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Engineering Notes

Energy-Optimal Solution to the Lambert Problem

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Introduction

THE Lambert problem offers a substantial way of determining the minimum energy transfer between two known points along a Keplerian orbit. Most of the analysis for this problem relies on a geometrical approach, since the problem's definition is attuned to the geometry [1,2]. The main idea of the Lambert minimum-energy problem starts by defining the length between two position vectors. Consequently, it states geometrically that the semimajor axis of the minimum-energy orbit is related to the chord length and length of the position vectors [1]. Currently, there is no other analytical approach besides the geometric analysis for solving the problem. In this note, however, an alternative analytical method for solving the Lambert minimum-energy problem is proposed. The minimum velocity at the initial position is obtained by applying a constrained optimization tool. As the initial position vector in the problem is fixed, it is apparent that determining the minimum initial velocity is the same as obtaining the minimum-energy orbit. Using the alternative technique could give us new insight into solving various orbital problems.

Problem Formulation

The geometrical concept of solving the Lambert problem is briefly reviewed in this section. The overall geometrical notations for the Lambert problem are given in Fig. 1. There are two known position vectors, $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$, so that the transfer angle between them is denoted as $\Delta\nu$. This figure illustrates how the intersections of the dashed circles can define the second focus, F' or F'', of a possible orbit. Therefore, there are two possible foci for a given value of semimajor axis a. The minimum-energy orbit can be calculated by minimizing the value of a. Note that the sum of the distances from the foci to any point on the orbit is equal to the twice the semimajor axis. It is a well-known fact that there is only one focus that determines the minimum-energy orbit, where the two dark circles illustrated in Fig. 1 just meet [1].

There are many orbital-element sets that describe an object in orbit. In this note, the f and g expressions are chosen to describe orbital maneuvers. The following relationship exists regarding two position vectors and one initial-velocity vector $\boldsymbol{v}_0$ [3]:

$$\boldsymbol{r}_1 = f\boldsymbol{r}_0 + g\boldsymbol{v}_0 \tag{1}$$

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where f and g are time-independent variables defined as

$$f = 1 - \frac{r_1}{p}(1 - \cos(\Delta\nu)) \tag{2}$$

$$g = \frac{r_1 r_0 \sin(\Delta\nu)}{\sqrt{\mu p}} \tag{3}$$

and the norms of $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$ are expressed as r_0 and r_1, respectively, p is the semiparameter, and μ denotes the gravitational constant. Next, note that the orbit energy is given by

$$\mathcal{E} = -\frac{\mu}{2a} \tag{4}$$

Now, the Lambert problem based on the constrained optimization technique is reformulated. For given $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$, there exists a semimajor axis that minimizes the performance index defined as

$$\mathcal{J} = \mathcal{E} \tag{5}$$

subject to

$$f\boldsymbol{r}_0 + g\boldsymbol{v}_0 - \boldsymbol{r}_1 = 0 \tag{6}$$

It is obvious that the smallest semimajor axis satisfying Eq. (6) is the optimal Lambert solution. The unknown variables to be obtained in this optimization problem are selected as $\boldsymbol{x} = [\,\boldsymbol{v}_0^T \quad p\,]^T$ by applying f and g solutions in Eqs. (2) and (3).

At this point, the orbit energy can be described as

$$\mathcal{E} = \frac{v_0^2}{2} - \frac{\mu}{r_0} \tag{7}$$

Note that because the second term of the right-hand side in Eq. (7) is known and constant, the performance index can be redefined without loss of generality as

$$\mathcal{J}(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{v}_0^T\boldsymbol{v}_0 \tag{8}$$

Minimum-Energy Orbit

The problem described in the above section is a well-formulated optimization problem with equality constraints. By adjoining the constraint with an undetermined Lagrange-multiplier vector, the Hamiltonian is defined as

$$L = \frac{1}{2}\boldsymbol{v}_0^T\boldsymbol{v}_0 + \boldsymbol{\lambda}^T(f\boldsymbol{r}_0 + g\boldsymbol{v}_0 - \boldsymbol{r}_1) \tag{9}$$

where $\boldsymbol{\lambda} \in \mathcal{R}^3$ is the Lagrange-multiplier vector. To minimize the performance index with respect to $\boldsymbol{x}$ constrained by Eq. (6), the optimality condition is expressed as [4]

$$\frac{\partial L}{\partial \boldsymbol{v}_0} = \boldsymbol{v}_0^T + g\boldsymbol{\lambda}^T = 0 \tag{10}$$

$$\frac{\partial L}{\partial p} = \boldsymbol{\lambda}^T\left(\frac{r_1}{p^2}(1 - \cos(\Delta\nu))\boldsymbol{r}_0 - \frac{r_1 r_0 \sin(\Delta\nu)}{2\sqrt{\mu p^3}}\boldsymbol{v}_0\right) = 0 \tag{11}$$

From Eqs. (2) and (3), the following two relationships can be readily obtained:

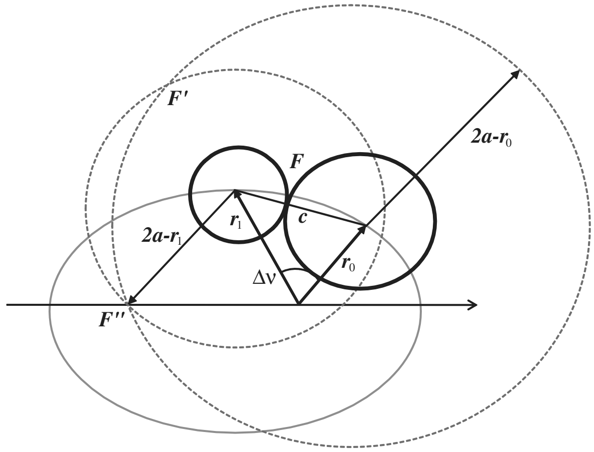

Fig. 1 Geometry for the Lambert problem.

$$\frac{r_1}{p^2}(1-\cos(\Delta\nu)) = \frac{1-f}{p} \tag{12}$$

$$\frac{r_1 r_0 \sin(\Delta\nu)}{2\sqrt{\mu p^3}} = \frac{g}{2p} \tag{13}$$

Then, the second necessary condition in Eq. (11) is rewritten as

$$\boldsymbol{\lambda}^T\left(\frac{1-f}{p}\boldsymbol{r}_0 - \frac{g}{2p}\boldsymbol{v}_0\right) = 0 \tag{14}$$

Next, the Lagrange-multiplier vector from Eq. (10) and the initial-velocity vector are given by

$$\boldsymbol{\lambda}^T = -\frac{1}{g}\boldsymbol{v}_0^T \tag{15}$$

$$\boldsymbol{v}_0 = \frac{\boldsymbol{r}_1 - f\boldsymbol{r}_0}{g} \tag{16}$$

Inserting the above two variables into Eq. (14) and multiplying pg on both sides yields

$$(\boldsymbol{r}_1^T - f\boldsymbol{r}_0^T)\left(\boldsymbol{r}_0 - f\boldsymbol{r}_0 - \frac{1}{2}(\boldsymbol{r}_1 - f\boldsymbol{r}_0)\right) = 0 \tag{17}$$

By manipulating the above equation, one can arrive at the second-order equation with respect to f, described as

$$f^2 - 2f + \frac{2\boldsymbol{r}_0^T\boldsymbol{r}_1 - r_1^2}{r_0^2} = 0 \tag{18}$$

It is straightforward to find the solution of the equation, which is

$$f = 1 \pm \sqrt{1 + \frac{r_1^2 - 2\boldsymbol{r}_0^T\boldsymbol{r}_1}{r_0^2}} \tag{19}$$

Because $f \leq 1$ for the case of eccentricity smaller than one, the final value of f is chosen as

$$f = 1 - \frac{\sqrt{r_0^2 + r_1^2 - 2\boldsymbol{r}_0^T\boldsymbol{r}_1}}{r_0} \tag{20}$$

Now, by equating Eqs. (2) and (20), one can determine the semiparameter so that

$$\frac{r_1}{p}(1-\cos(\Delta\nu)) = \frac{\sqrt{r_0^2 + r_1^2 - 2\boldsymbol{r}_0^T\boldsymbol{r}_1}}{r_0} \tag{21}$$

By arranging the above equation for p, the minimum-energy semiparameter is described as

$$p_{\min} = \frac{r_0 r_1}{\sqrt{r_0^2 + r_1^2 - 2\boldsymbol{r}_0^T\boldsymbol{r}_1}}(1-\cos(\Delta\nu)) \tag{22}$$

Note that the above semiparameter obtained by the optimization approach is identical to the previously studied geometrical minimum-energy solution in [2]. Finally, one can readily compute f and g using Eqs. (2) and (3) so that the minimum initial velocity is obtained as

$$\boldsymbol{v}_0 = \frac{\sqrt{\mu p_{\min}}}{r_0 r_1 \sin(\Delta\nu)}\left[\boldsymbol{r}_1 - \left(1 - \frac{r_1}{p_{\min}}(1-\cos(\Delta\nu))\right)\boldsymbol{r}_0\right] \tag{23}$$

Summary

The simple approach of computing the minimum-energy orbit given two position vectors is outlined here. In determining the transfer angle, one can use both the dot and cross products:

$$\cos(\Delta\nu) = \frac{\boldsymbol{r}_0^T\boldsymbol{r}_1}{r_0 r_1}, \qquad \sin(\Delta\nu) = \frac{\| \boldsymbol{r}_0 \times \boldsymbol{r}_1 \|}{r_0 r_1} \tag{24}$$

Next, compute the semiparameter using Eq. (22) and the minimum velocity using Eq. (23). Then, the minimum semimajor axis for the orbit is given by

$$a_{\min} = -\frac{\mu}{2}\bigg/\left(\frac{v_0^2}{2} - \frac{\mu}{r_0}\right) \tag{25}$$

and the eccentricity for the minimum energy is computed as

$$e_{\min} = \sqrt{1 - \frac{p_{\min}}{a_{\min}}} \tag{26}$$

Conclusions

The Lambert problem was reformulated using an alternative way that is different from the widely studied geometrical analysis. In this note, a constrained optimization technique was addressed to solve the Lambert minimum-energy problem. Finally, it was shown that the result from the proposed approach matched the solution of the geometrical approach. Applying the optimization approach to solve the Lambert problem could provide new insight into solving a variety of orbital problems.

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