Quantum Mechanics of Black Holes

I. Newtonian gravity

Given a black hole of mass M, the escape velocity a distance r from the point is determined by energy conservation:

$$\frac{1}{2} m v_{esc}^2 = \frac{GMm}{r}$$

Letting $v_{esc} \to c$ results in the radius $r = R$ of the event horizon (or the Schwarzschild radius)

$$\boxed{R = \frac{2GM}{c^2}} \qquad (1)$$

A sphere with this radius has a surface area $4\pi R^2$

$$\boxed{A = 16\pi \frac{G^2 M^2}{c^4}} \qquad (2)$$

The acceleration due to gravity at the Schwarzschild radius is $g = GM/R^2$, or

$$\boxed{g = \frac{c^4}{4GM}} \qquad (3)$$

Finally, the tidal force on an object of mass m a distance x from the 'center of mass' [this technically is the longitudinal tidal force (see Morin p 471)]

$$\boxed{F_{long} = \frac{2GM}{R^3} mx = \left(\frac{c^6}{4G^2M^2}\right) mx} \qquad (4a)$$

The transverse tidal force (squeezing) is just half that value (see Morin p 473)

$$\boxed{F_{trans} = -\left(\frac{c^6}{8G^2M^2}\right) my} \qquad (4b)$$

The above results are purely classical, except for (1) which really should be calculated using G.R. The lucky accident is that Newtonian gravity gives the same answer!

The scalings are key here. First, that $g \sim M^{-1}$ which means that a black hole of mass $\boxed{M = 3.09 \times 10^{42}\ kg} = 1.56 \times 10^{12} M_{\odot}$ has a "one-gee" acceleration at its event horizon, and the smaller the mass, the larger the acceleration.

In addition, the tidal forces scale as $F_{tide} \sim M^{-2}$ which means that super massive black holes are fairly benign. The old story of being ripped apart by tidal forces as you fall into a black hole is true, but this happens outside the event horizon only for small black holes. For large mass BHs - like the one at the center of the Milky Way - your passage through the event horizon would not be noticed by you.

For example, bone has a high compressive strength ~170 MPa, but not so good tensile strength ~110 MPa. Approximating a human as two 50-kg masses at the ends of a 1-m long bone with a cross-sectional area of $\pi (5\,cm)^2 = 7.85 \times 10^{-3}\ m^2$, then the differential tide factor is

$$\frac{F}{mx} = \frac{(110\ MPa)(7.85 \times 10^{-3}\ m^2)}{(50\ kg)(1\ m)} = 1.73 \times 10^{4}\ s^{-2}$$

The black hole mass needed to acheive this tidal force at the event horizion can be found from Eq (4a), or

$$\boxed{M = 1.54 \times 10^{33}\ kg} = 774\ M_{\odot}$$

with a radius of $R \cong 2300$ km. This is a fairly small BH. To be pulled apart outside the BH, it's mass (and radius) would need to be small. Falling into a large (supermassive) BH would be like a vacation.

II Hawking radiation

In 1975, Hawking predicted that black holes should have a temperature

$$\boxed{T = \frac{\hbar c^3}{8\pi k G M} = g\left(\frac{\hbar}{2\pi k c}\right)} \quad (5)$$

from Eq (3)

$$= 6.2\times10^{-8}\,\mathrm{K}\left(\frac{M_\odot}{M}\right)$$

(i.e, a black hole of one solar mass would have a thermodynamic temperature of 6.2×10^{-8} K) This means that as a "black body" it will emit radiation according to the Stefan-Boltzman law at a luminosity of $L = \sigma T^4) A$,

$$\boxed{L = \frac{\hbar c^6}{15360\,\pi G^2}\,\frac{1}{M^2}} \quad (6)$$

$$= 9\times10^{-29}\,\mathrm{W}\left(\frac{M_\odot}{M}\right)^2$$

This implies that as BH radiate, they lose energy; but $E_0 = mc^2$ means they lose mass, so this is a differential equation for the rate of mass loss

$$L = -\frac{dE_0}{dt} = -c^2\frac{dM}{dt}$$

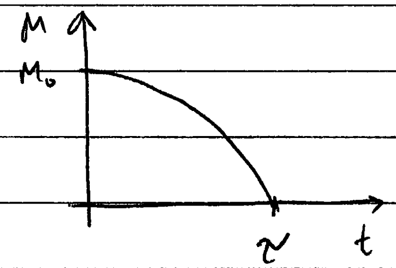
Along with (6) this gives

$$\boxed{\frac{dM}{dt} = -\frac{\alpha}{M^2}} \qquad \text{where } \alpha = \frac{\hbar c^4}{15360\pi G^2} \qquad (7a)$$

Applying separation of variables, this can be solved for $M(t)$ to give

$$\boxed{M^3(t) = M_0^3 - 3\alpha t} \qquad (7b)$$

Plotting gives



where the "lifetime" τ of the BH can be found by setting $M(\tau) = 0$, or

$$\boxed{\tau = \frac{M_0^3}{3\alpha} = \frac{5120\pi M_0^3 G^2}{\hbar c^4}} \qquad (8)$$

Again, scaling is important here: $\tau \sim M^3$, which means that while "mini-black holes" evaporate quickly, large black holes can last longer than the age of the universe. (8) can be written

$$\tau = 6.6\times10^{74}\,\text{s}\left(\frac{M}{M_\odot}\right)^3 = 2.1\times10^{67}\,\text{yr}\left(\frac{M}{M_\odot}\right)^3$$

Bekenstein's Law

Why should a black hole act like a blackbody and emit radiation? In the early 1970s, Hawking, Bekenstein, and others realized that the surface area of a black hole (ie, the surface area of the event horizon, Eq (2)) had properties similar to that of entropy.

For example, as matter falls into a BH, its mass M increases, and therefore so does the area A, Eq (2). Since we have lost information about that matter,* and since entropy is sometimes thought of as a measure of information that is inaccessible to an observer, the entropy of the black hole (plus the infalling matter) must have increased.

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* The "no hair" theorem states the "black holes have no hair." That is, the only properties of a black hole that are measurable by an external observer are its mass M, angular momentum $\vec{J}$, and electric charge Q. This means that all information concerning composition, etc, is lost when that matter falls in. A so-called "simple" black hole is neutral $Q=0$ and is not rotating $\vec{J}=0$.

A simple example is that of two black holes, of mass M_1 and M_2 and areas A_1 and A_2 that merge to form a single black hole of mass M_1+M_2. From Eq (2), the area of this new black hole is not A_1+A_2, but instead is

$$A = \frac{16\pi G^2}{c^4}(M_1+M_2)^2 = A_1+A_2+\frac{32\pi G^2}{c^4}M_1M_2$$
$$> A_1+A_2 \qquad (9)$$

The area has increased during this merger. This is very similar to the classical thermodynamic process in which two objects — at different temperatures — come into thermal contact and equilibrate. The total entropy of the system increases!

During a black hole merger, however, the system of accelerating masses will radiate energy in the form of gravitational waves. In order that the final area not decrease, Hawking (1971) showed that there's a maximum amount of energy that can be radiated. This amount E is determined by setting the new Area A equal to the old

$$A = \frac{16\pi G^2}{c^4}\left(M_1+M_2-\frac{E}{c^2}\right)^2 = A_1+A_1 \qquad (10)$$

Solving for the energy E gives

$$\frac{E}{c^2} = (M_1 + M_2) - \sqrt{M_1^2 + M_2^2} \qquad (11)$$

Hawking had done the calculation for two equal masses, and showed that the maximum energy radiated was $Mc^2(2-\sqrt{2})$, consistent with (11).

In 1973, Jacob Bekenstein showed that if the area A was to be a measure of the entropy of a black hole, then the entropy S must be a monotonically increasing function of A. Before deriving this function, he guessed a linear dependence

$$S = \gamma A \qquad (12)$$

and used dimensional analysis to determine the proportionality constant γ. The S.I units of entropy involve Boltzman's constant k $[k] = J/K$, which means that γ must equal k times a constant with dimensions of L^{-2}. There are two universal constants in classical general relativity, G and c, with dimensions

$$[G] = \frac{L^3}{MT^2} \qquad [c] = \frac{L}{T} \qquad (13a)$$

Using G and c, we can eliminate the dependence of η on either M or T, but not both.
Buckingham's π theorem shows that one more constant is needed, and the only one available is quantum in nature: $\hbar$

$$[\hbar] = \frac{M L^2}{T} \qquad (13b)$$

The only combination of these variables that results in the correct dimensions gives S up to a dimensionless constant

$$\boxed{S = \eta \frac{k c^3}{G \hbar} A} \qquad (14)$$

We will show below that $\eta = \frac{1}{4}$.
Bekenstein used quantum arguments to prove (14) and to calculate η, where he determined $\eta = \frac{\ln 2}{8\pi}$. He states that "our value ... might presumably be challenged ... if η is not exactly [$\ln 2/8\pi$] then it must be very close to this ... this value leads to no contradictions."
Hawking (1975) developed another method to determine S. He noted that classically, the first law of thermodynamics

can be written as

$$dU = TdS - pdV \qquad (15)$$

where $dQ = TdS$ comes from Clausius's definition of entropy. For a black hole, its mass M can be considered as its internal energy, and Bekenstein had shown that an infinitesimal change in mass dM was due to an area change dA, as well as changes in angular momentum $d\vec{L}$ and charge dQ. The terms in $d\vec{L}$ and dQ [4] "represent the work done on the black hole by an external agent who increases the black hole's angular momentum and charge."

For a simple black hole, the relation between M and A can be obtained by implicitly differentiating Eq (2)

$$dA = \frac{32\pi G^2}{c^4} M\,dM \qquad (16a)$$

or, using (3)

$$\boxed{dM = \left(\frac{g}{8\pi G}\right) dA} \qquad (16b)$$

Comparing this result with (15), and noting that no work is done on the black hole, we see that "if some multiple of A is regarded as being analogous to entropy, then some multiple of g is analogous to temperature." (Hawking, 1975, p 203) This is how Hawking obtained Eq (5) for the relationship between T and g.

Rewriting (16b) in order to incorporate this relationship gives

$$dU = c^2 dM = \underbrace{\left(\frac{g}{2\pi} \frac{\hbar}{kc}\right)}_{T} d\underbrace{\left(\frac{A}{4G} \cdot \frac{kc}{\hbar} c^2\right)}_{S} \qquad (17)$$

which means that the entropy of a black hole is

$$\boxed{S_{BH} = \frac{A k c^3}{4 G \hbar}} \qquad (18)$$

Originally, this definition of entropy was developed to describe the loss of information as black holes merged and matter fell in. In fact Hawking (1971) had shown that $\Delta S_{BH} \geq 0$ in all processes, which of cours implies $\Delta A \geq 0$. However, the realization that due to its temperature, the black hole also radiates, Hawking realized that a generalized second law of thermo-dynamics must hold. That is, if S_{BH} decreases, then the entropy of the surroundings must increase due to the increase in radiation.

Hawking, 1971, PRL 26 1344
"Gravitational Radiation from colliding Black Holes"
Hawking, 1975, Commun. math. Phys. 43 199-220
"Particle creation by Black Holes"
Bekenstein, 1973, Phys Rev D 7 2333-2346
"Black Holes and Entropy"

Some popular expositions are

Hawking "A Brief History of Time" Chap 7
Shipman "Black Holes, Quasars, and the Universe"
pp 116-121
Smolin "Three Roads to Quantum Gravity" this is the most extensive discussion, as he lists 'black hole thermodynamics' as one of the 3 roads, the other 2 being loop quantum gravity and string theory.

Ted Jacobson's page on Black Hole thermodynamics is good
www.physics.umd.edu/grt/taj/776b/texts.html
and his personal page /people/ted.html

Also see
Bardeen, Carter, and Hawking, 1973, Commun. math. Phys. 31 161-170 "The Four Laws of Black Hole Mechanics"