Centripetal Acceleration

see Tipler, "Physics"
Henry, "Circular Motion", AJP 68 (7) 637-639 (2000)
Landau, etal "General Physics"

History: uniform circular motion involves acceleration
(see Henry)
- realized by Galileo + Descartes
- solved by Huygens, 1658 "Horologium Oscillatorium"
(proof in "De Vi Centrifuga" 1703)

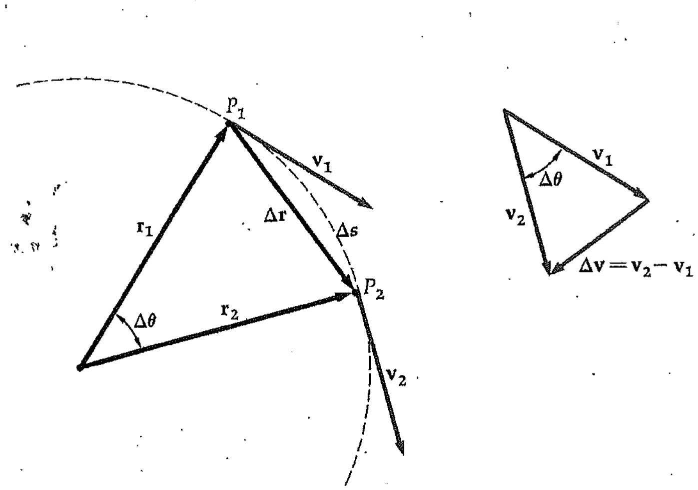
⇒ Halley, Wren, Hooke used $a_c = v^2/r$ + Kepler's 3rd law to deduce $F \sim 1/r^2$

Hooke claimed discovery, but...
Newton could deduce K3 from $F \sim 1/r^2$

I have found several methods to derive the formula (and direction) of the centripetal acceleration vector. Some are physical, some mathematical, some geometrical, some analytical. They all increase and deepen one's understanding of dynamical motion.

Method #1 graphical

Consider two points in a uniform, circular trajectory, p_1 and p_2.



The two similar, isosceles triangles, give, in the limit $\Delta t \to 0$

$$\frac{\Delta r}{r} = \frac{\Delta v}{v}$$ where $\Delta v = |\Delta \vec{v}|$ is the magnitude of the difference vector.

$$a_c = \frac{\Delta v}{\Delta t} = \frac{v}{r}\frac{\Delta r}{\Delta t} = \frac{v^2}{r} \;//$$ and, of course, the direction is toward the center of the circle

OR using the arc length of the circle Δs

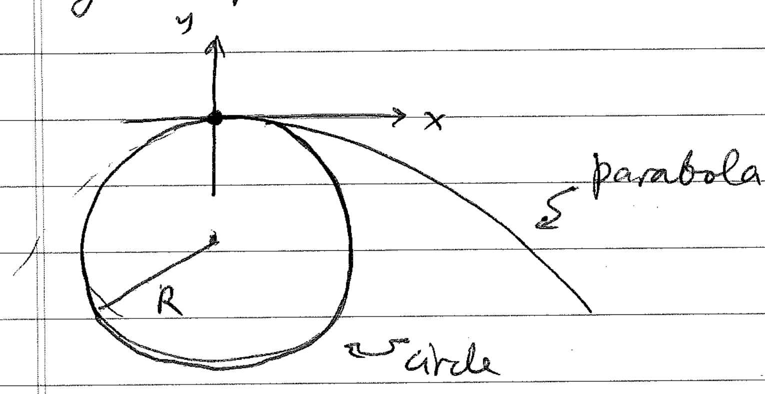
$$v = \frac{\Delta s}{\Delta t} = \frac{r\Delta\theta}{\Delta t} \left(= r\omega\right)$$ where $\Delta\theta = \frac{\Delta s}{r} \approx \frac{\Delta v}{v}$

$$\approx \frac{r}{v}\frac{\Delta v}{\Delta t} \Rightarrow \frac{\Delta v}{\Delta t} = \frac{v^2}{r} \;//$$

But! There wasn't a good vector theory in 1700.

Method #2   Huygens method

from Henry (2000), Huygens "constructed the largest circle that passes through the original position of" a "particle that is thrown sideways." "yet does not cut the parabola"



The circle must match the parabola in
1) position
2) slope, and
3) curvature

parabola: $x = v_{0x} t$, $y = -\frac{1}{2} g t^2$ } eliminate t { $y = -\dfrac{g}{2v_{0x}^2} x^2$

at $x=0$: $y = 0$

$$y' = -\frac{g}{v_{0x}^2} x \Big|_{x=0} = 0$$

$$y'' = -\frac{g}{v_{0x}^2}$$

circle: $x^2 + (y+R)^2 = R^2$ clearly $y(0) = y'(0) = 0$, so we only must match y''

implicit differentiation:

$$2x + 2(y+R)y' = 0$$

$$2 + 2(y+R)y'' + 2y'^2 = 0$$

$$y''\Big|_0 = -\frac{(1+y'^2)}{y+R}\Big|_{x=0} = -\frac{1}{R}$$ as expected.

Matching the curvature of both curves gives

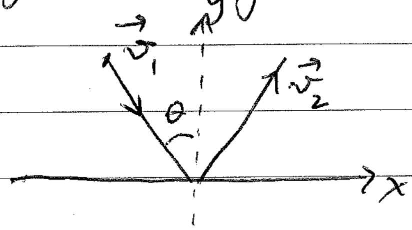
$$-\frac{g}{v_{0x}^{2}} = -\frac{1}{R}$$

Since the acceleration of the projectile is $\perp$ to its motion (at $x=0$), ie, it's instantaneously in circular motion, and the magnitude of that acceleration is g, we have, for the circle

$$a_c = g = \frac{v_{0x}^{2}}{R} \quad //$$

Method #3 Newton's first method (Henry, 2000)

Consider a perfectly elastic collision of a particle against a flat wall:



The $\hat{x}$ component of the velocity remains constant $v_x = v\sin\theta$ while the $\hat{y}$ component keeps the same magnitude $v\cos\theta$ but reverses direction. Therefore

$$\Delta\vec{v} = 2v\cos\theta\,\hat{y}$$

Now, apply this to "a particle that is crashing around a circle."

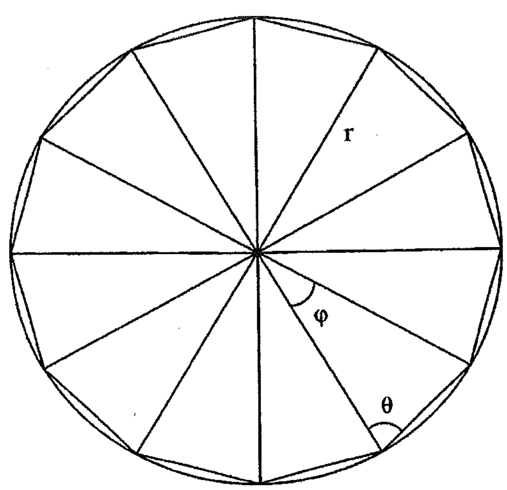


Fig. 2. A particle bounces around a circle, hitting the circle a total of 12 times, and each time bouncing at an angle θ to the normal to the circle.

From one of the triangles, we have

$$\varphi + 2\theta = \pi$$

and from all of them,

$$n\varphi = 2\pi$$

Solving for θ

$$\theta = \frac{\pi}{2} - \frac{\varphi}{2} = \frac{\pi}{2} - \frac{\pi}{n}$$

Hence, for one bounce

$$\Delta v = 2v\cos\theta$$
$$= 2v\sin\left(\frac{\pi}{n}\right) \approx \frac{2v\pi}{n}$$

as $n \to \infty$

The acceleration is (for one bounce) $a_c = \dfrac{\Delta v}{\Delta t} = \dfrac{2v\pi}{n}\left(\dfrac{n}{T}\right) = \dfrac{2v\pi}{\left(\frac{2\pi r}{v}\right)} = \dfrac{v^2}{r}$ //

Each bounces $\Delta\vec{v}$ is obviously centripetal!

Method #4 Newton's second method (see Tipler pp 67–68)

Again, like Method #2, we will compare circular motion to gravitational acceleration, but we will do it dynamically.

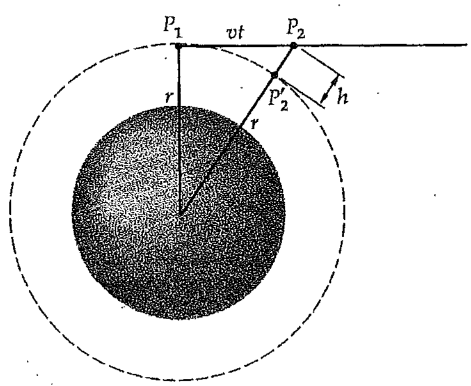


Figure 3-23
Satellite moving with speed v in a circular orbit of radius r about the earth. If the satellite did not accelerate toward the earth, it would move in a straight line from point P_1 to P_2 in time t. Because of its acceleration, it falls a distance h in this time. For small t, $h = \frac{1}{2}(v^2/r)t^2 = \frac{1}{2}at^2$.

Consider the object in the Earth's gravitational field. In a time t it will have "fallen" a distance h, which we can obtain from the right triangle:

$$(r+h)^2 = r^2 + (vt)^2$$

expanding $\quad r^2 + 2rh + h^2 = r^2 + v^2t^2$

ignoring h^2, then

$$h \approx \frac{1}{2}\left(\frac{v^2}{r}\right)t^2$$

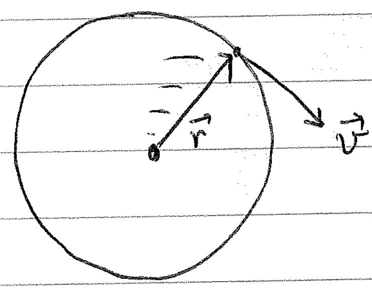
and we know the coefficient of $\frac{1}{2}t^2$ must be the acceleration

$$a = \frac{v^2}{r} \quad //$$

Method #5 Landau's method

Here is an interesting derivation of the centripetal acceleration vector from "General Physics" by Landau, Akhiezer, and Lifshitz p 10-12 (QC 21. L2713 1967)

The position vector $\vec{r}(t)$ of a particle undergoing uniform circular motion rotates uniformly about the origin, and perpendicular to that is the velocity vector $\vec{v}$, where

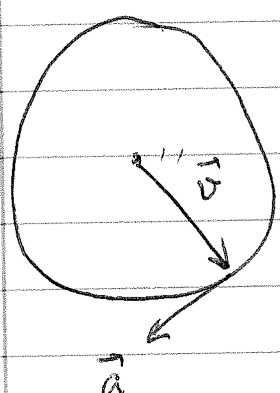


$$\vec{v} = \frac{d\vec{r}}{dt} \qquad (1)$$

The magnitude of $\vec{v}$ is constant and given by

$$v = \frac{2\pi r}{T} \qquad (2)$$

Similarly, the velocity vector must rotate uniformly, and, since the corrollary of (1) is



$$\vec{a} = \frac{d\vec{v}}{dt} \qquad (3)$$

the acceleration vector $\vec{a}$ must be perpendicular to $\vec{v}$.

The magnitude of $\vec{a}$ is constant, and by analogy with (2) we have

$$a = \frac{2\pi v}{T} \qquad (4)$$

Eliminating T from (2) and (4) gives

$$a = \frac{2\pi v}{2\pi r/v} = \frac{v^2}{r} \qquad (5)$$

which is just the radial, or centripetal acceleration!

Method #6 - Tulsian's method

simply write $\vec{r}(t) = r\cos\omega t\,\hat{x} + r\sin\omega t\,\hat{y}$
and take derivatives!

$$\vec{v}(t) = -r\omega\sin\omega t\,\hat{x} + r\omega\cos\omega t\,\hat{y}$$

$$\vec{a}(t) = -r\omega^2\cos\omega t\,\hat{x} - r\omega^2\sin\omega t\,\hat{y}$$
$$= -\omega^2\vec{r}$$

$$\boxed{\vec{a} = -\omega^2\vec{r}}$$

This also has the same information.
simple and straight forward!

"Proof" $F \sim 1/r^2$ from 1) Kepler's 3rd Law

2) Huygen's centripetal accel $a_c = v^2/r$

(from Henry "Circular Motion" AJP 68 (7) 637 (2000))

1. K3 $\quad T^2 \propto R^3 \quad$ (today: $T^2 = \left(\frac{4\pi^2}{GM}\right) R^3$)

consider two planets $\quad m \quad M$

in circular orbits of radius $\quad r \quad R$

and orbital period $\quad t \quad T \qquad t^2 = \left(\frac{4\pi^2}{GM}\right) r^3$

$$\Rightarrow \frac{T^2}{t^2} = \frac{R^3}{r^3}$$

NOTE: $F = ma$, Newton's 2nd Law, was already known by Galileo

2. Huygens $\quad \frac{f}{F} = \left(\frac{mv^2}{r}\right)\left(\frac{R}{MV^2}\right)$ $\quad$ of course, the force is centripetal in direction

$$= \frac{mr}{t^2}\frac{T^2}{MR} \qquad \text{and } v = \frac{2\pi r}{t}$$

3. eliminating the orbital periods t, T gives

$$\frac{f}{F} = \frac{(m/r^2)}{(M/R^2)} \Rightarrow \boxed{f \propto \frac{m}{r^2}}$$

Today, we know $GM_\odot$ is the proportionality constant

Note: This was done by Halley, Wren, Hooke, but only Hooke took credit for discovery. Newton could deduce K3 from $f \propto \frac{m}{r^2}$, which Halley & Wren recognized as true discovery.