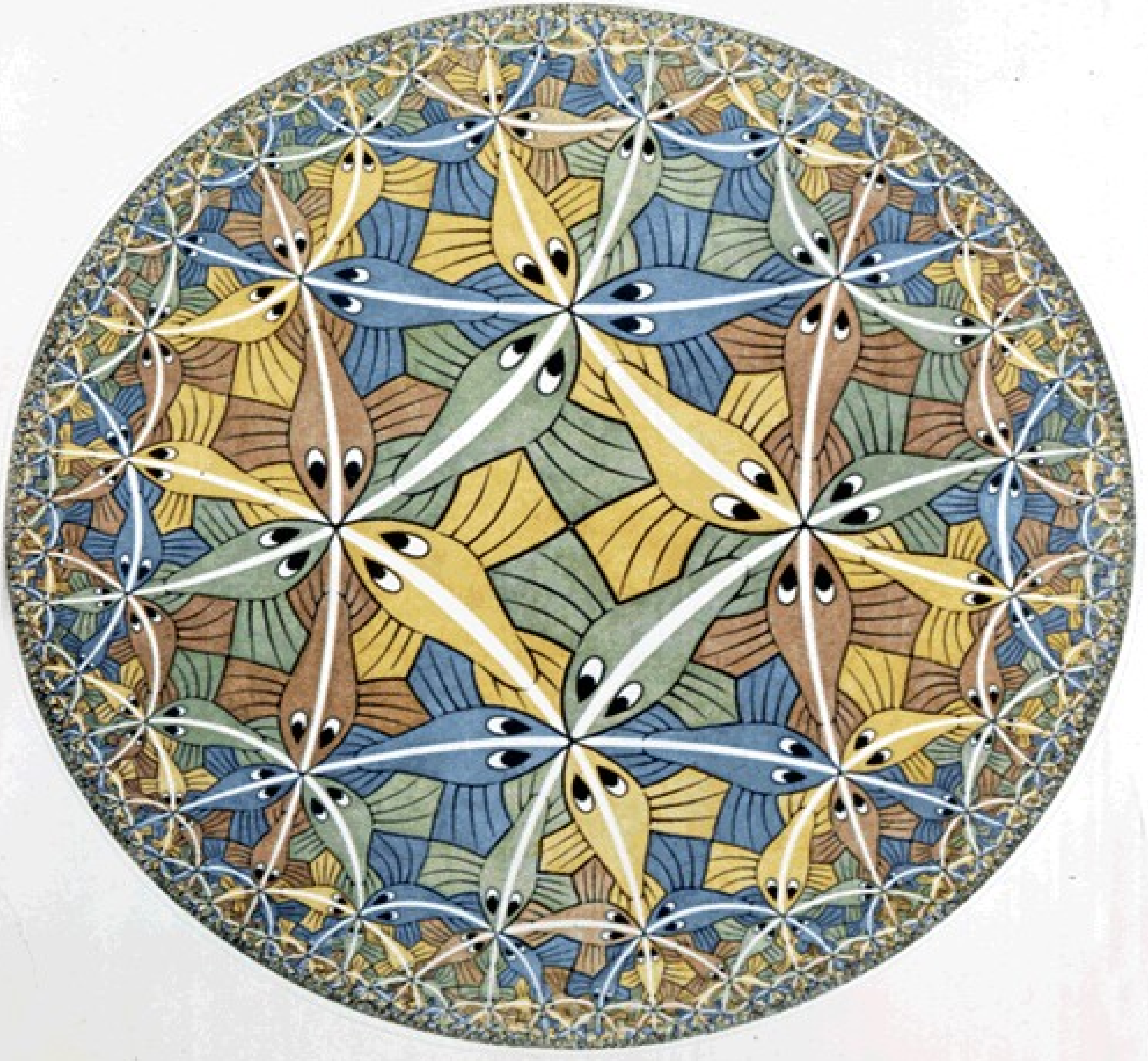


On the Quantumdynamics of Measurement

By Leon Hardy & Mohamed Elhamdadi

Topics

- Introduction to Geometric Algebra
- Spacetime Algebra
- Measurements in Quantum Mechanics
- Proper Time Observable, Proper Spin & Bell's Inequality
- Questions & Conclusions

Geometric Algebra I - Complex Numbers

Any *complex number* can be written as, $x+iy$, where x,y are real numbers and the imaginary unit i has the property $i^2 = -1$.

- How can the square of a quantity be negative?
- The question troubled many good Mathematicians for many years.
- The work of Hamilton, Grassmann and Clifford addresses the question with a surprising geometrical approach.

Development of Geometric Algebra

- **Hamilton** (1805-1865) discovered the quaternions and their relations $i^2=j^2=k^2=ijk=-1$.
- **Grassmann** (1809-1877) was a German schoolteacher who was disappointed in lack of interest in his mathematical ideas – turned to Sanskrit (dictionary still used).
- **Clifford** (1845-1879) Cambridge mathematician and philosopher who united Grassmann's ideas with the quaternions of **Hamilton**.

Hermann Grassmann

William Clifford

William Hamilton

The Language of Geometric Algebra

One of the major advantages of this new approach is that it provides a simple language:

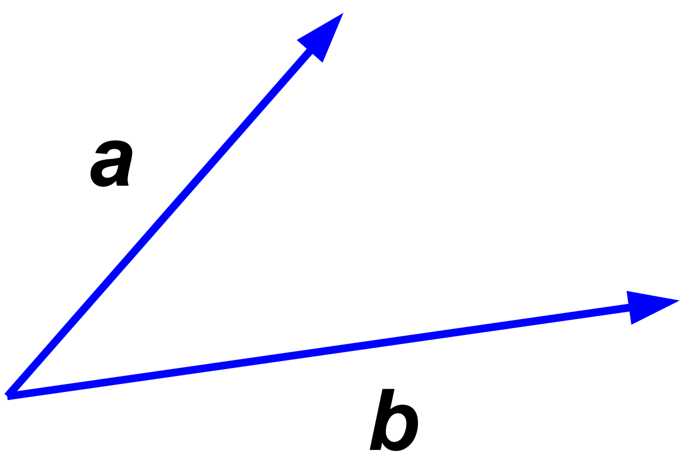
"words" $\leftrightarrow$ "geometrical objects"

"sentences" $\leftrightarrow$ "relations between the objects"

where one can carry out representation free computations, and it yields new geometric insights that are not available with other methods.

Geometric Algebra II -Vectors

Let two directions in space be represented by the vectors ***a*** and ***b***. Then if we had a language where we could string together *vectors*, can we make sense of sentences such as ***aaba*** and ***bbabbb***?



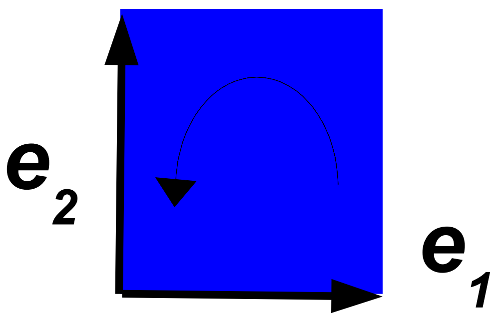
Geometric Algebra III – Two Rules

We introduce the two rules:

- If ***a*** and ***b*** are *perpendicular* then $\boldsymbol{ab} = -\boldsymbol{ba}$.
- If ***a*** and ***b*** are *parallel* then $\boldsymbol{ab} = |\boldsymbol{a}||\boldsymbol{b}|$, where $|\cdot|$ denotes the *length* of a vector.
- With two *orthogonal basis vectors* $\mathbf{e}_1$ and $\mathbf{e}_2$, the rules says: $\mathbf{e}_1^2 = \mathbf{e}_1\mathbf{e}_1 = 1$, $\mathbf{e}_2^2 = \mathbf{e}_2\mathbf{e}_2 = 1$, and $\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1$.

Geometric Algebra IV - Bivectors

- Compute $(\mathbf{e}_1\mathbf{e}_2)^2$.
- $(\mathbf{e}_1\mathbf{e}_2)^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\, \mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 = -1$. We have produced an object whose square is –1, which is like the complex number $x+(\mathbf{e}_1\mathbf{e}_2)y$.
- The object $\mathbf{e}_1\mathbf{e}_2$, sometimes written as the wedge product $\mathbf{e}_1 \wedge \mathbf{e}_2$, is called a *bivector*, which is represented as the oriented area of the parallelogram of the basis vectors.



Geometric Algebra V – Wedge Products

- Given k vectors, we define the *wedge product*, called a *k-blade*, as $\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_k$,where k is called the *grade*.
- Each *k-blade* can be viewed as a k-dimensional volume with orientation, i.e. a 0-blade is a scalar, a 1-blade is a vector, a 2-blade is a bivector, a 3-blade is a trivector, each respectively viewed as a length, directed length, directed area and directed volume.

Spacetime Algebra

Let the vector space M^4 denote the standard model for Minkowski spacetime. Then the spacetime algebra is generated by the set of standard basis $\{1, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_\mu i, i\}$, $\mu=1,...,4$, along with a geometric product of the vectors x and y defined by

$$xy = x \cdot y + x \wedge y,$$

where $\cdot$ and $\wedge$ denote the inner product and the wedge product.

The Spacetime Split

A ***spacetime split***, denoted by γ_0-split for an inertial observer γ_0 in the direction of the forward light cone, is defined as the geometric product

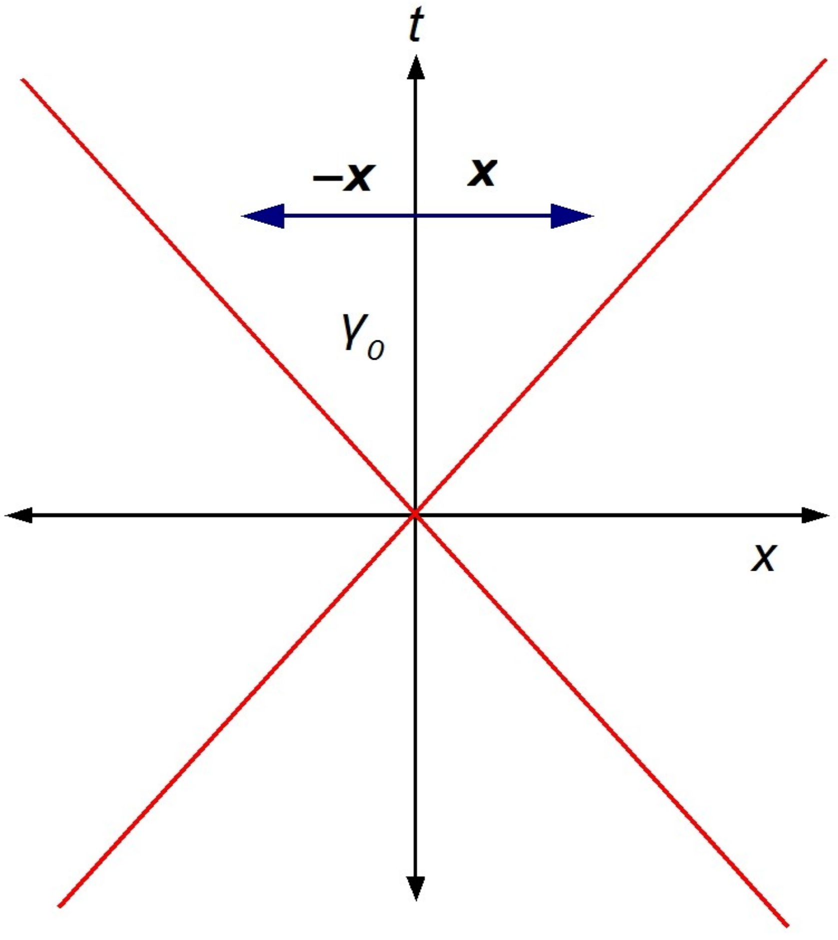
$$x\gamma_0 = t + \boldsymbol{x},$$

where the inner product is $t = x\cdot\gamma_0$ and the wedge product is $\boldsymbol{x} = x\wedge\gamma_0$.

The Reverse of the Spacetime Split

A ***reverse spacetime split*** for an inertial observer γ_0, is defined as the geometric product

$$\gamma_0 x = t - \mathbf{x}.$$



The Spacetime Invariant

Using the spacetime split, the ***spacetime invariant*** is defined as

$$x\gamma_0\gamma_0 x = (t + \boldsymbol{x})(t - \boldsymbol{x}) = t^2 - \boldsymbol{x}^2,$$

which can be spacelike, lightlike or timelike when the spacetime invariant is negative, vanishing or positive.

Equivalence of Observers

The inertial observers γ_0 and γ'_0 are said to be ***equivalent*** if

$$t^2 - \mathbf{x}^2 = t'^2 - \mathbf{x}'^2,$$

where equivalence is formally the relation $\{ (\gamma_0, \gamma'_0) \mid t^2 - \mathbf{x}^2 = t'^2 - \mathbf{x}'^2 \}$, and we write equivalent observers as $\gamma_0 \sim \gamma'_0$.

Introduction to Quantum Mechanics

The equation of motion for quantum phenomenon is Schrödinger's equation

$$i\hbar\partial\Psi/\partial t = -\hbar^2/2m\partial^2\Psi/\partial x^2 + V(x)\Psi$$

whose solutions are the wavefunctions $|\psi\rangle$, which provide expectation values yielding the measurement of physical quantities.

Measurement of Position

Let $|\psi\rangle$ denote the wavefunction of a quantum event for an inertial quantum observer γ_0. During the course of an experiment, a measurement of the position of a free particle is performed, which is assigned the (expectation) value $\langle X \rangle$ from within Quantum Mechanics.

$$\langle X \rangle = \int \psi^* x \psi \, dx$$

Measurement of Time

For an inertial quantum observer γ_0, the measurement of momentum of a free particle must be made over some time interval. From the velocity of the particle given by $v = x / t$, we obtain the time observable

$$T = (XP^{-1} + P^{-1}X) / 2 = \{X,P\} / 2|P|^2 ,$$

which is assigned the expectation value of the time of $\langle T \rangle$ by Quantum Mechanics.

Proper Time Observable

For an inertial quantum observer γ_0, the ***proper time observable*** is defined as

$$\langle \Delta^2 \rangle = \langle \{X,P\}^2 / 4|P|^4 \rangle - \langle X^2 \rangle,$$

where we remark that time is now viewed as a *manifestation* of the position and its rate of change!

Equivalence of Quantum Observers

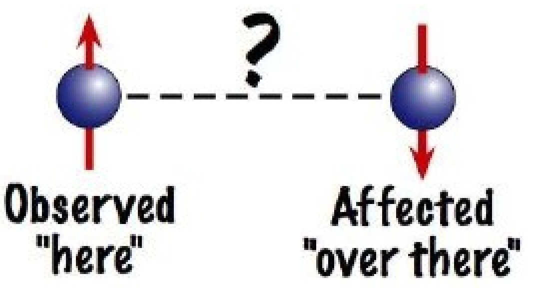
The inertial quantum observers γ_0 and γ'_0 are said to be ***equivalent*** if

$$\langle\{X,P\}^2 / 4|P|^4\rangle - \langle X^2\rangle = \langle\{X',P'\}^2 / 4|P'|^4\rangle - \langle X'^2\rangle,$$

where equivalence is formally the relation $\{ (\gamma_0, \gamma'_0) \mid \langle\{X,P\}^2 / 4|P|^4\rangle - \langle X^2\rangle = \langle\{X',P'\}^2 / 4|P'|^4\rangle - \langle X'^2\rangle\}$, and we write equivalent observers as $\gamma_0 \sim \gamma'_0$.

Local Realism of Intrinsic Spin

Consider two particles that can be in one of two quantum states: spin-up or spin-down. Suppose that one particle is measured by one quantum observers as spin-up. Since the total spin vanishes, the other quantum observer must measure spin-down without regard to the separation of observers. How can one observer's measurement effect the another observer's outcome?



Questions

- Can it be true that one quantum observers measurements effect another quantum observers experimental outcomes?
- Is Quantum Mechanics a non-local or local theory?
- Does Bell's inequality imply that no local theory of Quantum Mechanics is possible?

Intrinsic Spin of a Particle

Let σ_z denote the Pauli spin matrix. Then the equation of motion for the *z*-component of the intrinsic spin $S_z = \hbar\sigma_z / 2$ is

$$dS_z / dt = [S_z, H\,] / 2i\hbar + \partial S_z / \partial t,$$

where $[S_z, H\,] / 2 = S_z \wedge H$.

Proper Spin

The ***proper spin*** is defined as

$$\langle \Delta^2 \rangle = \langle \{ S_z, dS_z / dt \}^2 / 4 | dS_z / dt |^4 \rangle - \langle S_z^{\ 2} \rangle,$$

which can be spacelike, lightlike or timelike when the proper spin observable can be negative, vanishing or positive.

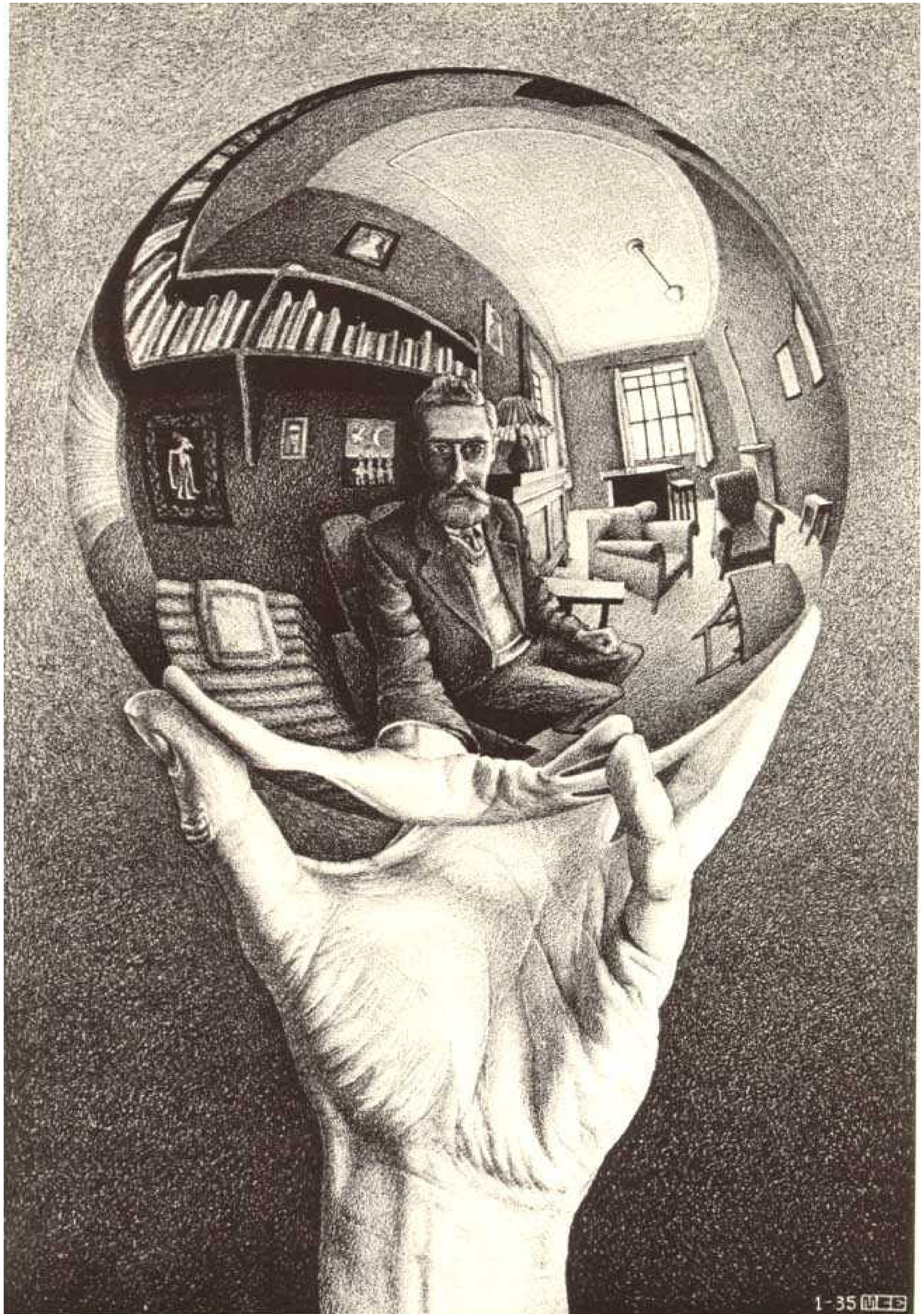
Reformulation of the Proper Spin

Using the identity $(\sigma\cdot a)(\sigma\cdot b)=a\cdot b + i\sigma\cdot(a\times b)$ from geometric algebra, the ***proper spin observable*** can be written as

$$\langle \Delta^2 \rangle = \langle (dS_z / dt \cdot dS_z / dt) / 4|dS_z / dt|^4 \rangle - \langle S_z{}^2 \rangle.$$

Conclusions

- The proper observable provides a local description of the experiments conducted by quantum observers and a notion of equivalence in the spirit of local realism.
- In doing so, one replaces the absolute description of a quantum event for a representation expressed as a relative wavefunction.
- Time is a manifestation of an observable and its rate of change. Time is an illusion!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