_y y + v_y \\
\sigma_3 &= k_\theta \theta + \omega.
\end{aligned} \tag{38}
$$

The objective is to drive the system (36) to the origin with an orientation of $\theta = 0$. Differentiating σ we have

$$\dot{\sigma} = B(X)u + G. \tag{39}$$

We used control (25), (27) with the following values:

$$
\begin{aligned}
[x, y, \theta] &= [2, -1.5, \frac{\pi}{4}] \\
C &= .3 \\
\mu &= .04 \\
k_x &= .12 \\
k_y &= .12 \\
k_\theta &= .1 \\
\mathbf{U}_0 &= .2\mathbf{I}
\end{aligned} \tag{40}
$$

In the Fig. 2 and Fig. 3 we demonstrate convergence to the sliding manifolds and initial part of the convergence for the variables x, y, θ.

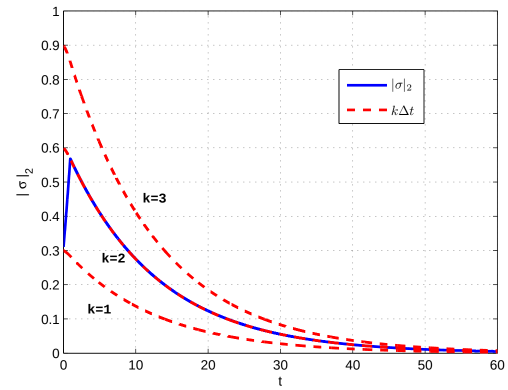

Fig. 2. Convergence of $|\sigma|_2$ to $k\Delta t$

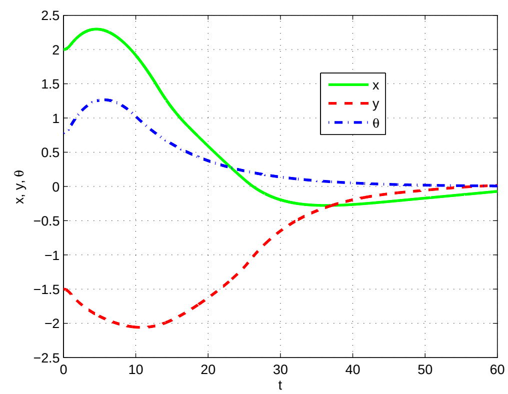

Fig. 3. Convergence of positions x, y, θ

B. Case II:

In our second numerical example we consider the 6-DOF motion of a rigid body using dual quaternions:

$$
\begin{aligned}
\dot{\hat{q}} &= \frac{1}{2}\hat{q} \circ [\hat{0}, \hat{\omega}] \\
\dot{\hat{\omega}} &= -\hat{\mathbf{M}}^{-1}(\hat{\omega} \times \hat{\mathbf{M}}\hat{\omega}) + \hat{\mathbf{M}}^{-1}\hat{\mathbf{f}}
\end{aligned} \tag{41}
$$

where $\hat{q}$ is the dual quaternion, $\hat{\omega}$ is the dual velocity vector and $\hat{\mathbf{f}} = [f_x, f_y, f_z]^T + \epsilon[\tau_x, \tau_y, \tau_z]^T$ is the generalized force vector that depends on the control $\hat{\mathbf{u}} \in \mathbb{DR}^3$. Let our sliding surface be $\hat{\sigma}$, where

$$\hat{\sigma} = (\omega + k_\theta \theta \mathbf{n}) + \epsilon(\mathbf{v} + k_p \mathbf{p}) \tag{42}$$

The objective is to drive the system (41) to the origin with an orientation of $\theta = 0$ or $q = [1, 0, 0, 0]$.

In Fig. 4 we show the convergence of to real and dual components of the sliding manifold. In the Fig. 5 and Fig. 6 we demonstrate convergence for the rotational and translational position respectively. For the purpose of this simulation the following values were used:

$$
\begin{aligned}
q(0) &= [.1739, .3392, -.8213, .4244] \\
\mathbf{p}(0) &= [2.5, 1.5, -1] \\
C &= .3 \\
\mu &= .04 \\
k_\theta &= .15 \\
k_p &= .51 \\
\mathbf{K_f} &= 5\mathbf{I} \\
\mathbf{K}_\tau &= 5\mathbf{I}
\end{aligned}
$$

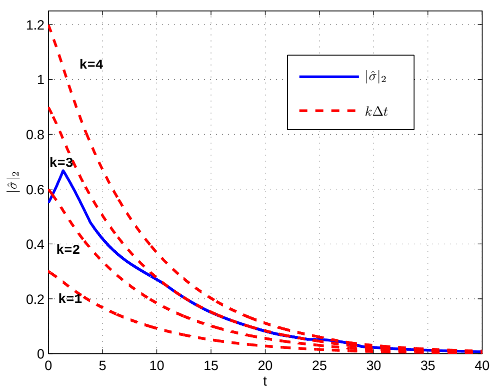

Fig. 4. Convergence of $|\hat{\sigma}|_2$ to $k\Delta t$

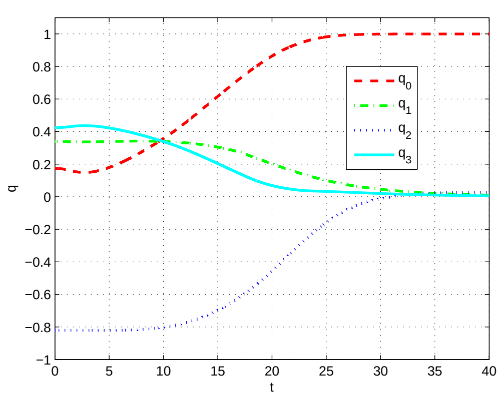

Fig. 5. Convergence of attitude q

VI. Conclusion

A novel sliding mode control law for dual quaternion representation for a class of system was presented, which achieve stability under unknown control direction. The algorithm is based on multiple manifolds in dual quaternions

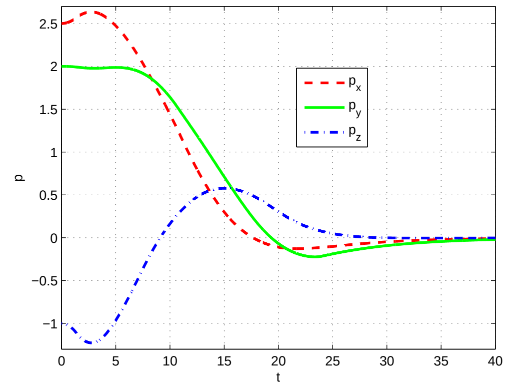

Fig. 6. Convergence of positions x, y, z

space that were created to adapt to the changing uncertainty in the control direction. The control algorithm is universal for a class of systems where the control where only partial knowledge of $B(x, t)$ is required.

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